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DISTRIBUTION OF ELECTRON SCATTERING RESONANCES IN A DISORDERED METALLIC GRAIN

(Bachelor's thesis)

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Abstract

The system under study consists of a dirty metallic grain and an M-channel clean wire connected to it. The reflection of waves incident on the metal is described by the S matrix. We are interested in the statistics of resonance widths (i.e., imaginary parts of the poles of the S matrix). Using the framework of a supersymmetric nonlinear sigma model we derive general formulas of the resonance widths distribution for all three Wigner-Dyson classes and calculate it in the zero-dimensional limit for the case of identical transmission coefficients.

Contents

1 Introduction

Wave scattering by a disordered medium with randomly distributed impurities has been actively studied for last decades both theoretically [1, 2, 3, 4, 5] and experimentally [6, 7, 8, 9]. The main object characterizing the process of scattering is the energy-dependent S matrix that relates the amplitudes of incoming and outgoing waves. Due to the random nature of the scattering potential, it makes sense to describe the S-matrix characteristics using some statistical measures. One of challenging problems in the area is to study statistics of the so-called resonance widths, i.e., imaginary parts of scattering matrix poles $E = E_n - i\Gamma_n$. Resonance widths are of a great interest because they determine temporal properties of wave scattering.

There are two complementary ways to describe chaotic scattering. The first one is the semiclassical approach. The primary paper in this direction is [10]. The peculiarity of the semiclassical approach is that it operates with the genuine microscopic Hamiltonians. It allows one to take into account characteristics of specific systems.

The second way is the stochastic approach. It is based on averaging over an ensemble of random Hamiltonians. Such ensemble averaging can be performed using the supersymmetry method as was demonstrated for the first time in this area of study in papers [11] and [12]. Authors of these papers used the standard random matrix theory (RMT) framework. The peculiarity of this approach is its universality. More precisely, statistical characteristics of closed systems do not depend on microscopic details. This is true only on energy scales much smaller in comparison with Thouless energy $E_{\mathrm{Th}} = 1/\tau_{\mathrm{erg}}$, where τ_{erg} is the characteristic time needed for a particle to pass through the sample by diffusion. Rigorous microscopic proof of the universality hypothesis for a metal grain was obtained by Efetov [13], while studying the pair correlator of energy levels. The RMT approach made it possible to study many properties of disordered systems, in particular scattering resonance statistics [2, 3, 4, 14, 15]. Some of theoretical results have been verified experimentally [7, 8, 9, 16] and numerically [17, 18].

But the supersymmetric approach does not stop there. The supersymmetric nonlinear sigma model formalism developed in the work of Efetov [19] is one of the most effective tools for describing a dirty metal. In particular, RMT corresponds to the so-called zero-dimensional limit of the sigma model. Using the sigma model allows one to extend the class of systems under study. For example, in a recent paper [1] the formula for the resonance density in a quasi-one-dimensional sample was obtained as well as pertubative formulas that are valid in any dimension. These results, like most of those obtained within RMT framework, were obtained for systems with broken time reversal invariance (class A). It is interesting to generalize the sigma model approach to systems with preserved time reversal invariance (classes AI and AII).

In the present work, we will develop the general method for calculating the resonance widths distribution in all Wigner-Dyson classes. Our formulas will be valid in any dimension and for any type of the interface between the disordered system and the wire. To obtain these formulas we will use the sigma model framework. But unlike most works in the area, we will use it to describe the whole system, not just the disordered part.

The structure of the work is the following. In Chapter 2 we formulate the problem and carry out some general calculations preceding the sigma model. In Chapter 3 we show that the resonance density can be expressed through the partition function of the sigma model with sources. To calculate the partition function it is convenient to use Fourier analysis on the sigma-model manifold [20, 21]. Using it, we obtain general formulas for all Wigner-Dyson classes in Chapter 4. The case of identical channels in the zero-dimensional limit is also considered there for all three classes. Main results and the limits of applicability are discussed in Chapter 5. Technical details of the Fourier analysis on the sigma model manifold are described in Appendix A.

2 Statement of the problem

We consider a dirty metallic grain and an M-channel ballistic wire connected to it.

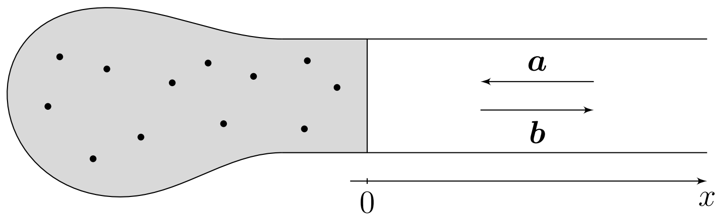

Figure 1: A sketch of the system under study. The right part is the wire, the left is the grain.

The wire is semi-infinite and one-dimentional. We will characterize the state of electrons in the basis of conducting channels using a $2M$-component wave function. The first and last M components refer to left- and right-propagating modes, respectively. Each channel is characterized by its own Fermi momentum p_i and Fermi velocity v_i. One can linearize the spectrum near points $p = \pm p_i$ for left- and right-propagating modes. We include oscillations with Fermi momentum into the definition of the channels' wave functions. The linearized Hamiltonian acts only on slow envelope amplitudes in these channels and has the form

$$\hat{H}_{\text{wire}} = v\hat{p}_x, \tag{1}$$

$$v = \operatorname{diag}(v_1, \dots, v_M, -v_1, \dots - v_M), \tag{2}$$

where v is the matrix of Fermi velocities for different channels for the waves that propagate both left and right.

The grain can be of any dimension, the Hamiltonian describing it contains the impurity potential

$$\hat{H}_{\text{grain}} = \frac{\hat{\boldsymbol{p}}^2}{2m} - \varepsilon_F + u(\boldsymbol{r}). \tag{3}$$

We assume that this potential is random and satisfies the standard Gaussian white-noise statistics

$$\langle u(\boldsymbol{r})\rangle = 0, \qquad \langle u(\boldsymbol{r}_1)u(\boldsymbol{r}_2)\rangle = \frac{1}{2\pi\nu\tau}\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2), \tag{4}$$

where ν is the density of states at the Fermi level and τ is the mean free time for electrons. One can find the spectrum $\{E_n\}$ of the Hamiltonian (3). Each real energy level E_n is converted to a complex resonance pole $E_n - i\Gamma_n$ after connecting the wire.

The most natural way to describe electron scattering is in terms of the S matrix defined as follows. Let the incident and reflected waves be characterized by a set of amplitudes $\boldsymbol{a} = (a_1, \dots, a_M)^T$ and $\boldsymbol{b} = (b_1, \dots, b_M)^T$. They are linearly related as $\boldsymbol{b} = S\boldsymbol{a}$. It is important that the S matrix describes the reflection from the whole grain. It should not be confused with the scattering on the boundary between the wire and the disordered medium. The latter is described by its own matrix, which has the form

$$S_{\text{boundary}} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}.$$

Here r, r', t, t' are $M \times M$ matrices corresponding to reflection and transmission at the interface between the ballistic wire and the disordered grain. We will only need T_i — eigenvalues of the matrix $tt^\dagger$, which we call transmission coefficients. These coefficients do not depend on the choice of the wave function basis inside the grain.

The S matrix has poles as a function of energy at $E = E_n - i\Gamma_n$. One can define the disorder-averaged density of resonance widths Γ_n as

$$\rho(y) = \Delta \sum_n \left\langle \delta(E - E_n)\delta\left(y - \frac{2\pi\Gamma_n}{\Delta}\right)\right\rangle, \tag{5}$$

where Δ is the mean level spacing of E_n and y has the meaning of a dimensionless resonance width. In the following we also use the notation $\eta = \frac{\Delta}{2\pi}y$.

From causality and unitarity of the S matrix it follows that its determinant is analytic in the upper complex half-plane of E with the absolute value equal to one. So, the general form of the S matrix determinant is

$$\det S(E) = \prod_n \frac{E - E_n - i\Gamma_n}{E - E_n + i\Gamma_n}. \tag{6}$$

Using this formula, one can express the resonance density in terms of a generating function $F(y)$

$$\rho(y) = -\frac{1}{2}\frac{\partial^2}{\partial y^2}F(y), \tag{7}$$

$$F(y) \stackrel{\text{def}}{=} \operatorname{tr}\left\langle \ln S^{-1}(E - i\eta)\, S(E + i\eta)\right\rangle. \tag{8}$$

It is true under the assumption that the dependence of $F(y)$ on the real part of the energy is slow near the Fermi surface. It will be seen later that this assumption is correct hence the energy is not included as an argument of the generating function. By definition of the matrix logarithm

$$F(y) = -\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr}\left\langle \left(1 - S^{-1}(E - i\eta)\, S(E + i\eta)\right)^n \right\rangle. \tag{9}$$

The S matrix is subunitary for $y > 0$, so the series converges. The terms of the series can be expressed via retarded and advanced Green's functions of the whole system in the wire [22]

$$F(y) = -\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr}\langle (G^R_{E+i\eta}(x_1, x_2) v G^A_{E-i\eta}(x_2, x_1) v)^n \rangle, \tag{10}$$

where x_1 and x_2 are arbitrary coordinates along the wire.

