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«Moscow Institute of Physics and Technology» (National Research University)
Landau Phystech School of Physics and Research
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Landau Institute for Theoretical Physics (Russian Academy of Sciences)

**Major of study**: 03.03.01 Applied Mathematics and Physics
**Academic specialization**: General and Applied Physics

# EFFECT OF SUPERCONDUCTIVITY ON NON-UNIFORM MAGNETIZATION IN DIRTY SF JUNCTIONS

(Bachelor's thesis)

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Moscow 2023

# Annotation

We consider a junction between a bulk superconductor and a thin ferromagnetic layer on its surface (SF junction) with nonuniform magnetization. We assume the junction is in the dirty limit, the ferromagnetic layer is sufficiently thin, and a tunnel boundary between the superconductor and the ferromagnet. These assumptions allow us to describe the hybrid system in the framework of the 2D Usadel equation. As a result of the competition of two effects — ferromagnet stiffness and penetration of Cooper pairs into the ferromagnet — a second-order phase transition occurs.

We have minimized the free energy of the system allowing for an arbitrary nonuniform magnetic state and constructed a Landau functional expanding the free energy in powers of magnetization gradients. This calculation establishes conditions for the phase transition between uniform and helical magnetic states. In particular, we have observed a quite unexpected "resonance" phenomenon: when the exchange energy of the ferromagnet equals the proximity-induced superconducting order parameter, transition to the nonuniform magnetic state occurs irrespective of the value of ferromagnetic stiffness.

In addition to describing the phase transition, our method also allows exploring a general case of arbitrary system parameters for helical state. In particular, we can determine the magnitude of the helical state wave vector far from the phase transition.

# Contents

# 1 Introduction

Nonuniform magnetic systems and in particular their dynamics is a fundamental topic of a great importance both scientifically and for countless practical applications. One of the most impressive examples in this field is the phenomenon of giant magnetoresistence [1, 2], an effect whose discovery was awarded with the Nobel Prize in 2007 and that lies at the core of the modern hard drive technology. Another and more recent breakthrough in the physics of dynamics of magnetic systems is the invention of the magnetic racetrack memory [3, 4].

Our work is devoted to the study of possible emergence of nonuniform magnetic order in hybrid structures involving ferromagnets and superconductors. A pristine ferromagnet is characterized by a uniform magnetic order of local moments while a superconductor hosts a condensate of Cooper pairs. When the two materials are brought into contact (SF junction) Cooper pairs enter into ferromagnet trying to disturb its uniform magnetic order in the vicinity of the interface. This phenomenon is known as the proximity effect. At the same time, a reverse effect of the ferromagnet on the superconductor leads to depletion of the condensate near the boundary. This is the essence of the inverse proximity effect. In the present work we will study a possible appearance of a nonuniform magnetization due to proximity to a superconductor while the inverse proximity effect will be neglected.

Microscopic description of superconductivity is conveniently provided by the Gor'kov equations [5] in terms of the Green functions. For metallic materials with high density of carriers, these equations can be simplified by taking advantage of the quasiclassical approximation. This leads to the Eilenberger equation [6] in terms of the quasiclassical Green function. Finally, if the material has a great amount of impurities and the underlying electron dynamics is diffusive the equation can be further simplified by averaging the quasiclassical Green function over the Fermi surface. This results in the Usadel equation [7]. Our work will be accomplished within this approximation. We thus assume the system is in the dirty limit $\Delta\tau \gg 1$, where $\Delta$ is the superconducting order parameter and $\tau$ is the electron mean free path. Dirty limit is exactly what is required for the validity of the Usadel equation. A weaker quasiclassical condition $E_F\tau \gg 1$ is fullfilled automatically since naturally $E_F \gg \Delta$, where $E_F$ is the Fermi energy.

Cooper pairs in the superconductor are in the singlet state with respect to electron spins. This means that two electrons forming a Cooper pair have antiparallel spin projections on to any direction. When such a Cooper pair enters the ferromagnet, electron spins participate in the exchange interaction with local magnetic moments of the ferromagnet. Since all such moments are naturally aligned in the same direction, exchange with Cooper pair electrons will necessarily lead to an attempt of spin flip. At the same time, the ferromagnet has its own magnetic stiffness that tends to align all localized moments in a single direction. Competition of these two effects may result in establishing a nonuniform magnetic if the proximity effect appears to be strong enough. Otherwise, magnetization will remain uniform. We thus anticipate some kind of a phase transition between these two states. Study of this possible phase transition is the main objective of our work.

A similar problem of the possible phase transition into a nonuniform magnetic state was considered in Ref. [8]. The authors of this work have indeed found that a second order phase transition occurs under certain conditions. However, in Ref. [8] the SF junction was considered close to the critical temperature of the superconductor. Moreover, inverse proximity effect was also taken into account while the overall electron dynamics was assumed ballistic. The latter condition required the use of Eilenberger equations which resulted in a very complicated model. We will study the same problem in a much simpler setting assuming the dirty limit, low temperatures, and disregarding the inverse proximity effect of the ferromagnet on the superconductor. As will be shown, these simplified conditions are sufficient to establish a similar phase transition into a nonuniformly ordered magnetic state. At the same time, the problem can be studied in greater detail also well beyond the transition and deep in the magnetically modulated phase.

Main material of the work is organized into three chapters. Chapter 2 contains a detailed formulation of the problem including the geometry of the SF junction and main equations describing it. In Chapter 3, we study a possible phase transition into nonuniform magnetic state assuming inhomogeneity of magnetization to be weak. Chapter 4 is devoted to the well developed nonuniform state far beyond the phase transition. We identify several limiting forms of the effect in this case. Finally, the work is concluded with Chapter 5 where all the results are summarized.

# 2 Statement of the problem

Let us consider a hybrid system consisting of a bulk superconductor with a ferromagnetic layer attached to its surface. We will assume ferromagnetic layer is thin enough to consider it as a two-dimensional metal (exact criteria will be given below). Ferromagnet is characterized by the local vector of its exchange field $\mathbf{h}$, which we will assume to be constant in absolute value. We assume tunneling boundary conditions between the superconductor and the ferromagnet, which allows us to disregard the inverse proximity effect of the ferromagnet on to the superconductor.

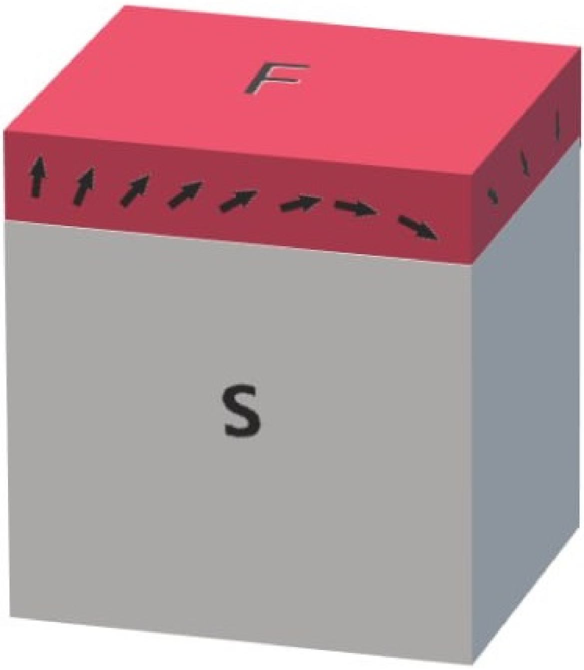


Figure 2.1: SF junction

To describe superconductivity, we will use the Gor'kov-Nambu matrix Green functions $g_F$ and $g_S$ (for the ferromagnet and superconductor, respectively), containing normal and anomalous components. Due to the influence of the exchange field, a triplet component appears in these Green functions, which is why the matrices $g_F$ and $g_S$ must be extended to the tensor product of the Gor'kov-Nambu space and spin space. For definiteness, we will denote the basis of Pauli matrices as $\boldsymbol{\tau}$ in the Gor'kov-Nambu space and $\boldsymbol{\sigma}$ in the spin space.

We can write the total free energy of the system as follows:

$$F_{\text{total}} = F_0 + F_S + F_{FS} + F_B, \tag{2.1}$$

where $F_0$ is the proper ferromagnet energy (without proximity effect), $F_S$ is the free energy of the bulk superconductor, $F_{FS}$ is the energy of induced superconductivity in the ferromagnet, and $F_B$ is the boundary energy.

As the free energy of the ferromagnet, we take an isotropic model with magnetic stiffness:

$$F_0 = \int d^2r\, \zeta\, (\nabla_i n_j)^2, \tag{2.2}$$

where $\mathbf{n} = \mathbf{h}/h$ is a unit vector in the direction of exchange field[1] and $\zeta > 0$ is a magnetic stiffness[2]. This energy favors a uniform order when the gradients of $\mathbf{n}$ vanish.

Since the inverse effect of the ferromagnet on the superconductor is weak, we can assume that the Green function $g_S$ of the superconductor is determined only by the minimum of $F_S$. In this formulation of the problem, $g_S$ can be fixed to be equal to the standard BCS expression

$$g_S = \frac{\epsilon\tau_z + \Delta_S(\tau_x\cos\phi + \tau_y\sin\phi)}{\sqrt{\epsilon^2+\Delta_S^2}}, \tag{2.3}$$

where $\epsilon$ is the Matsubara energy, $\Delta_S$ and $\phi$ are the modulus and the phase of the order parameter in the superconductor.