To calculate such expressions, it is convenient to use the framework of the supersymmetric nonlinear sigma model [1, 14]. The difference between our approach and the one outlined in these works is that we will use the sigma model to describe the whole system, not just the disordered medium.

3 General formalism

3.1 Expression of resonance density via the partition function of the sigma model

In this section we carry out some general calculations for the generating function $F(y)$. To facilitate this analysis, we will assume that the system belongs to the unitary symmetry class (i.e., class A). The final results can be easily generalized to other classes as will be shown later.

Disorder-averaged products of Green's functions as in the formula (10) for the generating function can be expressed through the supersymmetric nonlinear sigma model partition function with sources. The partition function is defined as

$$Z = \left\langle \int D\Phi^{\dagger} D\Phi \exp(-\mathcal{S}) \right\rangle, \tag{11}$$

$$\mathcal{S} = -i \int d\boldsymbol{r}\, \Phi^{\dagger}(\boldsymbol{r}) \hat{M} \Phi(\boldsymbol{r}), \tag{12}$$

where

$$\Phi(\boldsymbol{r}) = (S_R(\boldsymbol{r}), S_A(\boldsymbol{r}), \chi_R(\boldsymbol{r}), \chi_A(\boldsymbol{r}))^T, \tag{13}$$

$$\hat{M} = \begin{pmatrix} (\hat{G}^R_{E+i\eta})^{-1} & bv\delta(x-x_2) & 0 & 0 \\ bv\delta(x-x_1) & -(\hat{G}^A_{E-i\eta})^{-1} & 0 & 0 \\ 0 & 0 & (\hat{G}^R_{E+i\eta})^{-1} & fv\delta(x-x_2) \\ 0 & 0 & fv\delta(x-x_1) & -(\hat{G}^A_{E-i\eta})^{-1} \end{pmatrix}, \tag{14}$$

$$\hat{G}^{R/A}_{E+i\eta} = (E - \hat{H} \pm i\eta)^{-1}. \tag{15}$$

Here $S_{R/A}(\boldsymbol{r})$ and $\chi_{R/A}(\boldsymbol{r})$ are conventional and Grassmanian fields. In the wire they depend only on the x coordinate. We will describe both the grain and the wire in terms of the sigma model, so integration in the definition of the action (12) is carried out over the whole system. $\hat{M}$ is a 4×4 block matrix, the size of each block is $2M \times 2M$.

Let's show that

$$F(y) = \int_0^i db\, \left.\frac{\partial Z}{\partial b}\right|_{f=b}. \tag{16}$$

The functional integral is Gaussian, one can calculate it explicitly

$$Z = \left\langle \exp\left(-\operatorname{Str} \ln \hat{M}\right) \right\rangle. \tag{17}$$

Here Str is both the supertrace of the supermatrix as well as the trace in real space. Let's consider

$$\left.\frac{\partial Z}{\partial b}\right|_{f=b} = -\left\langle \exp\left(-\operatorname{Str} \ln \hat{M}\right) \operatorname{Str} \hat{M}^{-1} \left.\frac{\partial \hat{M}}{\partial b}\right|_{f=b} \right\rangle. \tag{18}$$

When $f = b$, the boson-boson and fermion-fermion blocks of $\hat{M}$ are the same. It is also true for the logarithm of this matrix, so its supertrace is equal to zero. Therefore, the expression under study simplifies

$$\begin{aligned} \left.\frac{\partial Z}{\partial b}\right|_{f=b} &= -\left\langle \operatorname{Tr} \begin{pmatrix} (\hat{G}^R_{E+i\eta})^{-1} & bv\delta(x-x_2) \\ bv\delta(x-x_1) & -(\hat{G}^A_{E-i\eta})^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 0 & v\delta(x-x_2) \\ v\delta(x-x_1) & 0 \end{pmatrix} \right\rangle = \\ &= 2\sum_{n=1}^{\infty} (-1)^n b^{2n-1} \left\langle \operatorname{Tr}\left(\hat{G}^R_{E+i\eta} v\delta(x-x_2) \hat{G}^A_{E-i\eta} v\delta(x-x_1)\right)^n \right\rangle. \end{aligned} \tag{19}$$

Here Tr is both the conventional trace of the $2M \times 2M$ matrix and the trace in real space. It is also important to note that delta functions must be understood as operators here. Directly $\delta(x - x_{1,2})$ is the coordinate representation of the corresponding operator $|x_{1,2}\rangle\langle x_{1,2}|$. Indeed

$$|x_{1,2}\rangle\langle x_{1,2}|\, x\rangle = \delta(x - x_{1,2})\, |x_{1,2}\rangle = \delta(x - x_{1,2})\, |x\rangle\,. \tag{20}$$

Integrating the equation (19) over b from 0 to i and taking the trace in real space, one can get the formula (16)

$$\begin{aligned}\int\limits_0^i db\, \frac{\partial Z}{\partial b}\bigg|_{f=b} &= \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\big\langle \big(\hat{G}^R_{E+i\eta}\,|x_2\rangle\, v\, \langle x_2|\, \hat{G}^A_{E-i\eta}\,|x_1\rangle\, v\, \langle x_1|\big)^n \big\rangle = \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}\big\langle \big(G^R_{E+i\eta}(x_1, x_2) v G^A_{E-i\eta}(x_2, x_1) v\big)^n \big\rangle = F(y).\end{aligned} \tag{21}$$

3.2 Calculation of the partition function

In order to express the partition function in terms of the sigma model, it is convenient to transform the fields over which the integral is taken so as to eliminate the sources

$$\Phi(\boldsymbol{r}) = \begin{pmatrix} U_1(\boldsymbol{r}) & 0 \\ 0 & U_3(\boldsymbol{r}) \end{pmatrix} \tilde{\Phi}(\boldsymbol{r}), \qquad \Phi^\dagger(\boldsymbol{r}) = \tilde{\Phi}^\dagger(\boldsymbol{r}) \begin{pmatrix} U_2(\boldsymbol{r}) & 0 \\ 0 & U_4(\boldsymbol{r}) \end{pmatrix}. \tag{22}$$

Here $\tilde{\Phi}(\boldsymbol{r})$ is the same vector of two conventional and two Grassmannian fields. Without loss of generality we set $0 < x_2 < x_1$. Since the sources are located in the two isolated points $x_{1,2}$ only, they can be eliminated by choosing

$$U_1(\boldsymbol{r}) = \begin{cases} \begin{pmatrix} \sqrt{1+b^2} & 0 \\ 0 & 1 \end{pmatrix}, & x < x_2 \\ \begin{pmatrix} \sqrt{1+b^2} & ib \\ 0 & 1 \end{pmatrix}, & x_2 < x < x_1 \\ \begin{pmatrix} \sqrt{1+b^2} & ib \\ -ib\sqrt{1+b^2} & 1+b^2 \end{pmatrix}, & y > x_1 \end{cases} \tag{23}$$

$$U_3(\boldsymbol{r}) = \begin{cases} \begin{pmatrix} -\sqrt{1+f^2} & 0 \\ 0 & \pm 1 \end{pmatrix}, & x < x_2 \\ \begin{pmatrix} -\sqrt{1+f^2} & if \\ 0 & 1 \end{pmatrix}, & x_1 < x < x_2 \\ \begin{pmatrix} -\sqrt{1+b^2} & ib \\ -if\sqrt{1+b^2} & 1+f^2 \end{pmatrix}, & y > x_1 \end{cases} \tag{24}$$

$$U_{2,4}(\boldsymbol{r}) = U^{-1}_{1,3}(\boldsymbol{r}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25}$$

Further it will be seen that it is convenient to parametrize sources as follows

$$b = \sinh(\theta_B/2), \qquad f = i\sin(\theta_F/2). \tag{26}$$

It is convenient for further calculations to reorder components of the vector field

$$\mathit{\Phi}(\boldsymbol{r}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tilde{\Phi}(\boldsymbol{r}) = (\tilde{S}_R(\boldsymbol{r}), \tilde{\chi}_R(\boldsymbol{r}), \tilde{S}_A(\boldsymbol{r}), \tilde{\chi}_A(\boldsymbol{r}))^T. \tag{27}$$

In the limit $x_{1,2} \to 0$ the partition function takes the form

$$Z = \left\langle \int D\Phi^\dagger D\Phi \exp\left(\int d\boldsymbol{r}\, (i\Phi^\dagger(\boldsymbol{r})(E - \hat{H})\Phi(\boldsymbol{r}) - \eta\Phi^\dagger(\boldsymbol{r})\tilde{\Lambda}(\boldsymbol{r})\Phi(\boldsymbol{r})) \right) \right\rangle, \tag{28}$$

where

$$\tilde{\Lambda}(\boldsymbol{r}) = \begin{cases} \Lambda = \mathrm{diag}(1, 1, -1, -1), & \text{in the grain,} \\ \begin{pmatrix} \cosh\theta_B & 0 & i\sinh\theta_B & 0 \\ 0 & \cos\theta_F & 0 & \sin\theta_F \\ i\sinh\theta_B & 0 & -\cosh\theta_B & 0 \\ 0 & \sin\theta_F & 0 & -\cos\theta_F \end{pmatrix}, & \text{in the wire.} \end{cases} \tag{29}$$

Next, we perform standard transformations for the derivation of the nonlinear supersymmetric sigma model. Averaging the partition function over the disorder leads to a fourth-order term in the action. To get rid of it, we perform the Hubbard-Stratonovich transformation introducing the functional integral over the supermatrix field Q. The integral over the original fields Φ becomes Gaussian and can be calculated. The remaining integral over the supermatrix Q is mainly accumulated near the saddle manifold. One can restrict Q to this manifold and expand the action in gradients

$$Z = \int DQ \exp(-S[Q] - S_\Gamma[Q(\pm 0)]), \tag{30}$$

$$S[Q] = \frac{\pi\nu}{4} \int d\boldsymbol{r}\; \mathrm{str}(D(\nabla Q)^2 - 4\eta\tilde{\Lambda}(\boldsymbol{r})Q), \tag{31}$$

$$S_\Gamma[Q(\pm 0)] = \frac{1}{2}\sum_{i=1}^{M} \mathrm{str}\ln\left(g_i + \frac{1}{2}\{Q(-0), Q(+0)\} \right), \tag{32}$$

where D is the diffusion coefficent. Here $S_\Gamma[Q(\pm 0)]$ is the part of the sigma-model action describing the boundary between the disordered medium and the wire. It is written in terms of $Q(\pm 0)$ at two sides of the interface and contains the parameters g_i related to the transmission coefficients as

$$g_i = \frac{2 - T_i}{T_i}. \tag{33}$$

For example, the case $g_i = 1$ corresponds to a perfectly open channel.

In the formula (31), $S[Q]$ is the standard expression for the action of nonlinear supersymmetric sigma model. The wire is clean, so the diffusion coefficient in it is formally infinite. This implies that the supermatrix Q is constant in the wire. Moreover, it is equal to $\tilde{\Lambda}(\boldsymbol{r})$. Indeed, in this case the action associated with the wire is zero, otherwise it is infinitely large since it is proportional to the length of the wire. Inside the grain, the supermatrix Q can take any values from the sigma-model manifold. This manifold is obtained from the saddle point Λ by the rotation

$$Q = T^{-1}\Lambda T. \tag{34}$$

Supermatrices T are elements of the group G, which is different for different symmetry classes.

Let's denote

$$\Psi(Q) = \int\limits_{Q'(-0)=Q} DQ' \exp(-S[Q']), \tag{35}$$

$$\Gamma(Q, Q_{\mathrm{wire}}) = \exp(-S_\Gamma[Q, Q_{\mathrm{wire}}]), \tag{36}$$

$$Z = \int dQ\, \Psi(Q)\Gamma(Q, Q_{\mathrm{wire}}). \tag{37}$$

Here $\Psi(Q)$ is the so-called order parameter function [23]. It is the partition function of the grain with a fixed value of the supermatrix $Q(-0)$ at the interface. The integral in (37) is no longer functional, it is carried out over the supermatrix $Q(-0)$. Although the derivation of the formulas (35) – (37) was conducted for the unitary symmetry class, they are valid for all Wigner-Dyson classes.

In order to take the integral (37) it is convenient to use the parametrization of the supermatrix by Efetov [19, 24]

$$Q = U^{-1}\Lambda e^{\theta} U, \tag{38}$$

This means that the supermatrix can be reduced to the form Λe^{θ}. There θ is from the abelian (Cartan) subalgebra of G, and the rotation U is from the subgroup $K \subset G$ whose elements commute with Λ:

$$[U, \Lambda] = 0, \qquad \Lambda e^{\theta} = e^{-\theta}\Lambda. \tag{39}$$

Since the order parameter function is invariant under such rotations $Q \mapsto U^{-1}QU$, it follows that it is a function of θ only

$$\Psi(Q) = \Psi(\theta). \tag{40}$$

The exponent of the boundary action is invariant under the simultaneous rotation of both of its arguments by any element of the group G. This means that

$$\Gamma(Q, Q_{\text{wire}}) = \Gamma(\tilde{\theta}), \tag{41}$$

where $\tilde{\theta}$ parameterizes the supermatrix

$$\tilde{Q} = e^{\theta/2} U e^{-\theta_{\text{wire}}/2} \Lambda e^{\theta_{\text{wire}}/2} U^{-1} e^{-\theta/2}. \tag{42}$$

This supermatrix is obtained from Q_{wire} by the rotation that takes Q to Λ.

In simple words it can be explained as follows. The function $\Psi(Q)$ depends only on the «distance» between Q and Λ and $\Gamma(Q, Q_{\text{wire}})$ depends only on the «distance» between Q_{wire} and Q. It is clear that after integration over Q in (37) the result must depend only on the «distance» between Q_{wire} and Λ, i.e., on θ_{wire}. Hence the convolution integral (37) will be much easier to take after the Fourier transform.

4 The resonance density in different Wigner-Dyson classes

In this chapter, we apply the general formula (37) to find the distributions of scattering resonances in each Wigner-Dyson symmetry class.

4.1 Unitary class

In the unitary class one can take the integral in formula (37) directly. Matrices U and θ in the parametrization (38) have the form

$$\theta = \begin{pmatrix} 0 & 0 & i\theta_1 & 0 \\ 0 & 0 & 0 & \theta_2 \\ -i\theta_1 & 0 & 0 & 0 \\ 0 & -\theta_2 & 0 & 0 \end{pmatrix}, \tag{43}$$

$$U = \begin{pmatrix} e^{-i\phi_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-i\phi_2} \end{pmatrix} \begin{pmatrix} 1-\frac{1}{2}\alpha^*\alpha & \alpha^* & 0 & 0 \\ -\alpha & 1+\frac{1}{2}\alpha^*\alpha & 0 & 0 \\ 0 & 0 & 1+\frac{1}{2}\beta^*\beta & i\beta^* \\ 0 & 0 & -i\beta & 1-\frac{1}{2}\beta^*\beta \end{pmatrix}, \tag{44}$$

where $\theta_1 \in (0,\infty)$, $\theta_2 \in (0,\pi)$, $\phi_{1,2} \in (0,2\pi)$, α and β are Grassmannian numbers. It is convenient to introduce "eigenvalues"

$$\lambda_1 = \cosh\theta_1, \qquad \lambda_2 = \cos\theta_2. \tag{45}$$

These are eigenvalues of the retarded-retarded block of the supermatrix Q. In particular, for the supermatrix in the wire

$$\lambda_1 = \cosh\theta_B \stackrel{\text{def}}{=} \lambda_B, \qquad \lambda_2 = \cos\theta_F \stackrel{\text{def}}{=} \lambda_F, \qquad U = 1. \tag{46}$$

In terms of eigenvalues, the formula (16) for the generating function $F(y)$ takes the form

$$F(y) = \int_{-1}^{1} d\lambda_B \left.\frac{\partial Z}{\partial \lambda_B}\right|_{\lambda_B=\lambda_F}. \tag{47}$$