Tunneling boundary conditions are described by adding to the free energy a term [10, 11]

$$F_B = -\frac{\pi G_T}{4}\int d^2r \int\limits_0^\infty \frac{d\epsilon}{\pi}\,\mathrm{tr}[g_F\, g_S], \tag{2.4}$$

where $G_T$ is the normal conductance per unit area of the interface measured in the units of $e^2/\hbar$. In terms of this boundary conductance, we will define a parameter

$$\Delta = \frac{G_T}{4\nu}, \tag{2.5}$$

that will play a role of the effective proximity-induced "order parameter". Here $\nu$ is the 2D density of states in the ferromagnet per one spin component.

We assume a "dirty limit" in the ferromagnet $\Delta\tau \ll 1$, where $\tau$ is the mean free time for electrons. In the dirty limit, the electronic system is governed by the Usadel equation [7, 12, 13] that minimizes the following free energy:

$$F_{FS} = \pi\nu\int d^2r\int_0^\infty\frac{d\epsilon}{\pi}\,\mathrm{tr}\left[\frac{D}{4}(\nabla g_F)^2 - (\epsilon\tau_z + i\mathbf{h}\boldsymbol{\sigma}\tau_z)g_F\right] \tag{2.6}$$

with the condition $g_F^2 = \mathbb{1}$. Adding the free energy (2.4) for the interface, we get:

$$S \equiv F_{FS} + F_B = \pi\nu\int d^2r\int_0^\infty\frac{d\epsilon}{\pi}\,\mathrm{tr}\left[\frac{D}{4}(\nabla g)^2 - (\epsilon\tau_z + i\mathbf{h}\boldsymbol{\sigma}\tau_z + \Delta g_S(\epsilon))g\right]. \tag{2.7}$$

As we will see below, the free energy is determined by energies less or of the order of $\Delta$. Hence in the limit $\Delta \ll \Delta_S$, we can set $g_S(\epsilon) \approx g_S(0)$. The proximity effect of the superconductor on the ferromagnet will then be described by the free energy functional

$$S = \pi\nu\int d^2r\int_0^\infty\frac{d\epsilon}{\pi}\,\mathrm{tr}\left[\frac{D}{4}(\nabla g)^2 - (\epsilon\tau_z + i\mathbf{h}\boldsymbol{\sigma}\tau_z + \Delta\tau_x\cos\phi + \Delta\tau_y\sin\phi)g\right] \tag{2.8}$$

which we will refer to as "action". Here $\Delta$ plays a role of $\Delta_S$ for the ferromagnet, which explains its name. Possible appearance of an inhomogeneous phase is precisely due to this

[1] Here and below, we use Einstein summation convention over repeated indices $(\nabla_i n_j)^2 = (\nabla n_x)^2 + (\nabla n_y)^2 + (\nabla n_z)^2$.

[2] The values of stiffness per unit volume $\zeta/d$ for various ferromagnets can be found, for example, in Ref. [9].

term. Since the system has only one bulk superconducting lead, we can further simplify the action by choosing the gauge with $\phi = 0$. That is, we assume the effective order parameter to be real.

In the following sections, we will study the total free energy $F = F_0 + S$ in more detail to describe the inhomogeneous magnetic phase and conditions for its occurrence. They will be formulated in terms of two dimensionless parameters of the system $h/\Delta$ and $\zeta/(\nu D\Delta)$.

# 3 Phase transition to inhomogeneous state

## 3.1 Rotation in the Gor'kov-Nambu space

The matrix Green function $g$ operates in the Gor'kov-Nambu and spin spaces and obeys the constraint $g^2 = \mathbb{1}$. Hence it has exactly two eigenvalues $+1$ and two $-1$ and can be represented as [6, 7, 12]

$$g = T^{-1}\tau_z T. \tag{3.1}$$

The matrix $T$ must be invertible and hence $T \in \mathrm{GL}(4, \mathbb{C})$ and contains 16 complex parameters. However, without loss of generality, we can restrict it to be unitary $T \in U(4)$ having 16 real parameters instead. If later the solution to the Usadel equation happens to be complex in some sectors it will automatically extend $T$ to the full general linear group.

Note that $\tau_z$ is invariant under rotations by block-diagonal matrices $T$ in the spin space. The matrix $g$ does not change if $T$ is multiplied from the left by such a block-diagonal matrix. Hence $g$ realizes a representation of the left coset space $\mathrm{U}(4)/\mathrm{U}(2) \times \mathrm{U}(2)$ and contains in general eight parameters (real, or complex if required by the Usadel equation).

We choose a gauge with $\phi = 0$ and $\tau_y$ no longer appears explicitly in the action (2.8). The only matrices from the Gor'kov-Nambu space that remain in the action ($\tau_z$ and $\tau_x$) anticommute with $\tau_y$. This means that the solution that minimizes the action will also anticommute with $\tau_y$. To demonstrate this, we decompose $g$ into two parts $g = g_+ + g_-$, where

$$g_\pm = \frac{g \mp \tau_y g \tau_y}{2} \quad \Rightarrow \quad \begin{aligned} [g_-, \tau_y] = 0 \\ \{g_+, \tau_y\} = 0 \end{aligned} \tag{3.2}$$

Substituting them into action (2.8), we use the following identities:

$$\mathrm{tr}[g_+ g_-] = \frac{1}{4}\mathrm{tr}[g^2 - (\tau_y g \tau_y)^2] = 0, \tag{3.3}$$

$$\mathrm{tr}[(\nabla(g_+ + g_-))^2] = \mathrm{tr}[(\nabla g_+)^2 + 2(\nabla g_+)(\nabla g_-) + (\nabla g_-)^2] = \mathrm{tr}[(\nabla g_+)^2 + (\nabla g_-)^2], \tag{3.4}$$

$$\mathrm{tr}[(\epsilon\tau_z + i\mathbf{h}\boldsymbol{\sigma}\tau_z + \Delta\tau_x)g_-] = \frac{1}{2}\mathrm{tr}[\{\epsilon\tau_z + i\mathbf{h}\boldsymbol{\sigma}\tau_z + \Delta\tau_x, \tau_y\}g_-\tau_y] = 0. \tag{3.5}$$

This shows that the terms $g_+$ and $g_-$ are separated in the action, and for $g_-$ the potential term also vanishes. In the corresponding solution we thus have $g_- = 0$ and hence $\{g, \tau_y\} = 0$.

To take advantage of this property, we will rotate the basis in the Gor'kov-Nambu space such that $\tau_y \mapsto \tau_z \mapsto \tau_x \mapsto \tau_y$. Then $\{g, \tau_z\} = 0$, which implies that $g$ has a block-off-diagonal form in the new basis. From the condition $g^2 = \mathbb{1}$ it follows that the blocks are inverse to each other:

$$g = \begin{pmatrix} 0 & U^{-1} \\ U & 0 \end{pmatrix}. \tag{3.6}$$

We have thus reduced the number of parameters from eight to only four [14]. They are contained in a smaller matrix $U \in \mathrm{U}(2)$, that operates only in the spin space.

Usadel action in terms of $U$ is

$$S = \pi\nu \int d^2r \int_0^\infty \frac{d\epsilon}{\pi} \operatorname{tr}\left[\frac{D}{2}(\nabla U^{-1})(\nabla U) - (\epsilon + i\mathbf{h}\boldsymbol{\sigma} - i\Delta)U - (\epsilon + i\mathbf{h}\boldsymbol{\sigma} + i\Delta)U^{-1}\right]. \quad (3.7)$$

## 3.2 Rotation in the spin space

In the case of homogeneous magnetization, we can choose the quantization axis in the spin space along $\mathbf{h}$. The Usadel equation is then factorized into two independent equations for the two spin projections. For a general $\mathbf{h}$, that varies in space, we introduce a unitary rotation matrix $W$ and define it such that it locally rotates the spin basis keeping the $z$ axis along $\mathbf{h}$.

$$\mathbf{h}\boldsymbol{\sigma} = hW^{-1}\sigma_z W, \qquad U = W^{-1}QW. \quad (3.8)$$

Here we also define the matrix $Q$ that differs from $U$ by the same rotation.

In these new variables, the term $\mathbf{h}\boldsymbol{\sigma}$ in the action (3.7) will be replaced by $h\sigma_z$. However, since the matrix $W$ depends on the point in space, the gradient terms in the action will also be modified. More specifically, we can rewrite the gradient in terms of the long derivative $\mathcal{D}$

$$\nabla\big(W^{-1}QW\big) = W^{-1}(\mathcal{D}Q)W, \quad (3.9)$$

defined as follows:

$$\mathcal{D} = \nabla + i[\mathbf{A}, \cdot\,], \qquad \mathbf{A} = i\nabla W W^{-1}. \quad (3.10)$$

Note that from the properties of the commutator, it follows that the action of $\mathcal{D}$ on matrices obeys the Leibniz rule. Also, since the action involves both the matrix trace and the real space integral, the terms with long derivatives $\mathcal{D}$ can be integrated by parts in the standard way.

Physically, the vector $\mathbf{A}$ (whose components are Hermitian matrices) plays the role of a non-Abelian vector potential. It characterizes the speed of rotation of the magnetization in space. Close to the phase transition into a nonuniform state, the magnitude of this vector is small.