The measure is given by

$$dQ = -d\lambda_1 d\lambda_2 J(\lambda_1,\lambda_2) d\phi_1 d\phi_2 d\alpha^* d\alpha d\beta^* d\beta, \tag{48}$$

$$J(\lambda_1,\lambda_2) = \frac{1}{(\lambda_1-\lambda_2)^2}. \tag{49}$$

As noted at the end of the last section, the function $\Psi(Q)$ depends only on the eigenvalues of the supermatrix Q. Exponent of the boundary action (36) takes the form

$$\Gamma(Q, Q_{\text{wire}}) = \Gamma(\tilde\lambda_1,\tilde\lambda_2) = \prod_{i=1}^{M} \frac{g_i+\tilde\lambda_2}{g_i+\tilde\lambda_2}, \tag{50}$$

where $\tilde\lambda_{1,2}$ are eigenvalues of the supermatrix $\tilde Q$, see (42). Hence the integrand of (37) contains Grassmannians only through $\tilde\lambda_{1,2}$. After integration over Grassmannians one can obtain

$$Z = \frac{\lambda_B-\lambda_F}{2\pi^2}\int_1^\infty d\lambda_1 \int_{-1}^{1} d\lambda_2 \frac{\Psi(\lambda_1,\lambda_2)}{\lambda_1-\lambda_2}\int_0^\pi d\phi_1 \int_0^\pi d\phi_2 \frac{1}{\tilde\lambda_1-\tilde\lambda_2}\Delta\Gamma(\tilde\lambda_1,\tilde\lambda_2). \tag{51}$$

Here Δ is the Laplace operator for variables $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, see (131). The notation $\tilde{\lambda}_{1,2}$ now means only the parts of these eigenvalues that are free of Grassmannian variables

$$\tilde{\lambda}_1 = \lambda_1\lambda_B - \sqrt{(\lambda_1^2-1)(\lambda_B^2-1)}\cos\phi_1, \tag{52}$$

$$\tilde{\lambda}_2 = \lambda_2\lambda_F + \sqrt{(1-\lambda_2^2)(1-\lambda_F^2)}\cos\phi_2. \tag{53}$$

Due to appearance of the Laplace operator in the remaining integral over $\lambda_{1,2}$ it turns out to be very convenient to use the basis of its eigenfunctions $L_{l,q}(\lambda_1,\lambda_2)$. We expand both functions $\Psi(\lambda_1,\lambda_2)$ and $\Gamma(\tilde{\lambda}_1,\tilde{\lambda}_2)$ in this basis. Structure of this basis, explicit form of the eigenfunctions, and the integration measure $\mu_{l,q}$ can be found in Appendix A. Fourier expansion of the functions in the integrand of (51) takes the form

$$\Psi(\lambda_1,\lambda_2) = \Psi_0 + \sum_{l=0}^{\infty}\int_0^{\infty} dq\,\mu_{l,q}\Psi_{l,q}L_{l,q}(\lambda_1,\lambda_2), \tag{54}$$

$$\Gamma(\tilde{\lambda}_1,\tilde{\lambda}_2) = 1 + \sum_{l=0}^{\infty}\int_0^{\infty} dq\,\mu_{l,q}\Gamma_{l,q}L_{l,q}(\tilde{\lambda}_1,\tilde{\lambda}_2). \tag{55}$$

Here Ψ_0 is the partition function of the grain with the boundary condition $Q(0)=\Lambda$. It corresponds to the absence of sources, so Ψ_0 is equal to one. Substituting (54) and (55) into (51), and using the properties of Legendre functions, we get a surprisingly simple result

$$Z = 1 + \sum_{l=0}^{\infty}\int_0^{\infty} dq\,\mu_{l,q}\Gamma_{l,q}\Psi_{l,q}L_{l,q}(\lambda_B,\lambda_F). \tag{56}$$

This remarkable formula could have actually been expected, since the Fourier component of the convolution of two functions is the product of their Fourier components. This will allow us to generalize the formula to other classes in the next section.

Let us now use the result (56) to compute the distribution of scattering resonances. We substitute (56) into (47) and then into (7) to obtain

$$\rho(y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\sum_{l=0}^{\infty}\int_0^{\infty} dq\,\mu_{l,q}\Gamma_{l,q}\Psi_{l,q}\int_{-1}^{1} d\lambda_B\,\left.\frac{\partial L_{l,q}(\lambda_B,\lambda_F)}{\partial\lambda_B}\right|_{\lambda_B=\lambda_F}. \tag{57}$$

Using the explicit eigenfunction $L_{l,q}(\lambda_1,\lambda_2)$ from Appendix A, we also find

$$\int_{-1}^{1} d\lambda_B\,\left.\frac{\partial L_{l,q}(\lambda_B,\lambda_F)}{\partial\lambda_B}\right|_{\lambda_B=\lambda_F} = \frac{(-1)^{l+1}\cosh(\pi q)}{\pi}. \tag{58}$$

Formulas (57) and (58) allow one to calculate the resonance density. But it requires the knowledge of the Fourier components $\Psi_{l,q}$ and $\Gamma_{l,q}$ and involves a summation over l and an integral over q. It turns out there is a way to further simplify this calculation using certain properties of the eigenfunctions.

4.1.1 Analytic continuation

The eigenfunctions $L_{l,q}(\lambda_1,\lambda_2)$ have a branch cut along the ray $(-\infty,-1]$ in the complex plane of the bosonic eigenvalue λ_1. The difference between eigenfunction's values above and below the cut is connected with the value of the same function with opposite arguments. From the explicit form of the eigenfunctions and properties of hypergeometric functions it follows that

$$L_{l,q}(-\lambda_1 + i0, -\lambda_2) - L_{l,q}(-\lambda_1 - i0, -\lambda_2) = -2\pi i L_{l,q}(\lambda_1, \lambda_2)\frac{(-1)^{l+1}\cosh(\pi q)}{\pi}. \tag{59}$$

Remarkably, the coefficient in this identity is equal to the integral (58) up to multiplication by a constant!

Using this identity one can return to the integral over the eigenvalues. For this we use the inverse Fourier transform (141) to express the Fourier component $\Psi_{l,q}$ through the function itself and substitute it into (57). Summation over l and integral over q are now straightforward and yield the result

$$\begin{aligned}\rho(y) &= \frac{1}{2}\frac{\partial^2}{\partial y^2}\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda_2\, J(\lambda_1,\lambda_2)\Psi(\lambda_1,\lambda_2)\sum_{l=0}^\infty \int_0^\infty dq\, \mu_{l,q}\Gamma_{l,q}L_{l,q}(\lambda_1,\lambda_2)\frac{(-1)^{l+1}\cosh(\pi q)}{\pi} \\ &= -\frac{1}{4\pi i}\frac{\partial^2}{\partial y^2}\int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda_2\, J(\lambda_1,\lambda_2)\Psi(\lambda_1,\lambda_2)(\Gamma(-\lambda_1+i0,-\lambda_2) - \Gamma(-\lambda_1-i0,-\lambda_2)).\end{aligned} \tag{60}$$

This expression for the distribution function is much easier to work with since it does not involve any Fourier transforms and does not require the knowledge of the Laplace operator eigenfunctions.

A similar trick can be also applied in other classes, as will be shown in the next sections. It will allow us to generalize the result for the distribution function of scattering poles.

In the special case of when all M transmission coefficients of the interface are equal, one can use (60) to reproduce the result originally obtained in the article [1]

$$\rho(y) = \frac{(-1)^{M-1}}{2(M-1)!}\frac{\partial^2}{\partial y^2}\int_{-1}^1 d\lambda_2\,(g-\lambda_2)^M \frac{\partial^{M-1}}{\partial\lambda_1^{M-1}}\frac{\Psi(\lambda_1,\lambda_2)}{(\lambda_1-\lambda_2)^2}\bigg|_{\lambda_1=g}. \tag{61}$$

4.1.2 0D limit

Now let us calculate the resonance density in the case of equal transmission coefficients for a zero-dimensional grain. In this limit the supermatrix is constant in the grain, so order parameter function is

$$\Psi(\lambda_1,\lambda_2) = e^{-y(\lambda_1-\lambda_2)}. \tag{62}$$

It can be seen that after taking the second derivative with respect to y, the Jacobian $J(\lambda_1,\lambda_2)$ in (60) will vanish. That will greatly simplify the remaining integration over $\lambda_{1,2}$.

In order to consider all possible values of M, we introduce the generating function (it should not be confused with the function $F(y)$)

$$\Gamma(\lambda_1,\lambda_2,\alpha) = \sum_{n=1}^\infty \frac{(g+\lambda_2)^n}{(\lambda_1+g)^n}(-1)^n\alpha^n = -\frac{(g+\lambda_2)\alpha}{\lambda_1+g+(g+\lambda_2)\alpha}. \tag{63}$$

The analytical continuation of this function is

$$\Gamma(-\lambda_1+i0,-\lambda_2,\alpha) - \Gamma(-\lambda_1-i0,-\lambda_2,\alpha) = (g-\lambda_2)\alpha\delta(\lambda_1-g-(g-\lambda_2)\alpha). \tag{64}$$

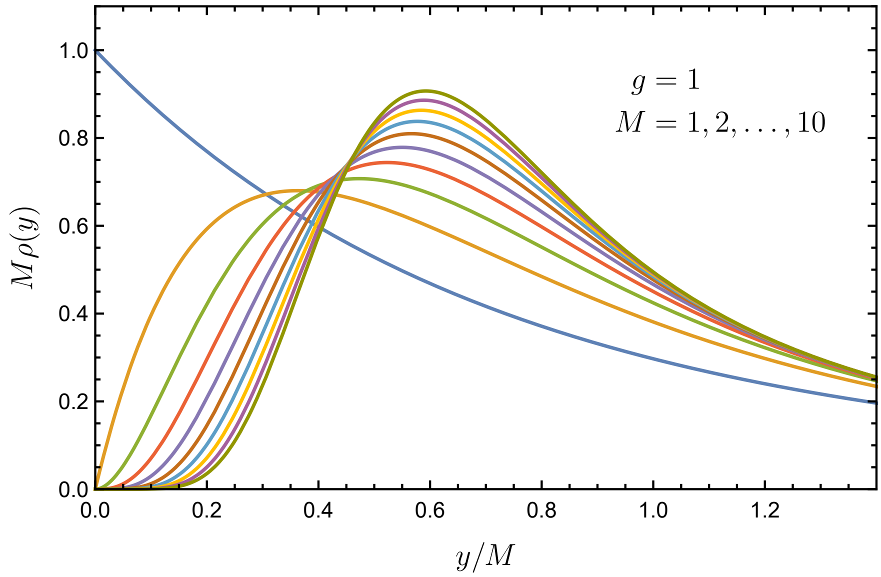

Figure 2: $\rho(y)$ in the unitary class for a 0D system with M perfectly open channels.

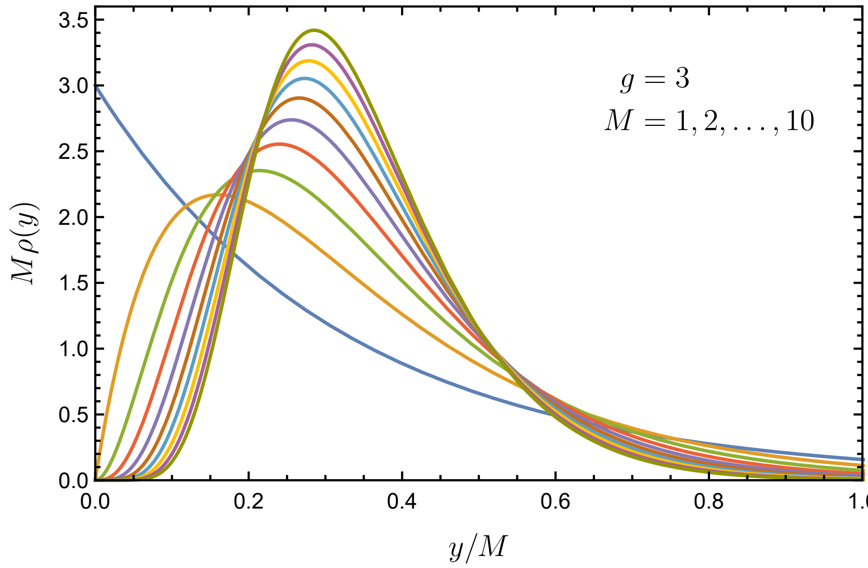

Figure 3: $\rho(y)$ in the unitary class for a 0D system with M identical channels with $g = 3$.

It allows us to get rid of the integral over λ_1

$$\rho(y) = \frac{1}{2}\frac{(-1)^{M-1}}{M!}\frac{\partial^M}{\partial\alpha^M}\int_{-1}^{1} d\lambda_2\, \Psi(g + (g-\lambda_2)\alpha, \lambda_2)(g-\lambda_2)\alpha, \tag{65}$$

The integral over λ_2 can be easily taken

$$\rho(y) = \frac{\partial}{\partial y}\frac{(-1)^M}{M!}\frac{\partial^M}{\partial\alpha^M}\frac{\alpha e^{-gy(1+\alpha)}\sinh(y(1+\alpha))}{y(1+\alpha)^2}\bigg|_{\alpha=0} = \frac{(-1)^M y^{M-1}}{(M-1)!}\frac{\partial^M}{\partial y^M}\frac{e^{-gy}\sinh y}{y}. \tag{66}$$

This formula was first obtained within RMT framework [14] and then later confirmed numerically in [17] and experimentally in microwave graphs [7]. The first ten functions are shown in Fig. 2 for ideal interface $g = 1$ and in Fig. 3 for the case $g = 3$ (transmission coefficient $T = 0.5$).

From the explicit formula (66) one can find asymptotics

$$\rho(y \to 0) = \frac{M\big((g+1)^{M+1} + (g-1)^{M-1}\big)}{2(M+1)!} y^{M-1}, \tag{67}$$

$$\rho(y \to \infty) = \begin{cases} \dfrac{(g-1)^M}{2(M-1)!} y^{M-2} e^{-(g-1)y}, & g > 1, \\ \dfrac{M}{2y^2}, & g = 1. \end{cases} \tag{68}$$

Let us note, that the power law decrease $1/y^2$ is typical for disordered systems strongly coupled to continua. There are semiclassical arguments that faithfully reproduce the same power law tail of the resonance density [14].

In the limit of small transmission coefficients (i.e., $g \gg 1$) the resonance density $\rho(y)$ takes the form of the χ^2-distribution with $2M$ degrees of freedom

$$\rho(y) = \frac{1}{(M-1)!} g^M y^{M-1} e^{-gy}. \tag{69}$$

It is also known as the Porter-Thomas distribution [25]. This result can also be obtained pertubatively [14].

Also there is an interesting limit of large number of channels $M \gg 1$. It is convenient to write the formula (66) in the form

$$\rho(y) = \frac{(-1)^M y^{M-1}}{2(M-1)!} \frac{\partial^M}{\partial y^M} \int_{g-1}^{g+1} dt\, e^{-ty} = \frac{y^{M-1}}{2(M-1)!} \int_{g-1}^{g+1} dt\, e^{-ty - M\log t}. \tag{70}$$

Let's make a change of variable $s = ty/M$ and expand the exponent near the saddle point $s = 1$. We also use the Stirling formula to estimate the factorial and obtain

$$\rho(y) = \frac{M}{2y^2}\sqrt{\frac{M}{2\pi}} \int_{(g-1)y/M}^{(g+1)y/M} ds\, e^{-\frac{M}{2}(s-1)^2} = \frac{M}{4y^2}\left(\operatorname{erf}\left(\frac{(g+1)y - M}{\sqrt{2M}}\right) - \operatorname{erf}\left(\frac{(g-1)y - M}{\sqrt{2M}}\right)\right). \tag{71}$$

The resonance density behaves as $\rho(y) \approx M/2y^2$ in the interval $\left(\frac{M}{g+1}, \frac{M}{g-1}\right)$ and exponentially decays otherwise.