In the introduced rotating frame, the action becomes

$$S = \pi\nu \int d^2r \int_0^\infty \frac{d\epsilon}{\pi} \operatorname{tr}\left[\frac{D}{2}(\mathcal{D}Q^{-1})(\mathcal{D}Q) - (\epsilon + ih\sigma_z - i\Delta)Q - (\epsilon + ih\sigma_z + i\Delta)Q^{-1}\right]. \quad (3.11)$$

## 3.3 Uniform state

In the uniform magnetic state, $Q$ is constant in space. Therefore, we can neglect the kinetic term completely. Since the remaining "potential" part of the action explicitly contains only the diagonal matrix $\sigma_z$, it will be minimized by a diagonal matrix $Q_0$. This matrix obeys the following equation:

$$Q_0(\epsilon + ih\sigma_z - i\Delta) = (\epsilon + ih\sigma_z + i\Delta)Q_0^{-1}. \quad (3.12)$$

The solution is

$$Q_0 = \begin{pmatrix} \sqrt{\dfrac{\epsilon + ih + i\Delta}{\epsilon + ih - i\Delta}} & 0 \\ 0 & \sqrt{\dfrac{\epsilon - ih + i\Delta}{\epsilon - ih - i\Delta}} \end{pmatrix}. \quad (3.13)$$

It is convenient to use the following parametrization:

$$Q_0 = \begin{pmatrix} e^{i\theta+m} & 0 \\ 0 & e^{i\theta-m} \end{pmatrix} \tag{3.14}$$

where $\theta$ and $m$ are

$$m = \frac{1}{2}\operatorname{arctanh}\left(\frac{2h\Delta}{\epsilon^2+h^2+\Delta^2}\right), \qquad \theta = \frac{1}{2}\arctan\left(\frac{2\epsilon\Delta}{\epsilon^2+h^2-\Delta^2}\right). \tag{3.15}$$

## 3.4 Perturbation theory: first and second order

Close to the transition into a nonuniform state, gradients of $Q$ are small. The vector potential $\mathbf{A}$ is also small in this case. Hence the whole kinetic term $(\mathcal{D}Q^{-1})(\mathcal{D}Q)$ in the action can be treated as a small perturbation near the anticipated phase transition. Since $Q$ belongs to the unitary group U(2) in the spin space, small deviation of $Q$ from $Q_0$ can be parameterized using a small matrix $X$ as

$$Q = Q_0 e^X = Q_0(\mathbb{1} + X + X^2/2 + O(X^3)). \tag{3.16}$$

As will be seen later, the matrix $X$ is of the order of $\mathbf{A}^2$. At the same time, acting by the long derivative $\mathcal{D}$ adds one more order of smallness in $\mathbf{A}$ to the expression. For this reason, we expand the action to the order $X^2$ and to the linear order in $X$ within the gradient term. Specifically, the gradient term gives

$$\mathcal{D}Q = (\mathcal{D}Q_0)(\mathbb{1} + X) + Q_0(\mathcal{D}X) + O(\mathbf{A}^4), \tag{3.17}$$

$$\mathcal{D}Q^{-1} = (\mathbb{1} - X)(\mathcal{D}Q_0^{-1}) - (\mathcal{D}X)Q_0^{-1} + O(\mathbf{A}^4), \tag{3.18}$$

hence

$$\begin{aligned}\mathcal{D}Q^{-1}\mathcal{D}Q = \mathcal{D}Q_0^{-1}\mathcal{D}Q_0 + [\mathcal{D}Q_0^{-1}\mathcal{D}Q_0, X] + (\mathcal{D}Q_0^{-1})Q_0(\mathcal{D}X) - \\ - (\mathcal{D}X)Q_0^{-1}(\mathcal{D}Q_0) + O(\mathbf{A}^6).\end{aligned} \tag{3.19}$$

Commutator term vanishes after taking a trace, which yields

$$\begin{aligned}\operatorname{tr}\bigl[\mathcal{D}Q^{-1}\mathcal{D}Q\bigr] = \operatorname{tr}\bigl[\mathcal{D}Q_0^{-1}\mathcal{D}Q_0 + (\mathcal{D}X)\bigl((\mathcal{D}Q_0^{-1})Q_0 - Q_0^{-1}(\mathcal{D}Q_0)\bigr)\bigr] = \\ = 2\operatorname{tr}\bigl[\mathbf{A}(\mathbf{A} - Q_0^{-1}\mathbf{A}Q_0) + i(\mathcal{D}X)(\mathbf{A} - Q_0^{-1}\mathbf{A}Q_0)\bigr].\end{aligned} \tag{3.20}$$

It is also necessary to expand the potential part:

$$\operatorname{tr}\bigl[-(\epsilon + ih\sigma_z - i\Delta)Q - (\epsilon + ih\sigma_z + i\Delta)Q^{-1}\bigr] = 2\operatorname{tr}\left[-\Lambda\left(\mathbb{1} + \frac{X^2}{2}\right)\right] \tag{3.21}$$

where

$$\Lambda = \operatorname{diag}\{\lambda, \lambda^*\} \equiv \sqrt{(\epsilon + ih\sigma_z)^2 + \Delta^2}. \tag{3.22}$$

Summarizing, the final action takes the form

$$S = \pi\nu \int d^2r \int_0^\infty \frac{d\epsilon}{\pi} \operatorname{tr}\bigl[D\mathbf{A}(\mathbf{A} - Q_0^{-1}\mathbf{A}Q_0) - iDX\mathcal{D}(\mathbf{A} - Q_0^{-1}\mathbf{A}Q_0) - X^2\Lambda\bigr], \tag{3.23}$$

where we have used integration by parts for the term linear in $\mathcal{D}X$. Minimizing this action with respect to $X$, we obtain a linear equation

$$\{X, \Lambda\} = -iD\mathcal{D}(\mathbf{A} - Q_0^{-1}\mathbf{A}Q_0). \tag{3.24}$$

Explicitly solving this equation in terms of matrix elements of $\mathbf{A}$, we get

$$X = D \begin{pmatrix} -\dfrac{\sinh(2m)}{\lambda}|\mathbf{A}_{12}|^2 & \dfrac{1-e^{-2m}}{\lambda+\lambda^*}(\mathbf{A}_{11}-\mathbf{A}_{22}-i\nabla)\mathbf{A}_{12} \\ \dfrac{e^{2m}-1}{\lambda+\lambda^*}(\mathbf{A}_{11}-\mathbf{A}_{22}+i\nabla)\mathbf{A}_{12}^* & \dfrac{\sinh(2m)}{\lambda^*}|\mathbf{A}_{12}|^2 \end{pmatrix}. \tag{3.25}$$

This solution is indeed of the second order in $\mathbf{A}$ and linear in $\nabla\mathbf{A}$.

### 3.4.1 Effective action for exchange field

Once the dependence $Q(\mathbf{A})$ is established, we will derive the free energy of the system as a functional of $\mathbf{A}$. To do this, we simplify the action (3.23) by using the equation of motion (3.24):

$$S = \pi\nu \int d^2r \int_0^\infty \frac{d\epsilon}{\pi} \operatorname{tr}\left[D\mathbf{A}(\mathbf{A}-Q_0^{-1}\mathbf{A}Q_0) + X(\mathbf{A})^2\Lambda\right]. \tag{3.26}$$

After substitution $X = X(\mathbf{A})$ from Eq. (3.25), we have:

$$S = \nu \int d^2r \left[-4C_1\Delta D|\mathbf{A}_{12}|^2 + 8C_2D^2|\mathbf{A}_{12}|^4 + 2C_3D^2|(\mathbf{A}_{11}-\mathbf{A}_{22}-i\nabla)\mathbf{A}_{12}|^2\right], \tag{3.27}$$

where

$$C_1 = \int\limits_0^{+\infty} \frac{d\epsilon}{\Delta}\sinh^2 m, \quad C_2 = \frac{1}{8}\int\limits_0^{+\infty} d\epsilon \sinh^2(2m)\left(\frac{1}{\lambda}+\frac{1}{\lambda^*}\right), \quad C_3 = 2\int\limits_0^{+\infty} d\epsilon \frac{\sinh^2 m}{\lambda+\lambda^*} \tag{3.28}$$

are dimensionless functions of $h/\Delta$.

We would like to rewrite the action in terms of magnetization gradients. Since our model does not include spin-orbit interaction, rotations in the spin space are decoupled from rotations in the coordinate space. Thus, the only symmetry-allowed quadratic term in gradients has the form $(\nabla_i n_j)^2$. In the next, fourth order there are two possible terms: the square of the quadratic term mentioned above and the term $(\nabla^2 n_j)^2$. All these terms can be expressed through $\mathbf{A}_{ij}$ as follows:

$$(\nabla_i n_j)^2 = \frac{1}{2}\operatorname{tr}\left[(\nabla_i \mathbf{n}\boldsymbol{\sigma})^2\right] = \frac{1}{2}\operatorname{tr}\left[(\mathcal{D}_i\sigma_z)^2\right] = 4|\mathbf{A}_{12}|^2, \tag{3.29}$$

$$(\nabla^2 n_j)^2 = \frac{1}{2}\operatorname{tr}\left[(\mathcal{D}^2\sigma_z)^2\right] = 16|\mathbf{A}_{12}|^4 + 4|(\mathbf{A}_{11}-\mathbf{A}_{22}-i\nabla)\mathbf{A}_{12}|^2. \tag{3.30}$$

This shows that the action (3.27) can be written directly as a functional of $\mathbf{n}$:

$$S = \int d^2r \left[-\nu D\Delta C_1(\nabla_i n_j)^2 + \frac{\nu D^2C_2}{2}(\nabla_i n_j)^4 + \frac{\nu D^2 C_3}{2}\left((\nabla^2 n_j)^2 - (\nabla_i n_j)^4\right)\right]. \tag{3.31}$$

Together with the proper ferromagnetic energy (2.2), this yields the total free energy of the system:

$$F = \int d^2r \left[(\zeta - \nu D\Delta C_1)(\nabla_i n_j)^2 + \frac{\nu D^2C_2}{2}(\nabla_i n_j)^4 + \frac{\nu D^2 C_3}{2}\left((\nabla^2 n_j)^2 - (\nabla_i n_j)^4\right)\right]. \tag{3.32}$$