4.2 Orthogonal class

4.2.1 Generalization of the approach

It was noted that formulas (35) – (42) are also applicable in other classes. In the orthogonal class (i.e., class AI) Q is an 8×8 supermatrix. It is also convenient to introduce eigenvalues

$$\lambda = \cos\theta \in [-1, 1], \qquad \lambda_{1,2} = \cosh\theta_{1,2} \in [1, \infty). \tag{72}$$

The supermatrix Q_{wire} contains the following source parameters

$$\lambda = \lambda_F, \qquad \lambda_1 = \lambda_B, \qquad \lambda_2 = 1, \qquad U = 1. \tag{73}$$

As before, the function $\Psi(Q)$ depends only on the eigenvalues of the supermatrix Q. Exponent of the boundary action (36) takes the form

$$\Gamma(Q, Q_{\text{wire}}) = \Gamma\big(\tilde\lambda, \tilde\lambda_1, \tilde\lambda_2\big) = \prod_{i=1}^{M} \frac{g_i + \tilde\lambda}{\big(g_i^2 + \tilde\lambda_1^2 + \tilde\lambda_2^2 + 2g_i\tilde\lambda_1\tilde\lambda_2 - 1\big)^{1/2}}. \tag{74}$$

In the orthogonal class it is difficult to take the integral (37) directly. However, it is clear that it has the meaning of convolution and in terms of the Fourier components should have the form similar to (56). Indeed, at the end of the first section it is noted that the partition function must depend only on the «distance» between Q_{wire} and Λ, i.e., on θ_{wire}. So we can expand it into eigenfunctions of the Laplace operator

$$Z = 1 + \sum_{l=0}^{\infty} \int_0^{\infty} dq_1 \int_0^{\infty} dq_2 \, \mu_{l,q_1,q_2} Z_{l,q_1,q_2} L_{l,q_1,q_2}(\lambda_F, \lambda_B, 1). \tag{75}$$

The Fourier components Z_{l,q_1,q_2} depend linearly on the Fourier components of the functions $\Psi(\lambda, \lambda_1, \lambda_2)$ and $\Gamma(\lambda, \lambda_1, \lambda_2)$. Due to the fact that indices l, q_1 and q_2 enumerate irreducible representations of the group G, the Fourier component Z_{l,q_1,q_2} can only depend on Fourier components with the same indices

$$Z_{l,q_1,q_2} = \#_{l,q_1,q_2} \Gamma_{l,q_1,q_2} \Psi_{l,q_1,q_2}. \tag{76}$$

The coefficient between them can be found from the special case $\Psi(Q) = \delta(Q - \Lambda)$. In this case $Z = \Gamma(\lambda_B, \lambda_F, 1)$. We choose the normalization of eigenfunctions so that the Fourier components of the delta function are equal to one, i.e., $\Psi_{l,q_1,q_2} = 1$. So the coefficient $\#_{l,q_1,q_2}$ is simply equal to one. Properly normalized eigenfunctions for the orthogonal class are given explicitly in Appendix A. As a result, the previously obtained formula (56) can be generalized to the orthogonal class

$$Z = 1 + \sum_{l=0}^{\infty} \int_0^{\infty} dq_1 \int_0^{\infty} dq_2 \, \mu_{l,q_1,q_2} \Gamma_{l,q_1,q_2} \Psi_{l,q_1,q_2} L_{l,q_1,q_2}(\lambda_F, \lambda_B, 1). \tag{77}$$

The formula (47) is also applicable in other classes. Substituting (77) into it, one can get

$$\rho(y) = \frac{1}{2} \sum_{l=0}^{\infty} \int_0^{\infty} dq_1 \int_0^{\infty} dq_2 \, \mu_{l,q_1,q_2} \Gamma_{l,q_1,q_2} \Psi_{l,q_1,q_2} \int_{-1}^{1} d\lambda_B \, \frac{\partial L_{l,q_1,q_2}(\lambda_F, \lambda_B, 1)}{\partial \lambda_B} \bigg|_{\lambda_B = \lambda_F}. \tag{78}$$

As in the unitary class, there is a linear relation between the value of the eigenfunction and its jump across the branch cut at the opposite arguments. The coefficient in this relation coincides with the integral of the eigenfunction from equation (78)

$$\int_{-1}^{1} d\lambda_B \, \frac{\partial L_{l,q_1,q_2}(\lambda_F, \lambda_B, 1)}{\partial \lambda_B} \bigg|_{\lambda_B = \lambda_F} = \frac{2(-1)^{l+1}}{\pi} \cosh(\pi q_1), \tag{79}$$

$$L_{l,q_1,q_2}(-\lambda, -\lambda_1 + i0, \lambda_2) - L_{l,q_1,q_2}(-\lambda, -\lambda_1 - i0, \lambda_2) = \\ = -\pi i L_{l,q_1,q_2}(\lambda, \lambda_1, \lambda_2) \frac{2(-1)^{l+1}}{\pi} \cosh(\pi q_1). \tag{80}$$

Using these identities and performing analytic continuation, we obtain the final formula for the density of resonances in the orthogonal class

$$\rho(y) = -\frac{1}{2\pi i} \frac{\partial^2}{\partial y^2} \int_{-1}^{1} d\lambda \int_1^{\infty} d\lambda_1 \int_1^{\infty} d\lambda_2 \, J(\lambda, \lambda_1, \lambda_2) \Psi(\lambda, \lambda_1, \lambda_2) \times \\ \times \left(\Gamma(-\lambda, -\lambda_1 + i0, \lambda_2) - \Gamma(-\lambda, -\lambda_1 - i0, \lambda_2) \right). \tag{81}$$

This is a very general integral representation of the distribution function that equally well applies for any orthogonal class system and any type of the interface.

In case of identical transmission coefficents at the interface, the distribution function takes the form

$$\rho(y) = -\frac{\partial^2}{\partial y^2}\int_{-1}^{1} d\lambda \int_{1}^{\infty} d\lambda_2 \int_{C} \frac{d\lambda_1}{2\pi i} J(\lambda,\lambda_1,\lambda_2)\Psi(\lambda,\lambda_1,\lambda_2)\frac{(g-\lambda)^M}{((a-\lambda_1)(b-\lambda_1))^{M/2}}, \tag{82}$$

where

$$a = g\lambda_2 - \sqrt{(g^2-1)(\lambda_2^2-1)}, \qquad b = g\lambda_2 + \sqrt{(g^2-1)(\lambda_2^2-1)}, \tag{83}$$

and C is a contour that goes around the interval (a, b) counterclockwise. If M is even the integral over λ_1 reduces to the sum of residues at a and b while for odd M there is also a contribution of the branch cut along (a, b).

A formula equivalent to (82) was obtained earlier in the paper [15] for the 0D case in a slightly different integral form.

4.2.2 0D limit

Let us calculate the resonance density in the case $g = 1$ for a zero-dimensional grain. In this case the interval (a, b) collapses into a point, and the order parameter function is

$$\Psi(\lambda,\lambda_1,\lambda_2) = e^{-y(\lambda_1\lambda_2-\lambda)}. \tag{84}$$

As it was done earlier for the unitary class, we introduce the generating function

$$\Gamma(\lambda,\lambda_1,\lambda_2,\alpha) = \sum_{n=1}^{\infty}\frac{(1+\lambda)^n}{(\lambda_1+\lambda_2)^n}(-1)^n\alpha^n = -\frac{(1+\lambda)\alpha}{\lambda_1+\lambda_2+(1+\lambda)\alpha}. \tag{85}$$

The analytical continuation of this function is

$$\Gamma(-\lambda,-\lambda_1+i0,\lambda_2) - \Gamma(-\lambda,-\lambda_1-i0,\lambda_2) = (1-\lambda)\alpha\delta(\lambda_1-\lambda_2-(1-\lambda)\alpha). \tag{86}$$

It allow us to take the integral over λ_1 in (81). It is convenient to introduce the following notation

$$\rho(y,\alpha) = -\frac{\partial^2}{\partial y^2}\int_{-1}^{1} d\lambda \int_{1}^{\infty} d\lambda_2\, J(\lambda,\lambda_2+(1-\lambda)\alpha,\lambda_2)\Psi(\lambda,\lambda_2+(1-\lambda)\alpha,\lambda_2)(1-\lambda)\alpha, \tag{87}$$

$$\rho(y) = \frac{(-1)^M}{M!}\frac{\partial^M}{\partial\alpha^M}\rho(y,\alpha)\bigg|_{\alpha=0}. \tag{88}$$

The remaining integral over λ and λ_2 in (87) is somewhat similar to the expression for the correlation of energy levels in RMT. We will apply the method described in [19] to compute this integral. First, we perform the inverse Laplace transform from y to t

$$\rho(t,\alpha) = \int_{\sigma-i\infty}^{\sigma+i\infty}\frac{dy}{2\pi i}e^{ty}\rho(y,\alpha). \tag{89}$$

This converts the exponential factor in the integrand into a delta function that resolves the integration over λ_2. The remaining integral over λ is elementary and yields

$$\rho(t,\alpha) = \begin{cases} \dfrac{\alpha t(\frac{t}{\sqrt{t+\alpha^2-1}} - \sqrt{1+t})}{2((1+t)\alpha^2-1)}, & t > 2+2\alpha, \\[2ex] \dfrac{\alpha t(\sqrt{1+t}-2-\alpha)}{2(1+\alpha)^2(1+\alpha\sqrt{1+t})}, & 0 < t < 2+2\alpha. \end{cases} \tag{90}$$

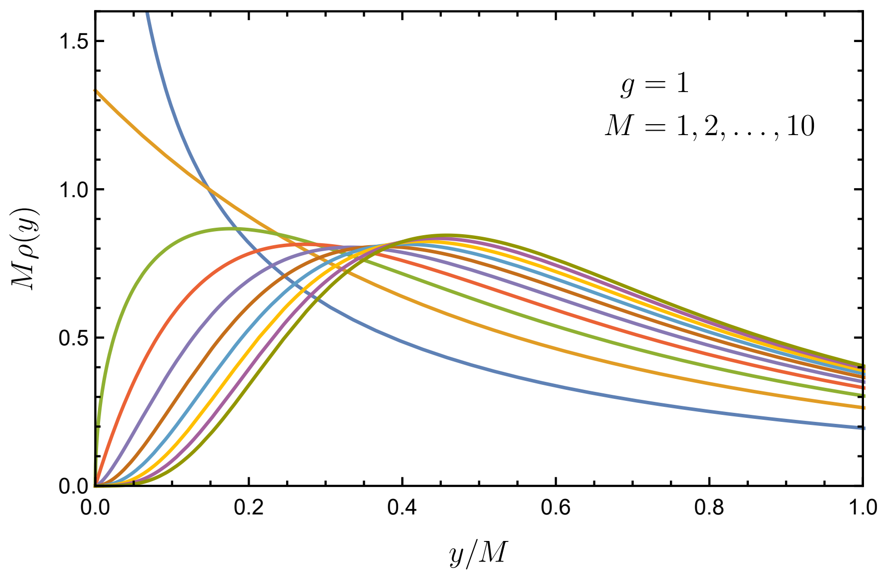

Figure 4: $\rho(y)$ in the orthogonal class for a 0D system with M perfectly open channels.

The direct Laplace transform to the variable y seems implausible. However, we can extract the resonance density for certain number of channels M directly from $\rho(t,\alpha)$

$$\rho(y) = \frac{(-1)^M}{M!} \int_0^\infty dt\, e^{-ty} \frac{\partial^M}{\partial \alpha^M} \rho(t,\alpha)\Big|_{\alpha=0}. \tag{91}$$

Here the integral over t is tractable for any chosen value of M. The result for first few values of M is

$$\rho_{M=1}(y) = -\frac{\partial}{\partial y}\left(\frac{1}{2y} - \frac{e^{-2y}}{2y} - \frac{\sqrt{\pi}}{4y^{3/2}}(e^y - (1+2y)e^{-y})\,\mathrm{erfc}\,\sqrt{y}\right), \tag{92}$$

$$\rho_{M=2}(y) = -\frac{\partial}{\partial y}\left(\frac{1}{y} - \frac{1}{2y^2} + \frac{e^{-2y}}{2y^2}\right), \tag{93}$$

$$\rho_{M=3}(y) = -\frac{\partial}{\partial y}\left(\frac{3}{2y} - \frac{3}{4y^2} - \left(\frac{1}{2} - \frac{3}{4y^2}\right)e^{-2y} - \frac{\sqrt{\pi}}{8y^{5/2}}(3e^y - (4y^3+6y^2+6y+3)e^{-y})\,\mathrm{erfc}\,\sqrt{y}\right), \tag{94}$$

$$\rho_{M=4}(y) = -\frac{\partial}{\partial y}\left(\frac{2}{y} - \frac{1}{y^2} - \frac{1}{y^3} + \left(\frac{1}{3} + \frac{2}{y} + \frac{3}{y^2} + \frac{1}{y^3}\right)e^{-2y}\right). \tag{95}$$

The resonance density for the cases $M = 1$ and $M = 2$ up to a change of variable coincides with the result of the paper [15]. The first ten functions are shown in the Fig. 4. From formula (91) it is easy to obtain the asymptotics for large y

$$\rho(y \to \infty) = \frac{M}{2y^2} - \frac{M}{2y^3} + \ldots \tag{96}$$

As noted above, the power law decrease $1/y^2$ is semiclassical, so the next term can be understood as a quantum correction.

The asymptotics for small y has the form

$$\rho(y \to 0) = \frac{\sqrt{\pi}M}{4\Gamma\left(\frac{M}{2} + \frac{3}{2}\right)}\left(y^{M/2-1} - y^{M/2} + \ldots\right). \tag{97}$$

We are not aware of any rigorous derivation of this result. It was guessed from the expansion at first few M's and then its validity was checked for M up to 100.

4.3 Symplectic class

Important feature of the symplectic class (i.e., class AII) is the Kramers degeneracy. It implies that the number of channels M is necessarily even and each transmission coefficient is doubly degenerate.

4.3.1 Generalization of the approach

In the symplectic class Q is also an 8×8 supermatrix and it has 3 eigenvalues

$$\lambda = \cosh\theta \in [1, \infty), \qquad \lambda_1 = \cos\theta_1 \in [-1, 1], \qquad \lambda_2 = \cos\theta_2 \in [0, 1]. \tag{98}$$

The supermatrix Q_{wire} contains the source parameters

$$\lambda = \lambda_B, \qquad \lambda_1 = \lambda_F, \qquad \lambda_2 = 1, \qquad U = 1. \tag{99}$$

Exponent of the boundary action (36) takes the form

$$\Gamma(Q, Q_{\text{wire}}) = \Gamma\big(\tilde\lambda, \tilde\lambda_1, \tilde\lambda_2\big) = \prod_{i=1}^{M} \frac{\big(\tilde\lambda_1^2 + \tilde\lambda_2^2 + g_i^2 + 2g_i\tilde\lambda_1\tilde\lambda_2 - 1\big)^{1/2}}{g_i + \tilde\lambda}. \tag{100}$$

The generalization of the formula (56) is completely analogous to the orthogonal class. The eigenfunctions of the Laplace operator are indexed by two integer numbers $l_{1,2} > 0$ with an additional requirement that $l_1 + l_2$ is even and by a single continuous variable q. Explicit expressions for the eigenfunctions and the integration measure μ_{q,l_1,l_2} can be found in Appendix A

$$Z = 1 + \sum_{\substack{l_1,l_2 \geq 0 \\ l_1+l_2 \text{ - even}}} \int_0^\infty dq\, \mu_{q,l_1,l_2} \Psi_{q,l_1,l_2} \Gamma_{q,l_1,l_2} L_{l_1,l_2,q}(\lambda_B, \lambda_F, 1). \tag{101}$$

The formula (47) is also applicable to the symplectic class

$$\rho(y) = \frac{1}{2}\frac{\partial^2}{\partial y^2} \sum_{\substack{l_1,l_2 \geq 0 \\ l_1+l_2 \text{ - even}}} \int_0^\infty dq\, \mu_{q,l_1,l_2} \Psi_{q,l_1,l_2} f_{q,l_1,l_2} \int_{-1}^{1} d\lambda_B \left.\frac{\partial L_{l,q_1,q_2}(\lambda_F, \lambda_B, 1)}{\partial \lambda_B}\right|_{\lambda_B=\lambda_F}. \tag{102}$$

However, unlike the unitary and orthogonal classes, the trick with analytical continuation is a little more complicated. One can show that

$$\int_{-1}^{1} d\lambda_F \left.\frac{\partial L_{l,q_1,q_2}(\lambda_F, \lambda_B, 1)}{\partial \lambda_B}\right|_{\lambda_B=\lambda_F} = \frac{(-1)^{l_1+1}}{\pi}\left(1 + \frac{l_2(l_2+1) - l_1(l_1+1)}{\frac{1}{4} + q^2}\right)\cosh(\pi q). \tag{103}$$

Substituting (103) into the formula (102) above, we get

$$\begin{aligned}\rho(y) = {}& \frac{1}{2}\frac{\partial^2}{\partial y^2} \sum_{\substack{l_1,l_2 \geq 0 \\ l_1+l_2 \text{ - even}}} \int_0^\infty dq\, \mu_{q,l_1,l_2} \Psi_{q,l_1,l_2} \Gamma_{q,l_1,l_2} \frac{(-1)^{l_1+1}}{\pi}\cosh(\pi q) + \\ & + \frac{1}{2}\frac{\partial^2}{\partial y^2} \sum_{\substack{l_1,l_2 \geq 0 \\ l_1+l_2 \text{ - even}}} \int_0^\infty dq\, \mu_{q,l_1,l_2} \Psi_{q,l_1,l_2} \Gamma_{q,l_1,l_2} \frac{(-1)^{l_1+1}}{\pi} \frac{l_2(l_2+1) - l_1(l_1+1)}{\frac{1}{4} + q^2}\cosh(\pi q).\end{aligned} \tag{104}$$

Let us note that the second term here vanishes. This follows from the symmetry of the Fourier components in the indices l_1 and l_2. Indeed, since both $J(\lambda, \lambda_1, \lambda_2)$ and $\Gamma(\lambda, \lambda_1, \lambda_2)$ are symmetric functions with respect to $\lambda_1 \leftrightarrow \lambda_2$, and the eigenfunction $L_{q,l_1,l_2}(\lambda, \lambda_1, \lambda_2)$ is invariant under the same transformation together with $l_1 \leftrightarrow \lambda_2$ it follows that the Fourier component Γ_{q,l_1,l_2} is also symmetric. The same is equally true for the Fourier component Ψ_{q,l_1,l_2}.