### 3.5 Functions $C_{1,2,3}$

We will now study in detail the functions $C_{1,2,3}$ defined by Eqs. (3.28). After substitution of $m$ and $\lambda$ from Eqs. (3.15) and (3.22), we have the following expressions:

$$C_1\left(\frac{h}{\Delta}\right) = \frac{1}{2}\int_0^{+\infty}\frac{d\epsilon}{\Delta}\left(\frac{\epsilon^2+\Delta^2+h^2}{\sqrt{(\epsilon^2+\Delta^2+h^2)^2-4h^2}}-1\right), \tag{3.33}$$

$$C_2\left(\frac{h}{\Delta}\right) = \frac{1}{16}\int_0^{+\infty}\frac{d\epsilon}{\Delta}\left(\frac{\epsilon^2+(\Delta+h)^2}{\epsilon^2+(\Delta-h)^2}+\frac{\epsilon^2+(\Delta-h)^2}{\epsilon^2+(\Delta+h)^2}-2\right)\cdot \mathrm{Re}\,\frac{\Delta}{\sqrt{(\epsilon+ih)^2+\Delta^2}}, \tag{3.34}$$

$$C_3\left(\frac{h}{\Delta}\right) = \frac{1}{2}\int_0^{+\infty}\frac{d\epsilon}{\Delta}\left(\frac{\epsilon^2+\Delta^2+h^2}{\sqrt{(\epsilon^2+\Delta^2+h^2)^2-4h^2}}-1\right)\cdot\frac{\Delta}{\mathrm{Re}\,\sqrt{(\epsilon+ih)^2+\Delta^2}}. \tag{3.35}$$

All of them are significantly simplified in terms of a new integration variable

$$z = \frac{1}{2h\Delta}\left(\epsilon^2+\Delta^2+h^2-\sqrt{(\epsilon^2+\Delta^2+h^2)^2-4h^2}\right). \tag{3.36}$$

Explicitly,

$$C_1(x) = \frac{x}{2}\int_0^{\min\left(x,\frac{1}{x}\right)}\frac{\sqrt{z}dz}{\sqrt{(z^2+1)x-(x^2+1)z}}, \tag{3.37}$$

$$C_2(x) = \sqrt{x}\int_0^{\min\left(x,\frac{1}{x}\right)}\frac{z\,dz}{(1-z^2)^2}\cdot\frac{1}{\sqrt{x-z}}, \tag{3.38}$$

$$C_3(x) = \frac{\sqrt{x}}{2}\int_0^{\min\left(x,\frac{1}{x}\right)}\frac{z\,dz}{1-zx}\cdot\frac{1}{\sqrt{x-z}} \tag{3.39}$$

The integration is now straightforward and yields the following results.

#### 3.5.1 $C_1$

The function $C_1$ can be expressed in terms of complete elliptic integrals of the first and second kind:

$$C_1(x) = \begin{cases} K(x)-E(x), & x<1,\\ x\left(K\left(\frac{1}{x}\right)-E\left(\frac{1}{x}\right)\right), & x>1,\end{cases} \tag{3.40}$$

where

$$K(x) = \int_0^1\frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}, \qquad E(x) = \int_0^1\frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}}dt. \tag{3.41}$$

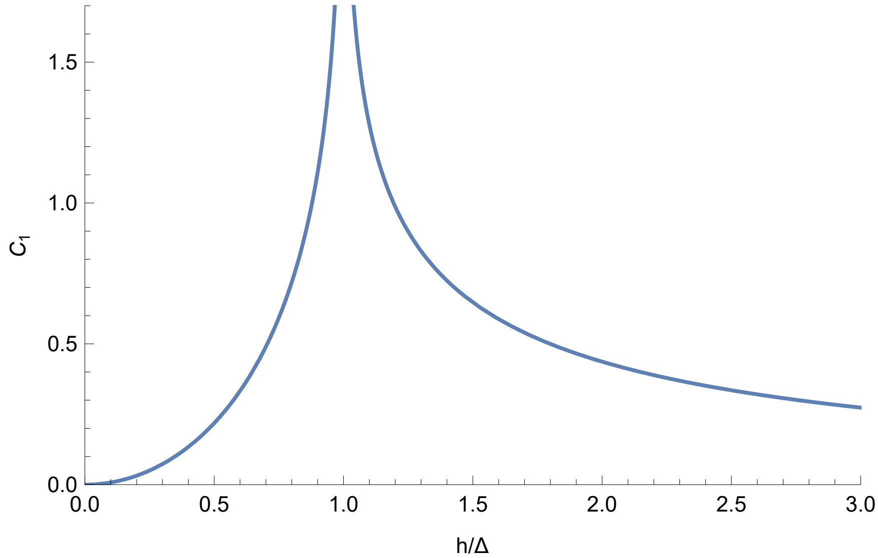


Figure 3.1: Dependence $C_1(x)$ where $x = h/\Delta$.

The result for $C_1$ is shown in Fig. 3.1. It has a logarithmic divergence at the point $h = \Delta$. Various asymptotics of $C_1$ are given by the following expressions:

$$C_1(x) \to \begin{cases} \frac{\pi}{4} x^2, & x \to 0 \\ -\frac{1}{2} \ln|1 - x|, & x \to 1 \\ \frac{\pi}{4} \frac{1}{x}, & x \to \infty \end{cases} \tag{3.42}$$

### 3.5.2 $C_2$

The function $C_2(x)$ can be expressed in terms of elementary functions

$$C_2(x) = \frac{1}{4} \begin{cases} \frac{x^2}{1-x^2} + \frac{\sqrt{x}}{2} \left( \frac{\arcsin\sqrt{x}}{(1-x)^{3/2}} - \frac{\operatorname{arcsinh}\sqrt{x}}{(1+x)^{3/2}} \right), & x < 1, \\ \frac{x}{x^2-1} \left( \frac{x^2+1}{\sqrt{x^2-1}} - x \right) \\ \quad + \frac{\sqrt{x}}{2} \left( \frac{1}{(x-1)^{3/2}} + \frac{1}{(1+x)^{3/2}} \right) \left( \operatorname{arccosh}\sqrt{x} - \operatorname{arcsinh}\sqrt{x} \right), & x > 1. \end{cases} \tag{3.43}$$

It is shown in Fig. 3.2. Different asymptotics of this function are

$$C_2(x) \to \begin{cases} \frac{2}{3} x^2, & x \to 0, \\ \frac{\pi}{16} \frac{1}{(1-x)^{3/2}}, & x \to 1 - 0, \\ \frac{\sqrt{2} - \operatorname{arcsinh}(1)}{8} \frac{1}{(x-1)^{3/2}}, & x \to 1 + 0, \\ \frac{1}{4} \frac{1}{x^2}, & x \to +\infty. \end{cases} \tag{3.44}$$

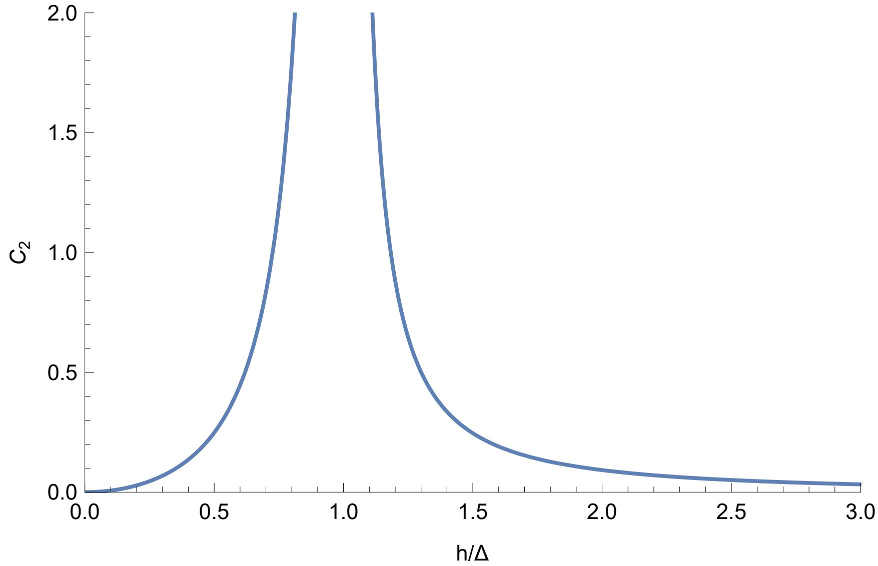


Figure 3.2: Dependence $C_2(x)$ where $x = h/\Delta$.

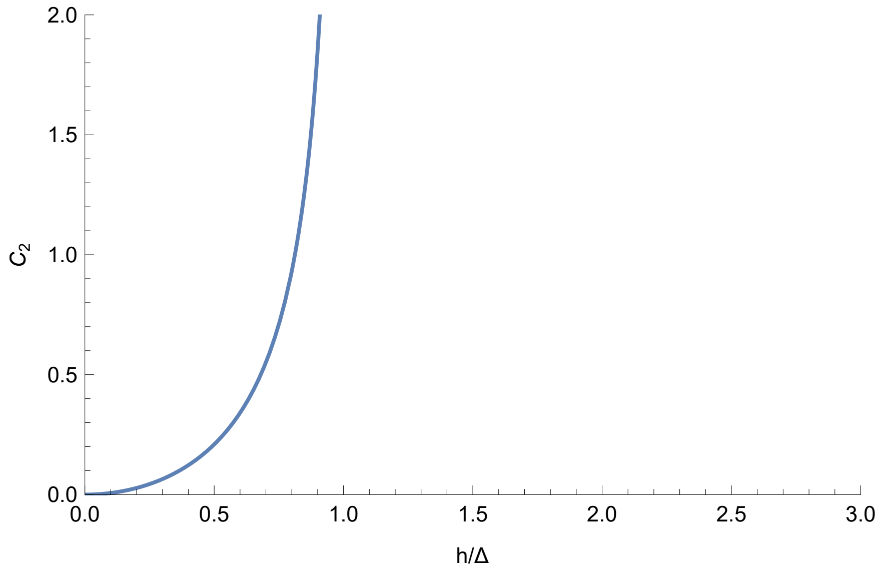


Figure 3.3: Dependence $C_3(x)$ where $x = h/\Delta$

### 3.5.3 $C_3$

Unlike $C_{1,2}$, the integral $C_3$ diverges everywhere in the region $h > \Delta$. This divergence is logarithmic and occurs at the lower limit of the integral. It can be removed if we generalize our theory to finite temperatures. In this case, integration over Matsubara energy $\epsilon$ will be replaced by a summation over discrete values $\epsilon = \pi T(2n + 1)$. Hence the lower limit will be effectively replaced by $\pi T$. The result of integration is thus

$$C_3(x) = \begin{cases} \dfrac{\arcsin x}{x\sqrt{1-x^2}} - 1, & x < 1, \\ \sim \ln\left(\dfrac{\Delta}{T}\right) \gg 1, & x > 1. \end{cases} \tag{3.45}$$

The function is shown in Fig. 3.3.

In what follows, we will show that for the most important case of the helical state, the value of $C_3$ does not play any significant role, so the apparent divergence does not have physical consequences.

Asymptotics of $C_3$ in the region $h < \Delta$ are

$$C_3(x) \to \begin{cases} \dfrac{2}{3}x^2, & x \to 0, \\ \dfrac{\pi}{2\sqrt{2}} \dfrac{1}{\sqrt{1-x}}, & x \to 1-0. \end{cases} \tag{3.46}$$

## 3.6 Helical state

Let us consider a particular case of a nonuniform magnetic state — the so called "helical state", in which the magnetization rotates in space with a constant velocity. The direction of magnetization can be parameterized as follows

$$\mathbf{n} = \begin{pmatrix} \sin\theta_h \cos(qx) \\ \sin\theta_h \sin(qx) \\ \cos\theta_h \end{pmatrix}, \tag{3.47}$$

where $\theta_h$ and $q$ are coordinate-independent parameters. We will determine their values from minimization of free energy. It should be clarified that $\theta_h$ parametrizes only the half-angle of the cone along which the magnetization vector rotates and is not related to the real direction of $\mathbf{h}$ in space. Substituting our helical state into Eq. (3.32), we express the free energy density in terms of $\theta_h$ and $q$:

$$\mathcal{F} = (\zeta - \nu D \Delta C_1) q^2 \sin^2\theta_h + \frac{\nu D^2}{2}\big(C_2 q^4 \sin^4\theta_h + C_3 q^4 \sin^2\theta_h \cos^2\theta_h\big). \tag{3.48}$$

This expression has the form of a Landau functional describing the second order phase transition into a helical state, with the order parameter $q$. A nonuniform state is established when $\nu D \Delta C_1 > \zeta$ and is characterized by:

$$q^2 = \frac{\Delta}{D} \frac{C_1 - \zeta/(\nu D \Delta)}{C_2 \sin^2\theta_h + C_3 \cos^2\theta_h}, \tag{3.49}$$

$$\mathcal{F} = -\frac{(\zeta - \nu D \Delta C_1)^2}{2\nu D^2} \frac{\sin^2\theta_h}{C_2 \sin^2\theta_h + C_3 \cos^2\theta_h}. \tag{3.50}$$

Minimizing this energy with respect to $\theta_h$, we conclude that $\theta_h = \pi/2$ regardless of the value of $C_3$. Moreover, if we had formulated the problem initially for a helical state rotating over a great circle ($\theta_h = \pi/2$), the possibly diverging term $C_3$ would not appear in the formalism.

Our final result for the order parameter $q$ is

$$\begin{cases} q = 0, & C_1 < \dfrac{\zeta}{\nu \Delta D}, \\ q = \sqrt{\dfrac{\Delta}{D}} \sqrt{\dfrac{C_1 - \zeta/\nu D \Delta}{C_2}}, & C_1 > \dfrac{\zeta}{\nu D \Delta}. \end{cases} \tag{3.51}$$

A remarkable feature of this result follows from the fact that $C_1$ diverges at $h = \Delta$. We observe that, irrespective of the value of magnetic stiffness $\zeta$, the phase transition into the helical state will always happen as long as the system is tuned close enough to the "resonance" point $h = \Delta$ in the parameter space.

# 4 Developed helical state

The theory constructed in the previous chapter makes it possible to describe the inhomogeneous ferromagnetic state in the neighbourhood of the phase transition. However, if we consider exclusively the helical state, the theory can be extended to the rest of the parameter range.

It is easy to verify that for the helical state, in the general case, one can seek a solution for $Q$ in the form Eq. (3.14), where $\theta$ and $m$ are constant in space. Using the identity

$$\frac{D}{2}\,\mathrm{tr}(\mathcal{D}Q^{-1}\mathcal{D}Q) = -\frac{D}{2}\,\mathrm{tr}\big([\mathbf{A},Q^{-1}][\mathbf{A},Q]\big) = \\ = D(Q_{11}-Q_{22})\big(Q_{11}^{-1}-Q_{22}^{-1}\big)|\mathbf{A}_{12}|^2 = -4D|\mathbf{A}_{12}|^2\cosh^2 m, \quad (4.1)$$

we rewrite Eq. (3.11) in terms of $\theta$ and $m$:

$$\mathcal{L} = -4\Big[D|\mathbf{A}_{12}|^2\cosh^2 m + \epsilon\cos\theta\cosh m + \Delta\sin\theta\cosh m - h\sin\theta\sinh m\Big], \quad (4.2)$$

where Lagrangian $\mathcal{L}$ is related to the action density $\mathcal{S}$ as follows:

$$S = \int d^2r\,\mathcal{S}(\mathbf{h},\Delta) = \nu\int d^2r\int_0^\infty d\epsilon\,\mathcal{L}(\epsilon,\mathbf{h},\Delta) \quad (4.3)$$

The angle $\theta$ can be easily found from the minimization condition

$$\frac{\partial\mathcal{L}}{\partial\theta} = 4\Big[\epsilon\sin\theta\cosh m - \Delta\cos\theta\cosh m + h\cos\theta\sinh m\Big] = 0. \quad (4.4)$$

Then the Langrangian becomes a function of $m$ only:

$$\mathcal{L} = -4D|\mathbf{A}_{12}|^2\cosh^2 m - 4\sqrt{(\epsilon^2+\Delta^2)\cosh^2 m + h^2\sinh^2 m - \Delta h\sinh 2m}, \quad (4.5)$$

The parameter $m$ is also determined from the condition on minimum of the Lagrange function. Taking derivative of $\mathcal{L}$ from Eq. (4.5) in $m$ and setting it to zero, we obtain

$$2a\sinh 2m\sqrt{x\cosh^2 m - h^2 - h\Delta\sinh 2m} = 2h\Delta\cosh 2m - x\sinh 2m, \quad (4.6)$$

where we denote $x \equiv \epsilon^2+\Delta^2+h^2$ and $a \equiv D|\mathbf{A}_{12}|^2$. The problem of solving this equation on $m$ is equivalent to finding a root of a sixth degree polynomial, which is not possible analytically.

Note, however, that Eq. (4.6) is solvable for $\epsilon$ (or $x$). Squaring both sides, we get a quadratic equation for $x$, the solution of which is given by the expression

$$x(m,a) = 2h\Delta\coth(2m) + 2a^2\cosh^2 m \pm 2a^2\sqrt{\cosh^4 m + \frac{h}{a^2}(\Delta\coth m - h)}. \quad (4.7)$$

Here we should choose "$-$" sign to ensure the right-hand side of Eq. (4.6) is not negative. Additionaly, the same condition requires

$$\tanh m \leq \frac{\Delta}{h}. \tag{4.8}$$

Let us point out that in the limit $\epsilon \to +\infty$, the value of $m$ is $m \sim h\Delta/\epsilon^2 \to 0$. In the opposite limit $\epsilon = 0$ the variable $m$ takes some non-trivial value. We will denote this value by $m_0$. When energy changes from 0 to $\infty$, $m$ varies in the interval $(0,\, m_0]$.

Consider now the original problem. To find the dependence of magnetization gradient $q$ on system parameters, it is necessary to minimize the total free energy with respect to $q$. This is equivalent to minimization with respect to $a = D|\mathbf{A}_{12}|^2 = D(\nabla_i n_j)^2/4 = Dq^2/4$ of the following expression:

$$\mathcal{F} = \nu \int\limits_0^\infty d\epsilon\, \mathcal{L}(a, \epsilon, m(a, \epsilon)) + \frac{4\zeta}{D} a. \tag{4.9}$$

Taking derivative in $a$, we find

$$\frac{d\mathcal{F}}{da} = \nu \int\limits_0^\infty d\epsilon \left( \frac{\partial \mathcal{L}}{\partial a} + \cancelto{0}{\frac{\partial \mathcal{L}}{\partial m}} \frac{dm}{da} \right) + \frac{4\zeta}{D} = 0, \tag{4.10}$$

where term with $\partial\mathcal{L}/\partial m$ vanishes due to the extremity condition (4.6). So the equation for $a$ takes the form:

$$\int\limits_0^\infty d\epsilon\, \sinh^2 m(a, \epsilon) = \frac{\zeta}{\nu D}. \tag{4.11}$$

We now perform a change of variable under the integral and go from integration in $\epsilon$ over to integration in $m$. Subsequent integration by parts yields

$$\int\limits_{m(a,0)}^{m(a,\infty)} dm\, \frac{\partial \epsilon(m, a)}{\partial m} \sinh^2 m = \epsilon \sinh^2 m \Bigg|_{m=m_0}^{m=0} - \int\limits_{m_0}^{0} dm\, \epsilon(m, a) \sinh 2m. \tag{4.12}$$

At the lower limit, the boundary term vanishes due to zero energy, and at the upper limit, due to the identity

$$\lim_{\epsilon\to\infty} \epsilon m^2(a, \epsilon) = \lim_{\epsilon\to\infty} \frac{(h\Delta)^2}{\epsilon^3} = 0. \tag{4.13}$$

In summary, the value of $a$ is to be found from the following equations:

$$\epsilon^2(m, a) = -\Delta^2 - h^2 + 2h\Delta \coth 2m + 2a^2 \left[ \cosh^2 m - \sqrt{\cosh^4 m + \frac{h}{a^2}(\Delta \coth m - h)} \right], \tag{4.14}$$

$$\frac{d\mathcal{S}}{da} = -4\nu \int\limits_0^{m_0} dm\, \epsilon(m, a) \sinh 2m = -\frac{4\zeta}{D}. \tag{4.15}$$

In the subsequent sections, we analyze various limiting cases of these equations for the parameters $a/\Delta$, $h/\Delta$, and $\zeta/(\nu D\Delta)$.

## 4.1 Branch changing

We begin with establishing the dependence of $m_0$ on the other parameters of the problem. By definition, $m_0$ is the value attained by $m$ in the limit $\epsilon \to 0$ from Eq. (4.14). It seems unfeasible to resolve this equation with respect to $m$. However, we can set $\epsilon = 0$ and solve the linear equation (4.6) for $a$. This yields

$$a(m_0, h) = \frac{h}{2\sinh m_0} - \frac{\Delta}{2\cosh m_0}. \tag{4.16}$$

In the case $h < \Delta$, this equation has a unique positive solution for $m_0$ and the inequality (4.8) is always valid. In the opposite case $h > \Delta$, Eq. (4.16) also has a unique positive solution but the inequality (4.8) is fulfilled only for sufficiently large values of $a$. If, however, $a$ is not large enough, an alternative solution

$$\tanh m_0 = \frac{\Delta}{h} \tag{4.17}$$

takes over. This is easy to see that for such a value of $m_0$, the energy in Eq. (4.14) vanishes for any $a$. At the same time, the inequality (4.8) saturates but is not violated.

Switching between the two branches of $m_0$ occurs when both solutions coincide. Substituting $m_0$ from Eq. (4.17) into Eq. (4.16), we find the switching value of $a$:

$$a_{\text{min}} = \frac{(h^2 - \Delta^2)^{3/2}}{2h\Delta}. \tag{4.18}$$

Summarizing, there are two different branches of $m_0$ as a function of $h$, $\Delta$, and $a$. They are determined by the following relations:

$$\begin{cases} a = \dfrac{h}{2\sinh m_0} - \dfrac{\Delta}{2\cosh m_0}, & h < \Delta \quad \text{or} \quad a > \dfrac{(h^2 - \Delta^2)^{3/2}}{2h\Delta}, \\ \tanh m_0 = \dfrac{\Delta}{h}, & \text{otherwise}. \end{cases} \tag{4.19}$$

We illustrate the dependence of $m_0$ on the ratio $h/\Delta$ for a fixed $a$ in Fig. 4.1.

## 4.2 Asymptotic solutions for helical state

In this section, we will analyze various asymptotic forms for the helical state wave vector $q$ as a function of system parameters. One such limiting case corresponds to the vicinity of the phase transition studied in Chapter 3, when $q$ itself is small. To simplify intermediate formulas, we will measure $h$ and $a$ in units of $\Delta$, which will make them dimensionless. In the final answers, the proper dimension will be restored.

### 4.2.1 Small exchange field

Consider first the case when the exchange field $h$ is relatively small, $h \ll 1$. This corresponds to the first case of Eq. (4.19) and the value of $m_0$ is also small,

$$m_0 \approx \frac{h}{1 + 2a} \ll 1. \tag{4.20}$$

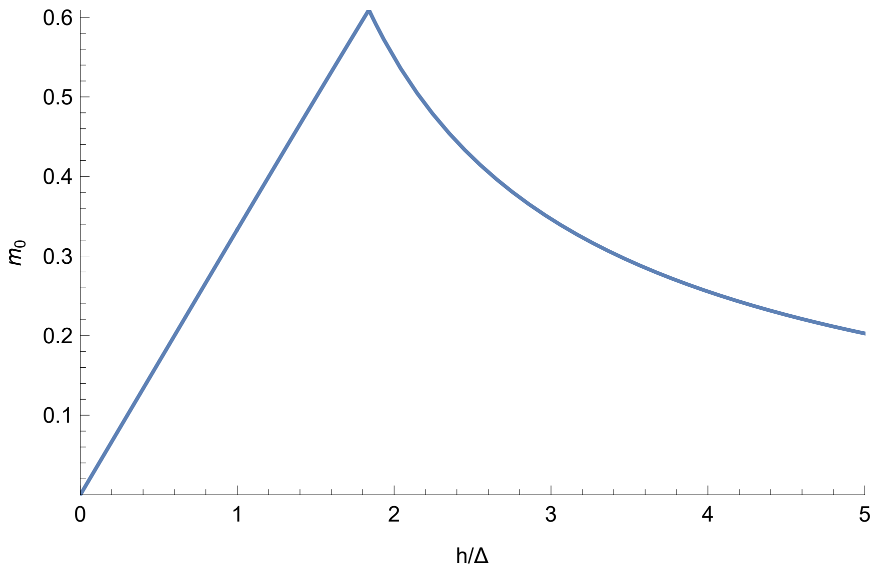


Figure 4.1: Solution for $m_0$ as a function of $h/\Delta$ for $a/\Delta = 1$. Switching of the root is clearly seen.

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We expand Eq. (4.14) to the order $O(1)$ in $h$ and $m$ without making any assumptions about the value of $a$:

$$\epsilon^2 \approx -1 + \frac{h}{m} + 2a^2 - 2a\sqrt{a^2 + \frac{h}{m}} = \left(\sqrt{a^2 + \frac{h}{m}} - a\right)^2 - 1. \tag{4.21}$$

When calculating the free energy, we also make an approximation $\sinh 2m \approx 2m$ and change integration variable to $z = \sqrt{a^2 + h/m} - a$:

$$\frac{1}{\nu}\frac{d\mathcal{S}}{da} \approx -8 \int_0^{h/(2a+1)} \epsilon m \, dm = -4 \int_\infty^1 \sqrt{z^2 - 1}\, d\left(\frac{h}{z(z+2a)}\right)^2 = $$
$$= 16h^2 \int_1^\infty dz\, \sqrt{z^2-1}\frac{z+a}{z^3(z+2a)^3}. \tag{4.22}$$

Taking this integral and substituting the result into Eq. (4.15) leads to the following equation for $a$:

$$f_1\left(\frac{2a}{\Delta}\right) = \frac{\zeta}{\nu \Delta D}\left(\frac{\Delta}{h}\right)^2, \tag{4.23}$$

where we have introduced the notation

$$f_1(x) \equiv \frac{\pi}{2x^2} - \frac{1}{x(1-x^2)} - \frac{1-2x^2}{x^2(1-x^2)} \times \begin{cases} \dfrac{\arccos x}{\sqrt{1-x^2}}, & x < 1, \\ \dfrac{\operatorname{arccosh} x}{\sqrt{x^2-1}}, & x > 1. \end{cases} \tag{4.24}$$

In the limiting cases of large and small $a$, the asymptotics of $f_1$ are

$$f_1(x) \to \begin{cases} \dfrac{\pi}{4} - \dfrac{4}{3}x, & x \to 0, \\ \dfrac{\pi}{2x^2}, & x \to \infty. \end{cases} \tag{4.25}$$

They correspond to the following limiting values of the wave vector $q$:

$$q^2 = \frac{\Delta}{D} \times \begin{cases} \dfrac{3\pi}{8} - \dfrac{3\zeta}{2\nu\Delta D}\left(\dfrac{\Delta}{h}\right)^2, & 0 < \dfrac{\pi}{4} - \dfrac{\zeta}{\nu\Delta D}\left(\dfrac{\Delta}{h}\right)^2 \ll 1 \\ \sqrt{2\pi\dfrac{\nu\Delta D}{\zeta}\dfrac{h}{\Delta}}, & \left(\dfrac{h}{\Delta}\right)^2 \gg \dfrac{\zeta}{\nu\Delta D} \end{cases} \tag{4.26}$$

The first case of this result is in complete agreement with the small $q$ expansion in the previous Chapter, cf. Eq. (3.51) and small $x$ asymptotics of Eqs. (3.42) and (3.43).

### 4.2.2 Large exchange field far from branch switching

Now we will consider the region of high magnetization, which is significantly remote from the root change curve (4.18).This region is characterized as $h^2 \gg \{1, a\}$. Note that terms of the form $a/h$ can still be comparable to 1 or even much greater than 1. We expand the energy (4.14) keeping only the terms of the order $h^2$ and $a^2$:

$$\epsilon^2 \approx -h^2 + \frac{h}{m} + 2a^2 - 2a\sqrt{a^2 + h^2\left(\frac{1}{mh} - 1\right)} = \left(\sqrt{a^2 - h^2 + \frac{h}{m}} - a\right)^2 \tag{4.27}$$

Convenient integration variable in this case is $z = \sqrt{a^2 - h^2 + h/m}$, so

$$\frac{1}{\nu}\frac{d\mathcal{S}}{da} \approx -8\int_0^{1/h}\left(\sqrt{a^2 - h^2 + \frac{h}{m}} - a\right) m\, dm = -16h^2\int_a^{\infty}\frac{z(z-a)dz}{(z^2 + h^2 - a^2)^3}. \tag{4.28}$$

The remaining integral is straightforward and the resulting equation for $a$ is (dimension restored)

$$f_2\left(\frac{a}{h}\right) = \frac{2\zeta}{\nu\Delta D}\frac{h}{\Delta}, \tag{4.29}$$

where

$$f_2(x) = \frac{x}{x^2 - 1} - \frac{1}{x^2 - 1} \times \begin{cases} \dfrac{\arccos(x)}{\sqrt{1 - x^2}}, & x < 1, \\ \dfrac{\operatorname{arccosh}(x)}{\sqrt{x^2 - 1}}, & x \geqslant 1. \end{cases} \tag{4.30}$$

Asymptotics of this function are

$$f_2(x) \to \begin{cases} \dfrac{\pi}{2} - x, & x \to 0, \\ \dfrac{1}{x}, & x \to \infty. \end{cases} \tag{4.31}$$

We thus have the following limiting expressions for the helical state wave vector:

$$q^2 = \frac{\Delta}{D}\begin{cases} \dfrac{\pi h}{\Delta} - \dfrac{4\zeta}{\nu\Delta D}\left(\dfrac{h}{\Delta}\right)^2, & 0 < \dfrac{\pi}{4} - \dfrac{\zeta}{\nu\Delta D}\dfrac{h}{\Delta} \ll 1 \\ \dfrac{2\nu\Delta D}{\zeta}, & \dfrac{\Delta}{h} \gg \dfrac{\zeta}{\nu\Delta D} \end{cases} \tag{4.32}$$

The first of these two asymptotics is also consistent with Eq. (3.51) and the limits of large $x$ in Eqs. (3.42) and (3.43).

### 4.2.3 Large exchange field near branch switching

Now we consider the area of large exchange fields in the immediate vicinity of the root change (4.18). We assume $a \sim h^2 \gg 1$ and derive the following asymptotic form of Eq. (4.14) up to $O(1)$:

$$\epsilon^2 \approx (mh-1)^2\left(\frac{h^2}{(2ma)^2}-1\right). \tag{4.33}$$

It is obvious that the minimum positive root $m_0$ is given by $m_0 = \min\{1/h,\, h/2a\}$ in accordance with Eq. (4.19). The integral in Eq. (4.15) naturally splits into two cases. Using the variable $z = 2ma/h$, we have

$$\frac{1}{\nu}\frac{d\mathcal{S}}{da} \approx -\frac{2h^2}{a^2}\int\limits_0^{\min\{2a/h^2,1\}}\left(1-\frac{h^2}{2a}z\right)\sqrt{1-z^2}\,dz. \tag{4.34}$$

The final equation for $a$ is

$$f_3\left(\frac{2a\Delta}{h^2}\right) = \frac{\zeta}{\nu\Delta D}\left(\frac{h}{\Delta}\right)^2, \tag{4.35}$$

where

$$f_3(x) \equiv -\frac{2}{3x^3} + \begin{cases} \dfrac{2(1-x^2)^{3/2}}{3x^3} + \dfrac{\sqrt{1-x^2}}{x} + \dfrac{\arcsin x}{x^2}, & x<1, \\ \dfrac{\pi}{2x^2}, & x \geqslant 1. \end{cases} \tag{4.36}$$

Limiting forms of this function are

$$f_3(x) \to \begin{cases} \dfrac{1}{x}, & x \to 0, \\ \dfrac{\pi}{2x^2}, & x\to\infty. \end{cases} \tag{4.37}$$

They provide the following results for the wave vector:

$$q^2 = \frac{\Delta}{D}\begin{cases} \sqrt{2\pi\dfrac{\nu\Delta D}{\zeta}\dfrac{h}{\Delta}}, & \left(\dfrac{\Delta}{h}\right)^2 \ll \dfrac{\zeta}{\nu\Delta D} \\ \dfrac{2\nu\Delta D}{\zeta}, & \left(\dfrac{\Delta}{h}\right)^2 \gg \dfrac{\zeta}{\nu\Delta D} \end{cases} \tag{4.38}$$

Note that these asymptotic solution exactly coincide with the corresponding results (4.26) and (4.32). This means that the solutions we found seamlessly transform into each other.

### 4.2.4 Resonant exchange field

In this final section, we consider the case of resonance when the exchange field is exactly equal to the induced order parameter $h = 1$ and $a \ll 1$. Using the first case of Eq. (4.19), we have an estimate $a \approx 2e^{-3m_0}$. This shows that $m_0$ is logarithmically large and the integral in Eq. (4.15) is taken over a parametrically large range of $m$. To estimate this integral, we will split integration domain into two asymptotic regions

$$\frac{d\mathcal{S}}{da} = \left(\frac{d\mathcal{S}}{da}\right)_{m<\mu} + \left(\frac{d\mathcal{S}}{da}\right)_{m>\mu} = -4\nu\left(\int_0^\mu + \int_\mu^{m_0}\right)dm\,\epsilon\sinh 2m. \tag{4.39}$$

Here $\mu$ is some intermediate scale chosen from the interval $1 \ll \mu \ll m_0$.

In the region $m < \mu$, we can neglect terms with $a$ in Eq. (4.14), so

$$\epsilon^2 \approx 2(\coth 2m - 1). \tag{4.40}$$

With this value of $\epsilon$, we calculate the integral in Eq. (4.15) and expand the result in the limit $\mu \gg 1$:

$$-\frac{1}{\nu}\left(\frac{d\mathcal{S}}{da}\right)_{m<\mu} \approx 4\int_0^\mu \sqrt{2(\coth 2m - 1)}\sinh 2m\, dm = 4\int\limits_0^\mu \sqrt{1-e^{-4m}}\, dm$$
$$= 2\,\mathrm{arctanh}\sqrt{1-e^{-4\mu}} - 2\sqrt{1-e^{-4\mu}} \approx 4\mu + 2\ln 2 - 2. \tag{4.41}$$

In the second region $\mu < m < m_0$, we substitute $a = 2e^{-3m_0}$ in Eq. (4.14) and keep only the leading order in the exponential $e^{-m} \sim e^{-m_0}$,

$$\epsilon^2 \approx 4e^{-4m} + \frac{a^2 e^{2m}}{2}\left(1 - \sqrt{1+\frac{32}{a^2}e^{-6m}}\right) =$$
$$= \frac{e^{2m-6m_0}}{2}\left(\sqrt{1+8e^{6(m_0-m)}} - 3\right)\left(\sqrt{1+8e^{6(m_0-m)}} - 1\right). \tag{4.42}$$

In the integral (4.15), we also approximate $\sinh 2m \approx (e^{2m})/2$ and introduce a new integration variable $z = \sqrt{1+8e^{6(m_0-m)}}$,

$$-\frac{1}{\nu}\left(\frac{d\mathcal{S}}{da}\right)_{m>\mu} \approx 2\int_\mu^{m_0} dm\, \epsilon\, e^{2m} = \frac{4}{3}\int\limits_3^{\sqrt{1+8e^{6(m_0-\mu)}}} \frac{\sqrt{z-3}\, z\, dz}{(z-1)(z+1)^{3/2}}. \tag{4.43}$$

Changing integration variable once more to $t = \sqrt{(z-3)/(z+1)}$, we obtain the following result:

$$-\frac{1}{\nu}\left(\frac{d\mathcal{S}}{da}\right)_{m>\mu} = \frac{4}{3}\int\limits_0^{t(\mu)} dt\left(\frac{2}{1-t^2} - \frac{1}{1+t^2} - 1\right) = \frac{4}{3}\Big[2\,\mathrm{arctanh}\, t(\mu) - \arctan t(\mu) - t(\mu)\Big]. \tag{4.44}$$

Using the inequality $\mu \ll m_0$, we estimate the value of $t$ at the upper limit of the integral as

$$t(\mu) = \sqrt{\frac{\sqrt{1+8e^{6(m_0-\mu)}} - 3}{\sqrt{1+8e^{6(m_0-\mu)}} + 1}} \approx 1 - \frac{e^{3\mu-3m_0}}{\sqrt{2}}. \tag{4.45}$$

This value is less than 1 by a very small amount. Expanding Eq. (4.44) in this limit we obtain the final result for the second region:

$$-\frac{1}{\nu}\left(\frac{d\mathcal{S}}{da}\right)_{m>\mu} \approx 4m_0 - 4\mu + 2\ln 2 - \frac{4+\pi}{3}. \tag{4.46}$$

Adding together this result and Eq. (4.41), we observe that the scale $\mu$ cancels as it should be. The final estimate for the action derivative is

$$-\frac{1}{\nu}\frac{d\mathcal{S}}{da} \approx 4m_0 + 4\ln 2 - \frac{10+\pi}{3}. \tag{4.47}$$

Using the relation $a \approx 2e^{-3m_0}$ and Eq. (4.15), we find the value of $a$ at resonance:

$$a = 16 \exp\left(-\frac{3\zeta}{\nu\Delta D} - \frac{10+\pi}{4}\right). \tag{4.48}$$

In terms of the wave vector $q$, we have

$$q = 8\sqrt{\frac{\Delta}{D}}\, \exp\left(-\frac{3\zeta}{2\nu\Delta D} - \frac{10+\pi}{8}\right) \approx 1.55\sqrt{\frac{\Delta}{D}}\, \exp\left(-\frac{3\zeta}{2\nu\Delta D}\right). \tag{4.49}$$

We have thus found a nonzero but exponentially small value of $q$ at the point of resonance $h = \Delta$. Expansion of $\mathcal{S}$ in small $q$ at this point contains logarithmic terms [cf. Eq. (4.47)] hence the simple power series expansion of the Landau functional developed in Chapter 3 fails at the resonance.

## 4.3 Summary and numerical solution

Let us summarize all limiting cases of the problem studied in this and previous Chapter.

In Fig. 4.2, we present a 2D density plot of $d\mathcal{S}/da$ as a function of two dimensionless parameters $h/\Delta$ and $a/\Delta$. This plot is a result of numerical computation based on Eqs. (4.14) and (4.15). According to Eq. (4.15), the wave vector of the helical state is determined by equating this function to $\zeta/(\nu D\Delta)$.

The whole parameter range is separated into four asymptotic regions enumerated by roman numbers. The value of $q$ in these limits are

$$q^2 = \frac{\Delta}{D}\begin{cases} \dfrac{3}{2}\left(\dfrac{\pi}{4} - \dfrac{\zeta}{\nu\Delta D}\left(\dfrac{\Delta}{h}\right)^2\right), & \text{region I: } \left\{\dfrac{\pi}{4} - \dfrac{\zeta}{\nu\Delta D}\left(\dfrac{\Delta}{h}\right)^2, \dfrac{h}{\Delta}\right\} \ll 1, \\ \sqrt{2\pi\dfrac{\nu\Delta D}{\zeta}\dfrac{h}{\Delta}}, & \text{region II: } \left\{\left(\dfrac{h}{\Delta}\right)^2, \left(\dfrac{\Delta}{h}\right)^2\right\} \gg \dfrac{\zeta}{\nu\Delta D} \\ 2\dfrac{\nu\Delta D}{\zeta}, & \text{region III: } \left(\dfrac{\Delta}{h}\right)^2 \ll \dfrac{\zeta}{\nu\Delta D} \ll \dfrac{\Delta}{h} \ll 1 \\ 4\left(\dfrac{h}{\Delta}\right)\left(\dfrac{\pi}{4} - \dfrac{\zeta}{\nu\Delta D}\dfrac{h}{\Delta}\right), & \text{region IV: } \left\{\dfrac{\pi}{4} - \dfrac{\zeta}{\nu\Delta D}\dfrac{h}{\Delta}, \dfrac{\Delta}{h}\right\} \ll 1 \end{cases} \tag{4.50}$$

Pairs of adjacent regions are covered by unified expressions denoted by arabic numbers. The corresponding formulas are referenced in the caption of Fig. 4.2. Finally, a special resonant case $h = \Delta$ occurs near the point denoted by the number 4. The value of $q$ is exponentially small in this case, see Eq. (4.49).

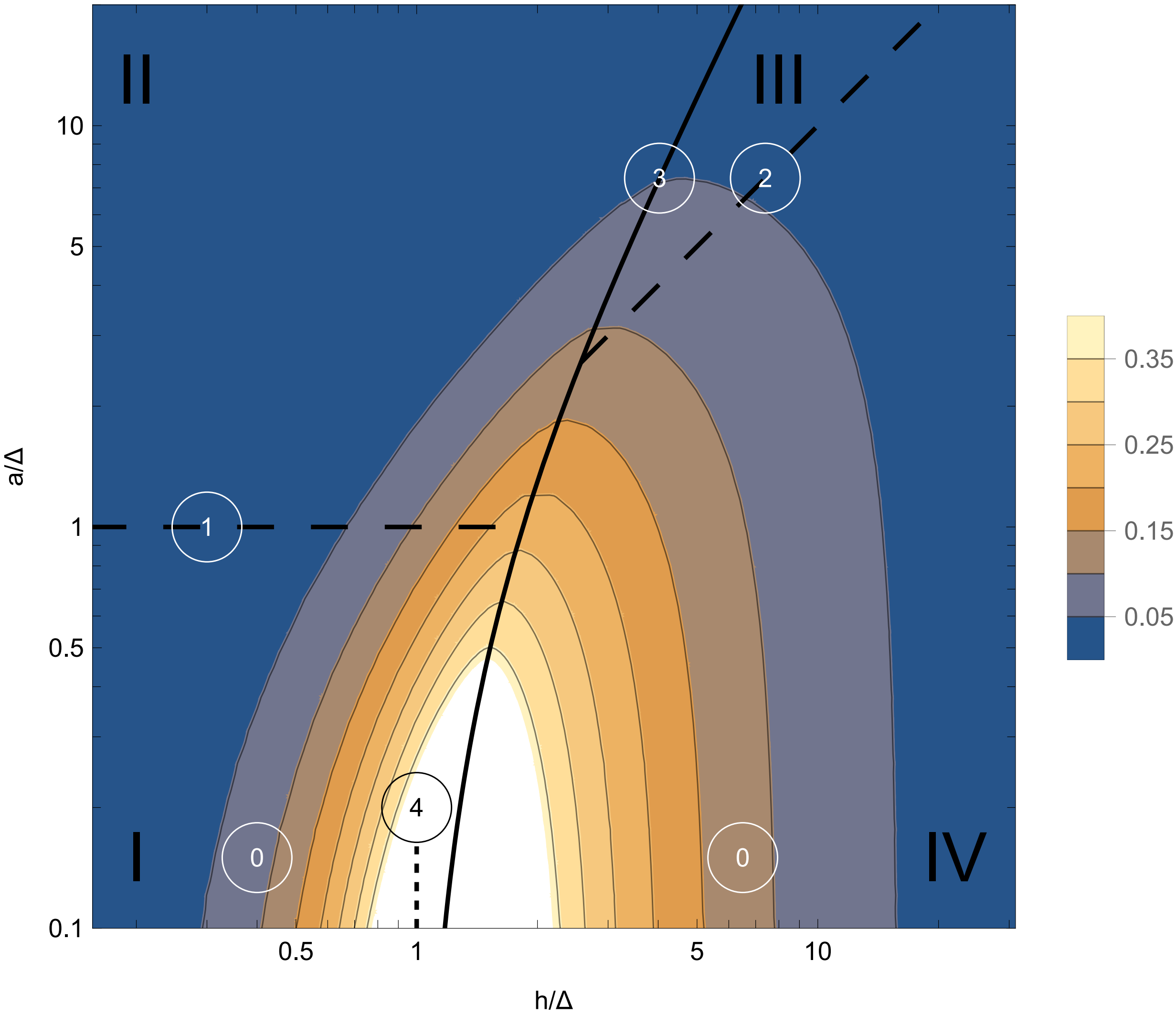


Figure 4.2: Dependence of $d\mathcal{S}/da$ (in color) on $h/\Delta$ (horizontal axis) and $a/\Delta$ (vertical axis) on a logarithmic scale. The solid black line indicates switching of the root for $m_0$ according to Eq. (4.19). Dashed lines separate the regions of small and large $a/\Delta$. Four sectors of the parameter plane correspond to unified asymptotic expressions. The corresponding values of $q$ are summarized in Eq. (4.50). Arabic numbers 1, 2, 3 correspond to the previously found solutions (4.23), (4.29), and (4.35), respectively. Each of these solutions covers two adjacent asymptotic regions. The case number 0 covers regions I and IV where the Landau-type expansion in small $q$ is valid, as discussed in the previous Chapter. The corresponding solution is given by Eq. (3.51). Finally, number 4 denotes the point of resonance $h = \Delta$. Corresponding value of $q$ is given by Eq. (4.49). As can be clearly seen from the color plot, the function $d\mathcal{S}/da$ takes its highest values near the resonance.

# 5 Conclusion

In this work, we investigated the possibility of inhomogeneously magnetized phase of a thin disordered ferromagnetic layer brought into contact with a bulk superconductor. We assumed the limit of tunnel junction and strong disorder which allowed us to solve the problem in the framework of the Usadel equation for the ferromagnetic part of the junction.

The results can be divided into two main parts. In the first part contained in Chapter 3, we have derived an effective Landau functional (3.32) expanding the free energy of the system in powers of gradients of magnetization. This functional predicts a second order phase transition from the uniform into a helical magnetic state. It turns out that the most energetically favorable state beyond the transition corresponds to the magnetization rotating in space with a constant velocity over a great circle in the spin space. The wave vector of this helical state immediately beyond the transition is small and given by Eq. (3.51). In addition, we have also found quite an unexpected resonant behavior of the system near the point $h = \Delta$. Namely, transition to a helical state does occur close to this point irrespective of the value of magnetic stiffness $\zeta$. For other values $h \neq \Delta$, transition is only possible if $\zeta$ is less than a certain threshold value.

In the second part, the helical state was studied in the entire space of parameters. Explicit system of equations for the helical state wave vector (4.14) and (4.15) was both solved numerically and thoroughly investigated analytically in various asymptotic regimes (see Section 4.3 for a concise list of all cases). In particular, the resonant case $h = \Delta$ was studied in more detail and an exponential dependence of the wave vector (4.49) was predicted with the accuracy up to and including a numerical prefactor.

Simple model of an SF junction studied in this work can be extended in several important directions. To name a few, it can include possible spin-orbit interaction, inverse proximity effect in the superconductor, general boundary conditions, SFS geometry and possible interplay between the Josephson current and nonuniform magnetization, real-time dynamics of transitions between different magnetic textures. Possible extensions of the theory developed here will be the subject of our future studies.

# Acknowledgements

I express my gratitude to Ms. Aisha Idiatullina for creating a magnificent drawing 2.1.

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