The jump of an eigenfunction across the branch cut of λ is

$$L_{q,l_1,l_2}(-\lambda + i0, -\lambda_1, \lambda_2) - L_{q,l_1,l_2}(-\lambda - i0, -\lambda_1, \lambda_2) = \\ = -2\pi i L_{q,l_1,l_2}(\lambda, \lambda_1, \lambda_2) \frac{(-1)^{l_1+1}}{\pi} \cosh(\pi q). \quad (105)$$

This coincides with the factor in the first (and only relevant) term of (104) which allows us to apply the analytic continuation trick

$$\rho(y) = -\frac{1}{4\pi i} \frac{\partial^2}{\partial y^2} \int_1^\infty d\lambda \int_{-1}^1 d\lambda_1 \int_0^1 d\lambda_2 \, J(\lambda, \lambda_1, \lambda_2) \Psi(\lambda, \lambda_1, \lambda_2) \times \\ \times (\Gamma(-\lambda + i0, -\lambda_1, \lambda_2) - \Gamma(-\lambda - i0, -\lambda_1, \lambda_2)). \quad (106)$$

This is the most general result for the distribution function that equally well applies for any symplectic class system and any type of the interface.

In case of identical transmission coefficients at the interface the analytical coninutation of the exponent of the boundary action turns into

$$\Gamma(-\lambda + i0, -\lambda_1, \lambda_2) - \Gamma(-\lambda - i0, -\lambda_1, \lambda_2) = \\ = -(\lambda_1^2 + \lambda_2^2 + g^2 - 2g\lambda_1\lambda_2 - 1)^{M/2} \frac{2\pi i}{(M-1)!} \delta^{(M-1)}(\lambda - g). \quad (107)$$

It allow us to take the integral over the bosonic eigenvalue λ

$$\rho(y) = -\frac{1}{2(M-1)!} \frac{\partial^2}{\partial y^2} \int_{-1}^1 d\lambda_1 \int_0^1 d\lambda_2 \, (\lambda_1^2 + \lambda_2^2 + g^2 - 2g\lambda_1\lambda_2 - 1)^{M/2} \times \\ \times \frac{\partial^{M-1}}{\partial \lambda^{M-1}} (\Psi(\lambda, \lambda_1, \lambda_2) J(\lambda, \lambda_1, \lambda_2)) \Big|_{\lambda = g}. \quad (108)$$

This is an analogue of the formula (61) from the unitary class.

4.3.2 0D limit

Let us calculate the resonance density in the case $g = 1$ for a zero-dimensional grain. In the symplectic class for a zero-dimensional system

$$\Psi(\lambda, \lambda_1, \lambda_2) = e^{-y(\lambda - \lambda_1\lambda_2)}. \quad (109)$$

It is convenient to introduce the generating function

$$\Gamma(\lambda, \lambda_1, \lambda_2, \alpha) = \sum_{n=1}^\infty \frac{(\lambda_2 + \lambda_1)^n}{(\lambda + 1)^n} (-1)^n \alpha^n = -\frac{(\lambda_2 + \lambda_1)\alpha}{\lambda + 1 + (\lambda_2 + \lambda_1)\alpha}. \quad (110)$$

The analytical continuation of this function is

$$\Gamma(-\lambda,-\lambda_1+i0,\lambda_2)-\Gamma(-\lambda,-\lambda_1-i0,\lambda_2)=(\lambda_2-\lambda_1)\alpha\delta(\lambda-1-(\lambda_2-\lambda_1)\alpha). \tag{111}$$

It allow us to take the integral over λ in (106). It is also convenient to introduce the notation as in the orthogonal class

$$\rho(y,\alpha)=-\frac{1}{2}\frac{\partial^2}{\partial y^2}\int\limits_{-1}^{1}d\lambda_1\int\limits_{\lambda_1}^{1}d\lambda_2\,J(1+(\lambda_2-\lambda_1)\alpha,\lambda_1,\lambda_2)\Psi(1+(\lambda_2-\lambda_1)\alpha,\lambda_1,\lambda_2), \tag{112}$$

$$\rho(y)=\frac{1}{M!}\frac{\partial^M}{\partial\alpha^M}\rho(y,\alpha)\bigg|_{\alpha=0}. \tag{113}$$

Let's perform the inverse Laplace transform for $\rho(y,\alpha)$. As before, it will convert the exponential factor to a delta function which allows to integrate over λ_2. The remaining integral over λ_1 is straightforward and yields

$$\rho(t,\alpha)=\begin{cases}\dfrac{t}{2}\left(\dfrac{1-t-\alpha\sqrt{1-t}}{1-t-\alpha^2}-\dfrac{1+2\alpha}{(1+\alpha)^2}\right), & 0<t<1,\\ \dfrac{\alpha^2t(2+2\alpha-t)}{2(1+\alpha)^2(1-t-\alpha^2)}, & 1<t<2+2\alpha,\\ 0, & t>2+2\alpha.\end{cases} \tag{114}$$

As it was in the case of orthogonal class, direct Laplace transform to the original variable y is implausible. Instead we would like expand the function $\rho(t,\alpha)$ in powers of α first and then perform the Laplace transform. However, it turns out that the Laplace integral can not be expanded in powers of α since the singular point $t=1-\alpha^2$ of the integrand in the interval $(1,2+2\alpha)$ approaches the edge of this interval at small α.

This problem can be solved by deforming the contour. Let us analytically continue $\rho(t,\alpha)$ from the interval $0<t<1$ to larger values of t. This function has a branch cut along the ray $[1,\infty)$. Note that the value of $\rho(t,\alpha)$ in the interval $(1,2+2\alpha)$ is the half-sum of analytical continuations from the two sides of the cut. This means that that the whole integral can be represented as the half-sum of integrals along the contours $[\pm i0,2+2\alpha\pm i0]$. Since both upper and lower contour can be further deformed to go far from the singular point $t=1-\alpha^2$, it is now safe to expand in powers of α under the integral.

The number of channels is even, so in $\rho(t,y)$ we can keep only the terms of even power in α. As a result, we get the formula

$$\begin{aligned}\rho(y,\alpha)&=\frac{\alpha^2}{2(1+\alpha)^2}\int_0^{2+2\alpha}dt\,e^{-ty}\frac{t(2+2\alpha-t)}{(1-t-\alpha^2)}=\\ &=-\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{1-e^{-2y(1+\alpha)\alpha^2}}{y(1+\alpha)^2}-e^{-y(1-\alpha^2)}\alpha^2\left(\mathrm{Ei}(-y(1+\alpha^2))-\mathrm{Ei}(y(1-\alpha^2))\right)\right).\end{aligned} \tag{115}$$

Remarkably, unlike in the orthogonal class, we have found an explicit expression for the generating function $\rho(y,\alpha)$.

The result for first few values of M is

$$\rho_{M=2}(y)=-\frac{\partial}{\partial y}\left(\frac{1}{2y}-\frac{e^{-2y}}{2y}+e^{-y}\,\mathrm{Shi}\,y\right), \tag{116}$$

$$\rho_{M=4}(y)=-\frac{\partial}{\partial y}\left(\frac{3}{2y}-\left(\frac{3}{2y}+\frac{3}{2}\right)e^{-2y}+ye^{-y}\,\mathrm{Shi}\,y\right), \tag{117}$$

$$\rho_{M=6}(x)=-\frac{\partial}{\partial y}\left(\frac{5}{2y}-\frac{y}{4}-\left(\frac{5}{2y}+\frac{15}{4}+\frac{9y}{4}+\frac{y^2}{3}\right)e^{-2y}+\frac{y^2}{2}e^{-y}\,\mathrm{Shi}\,y\right). \tag{118}$$

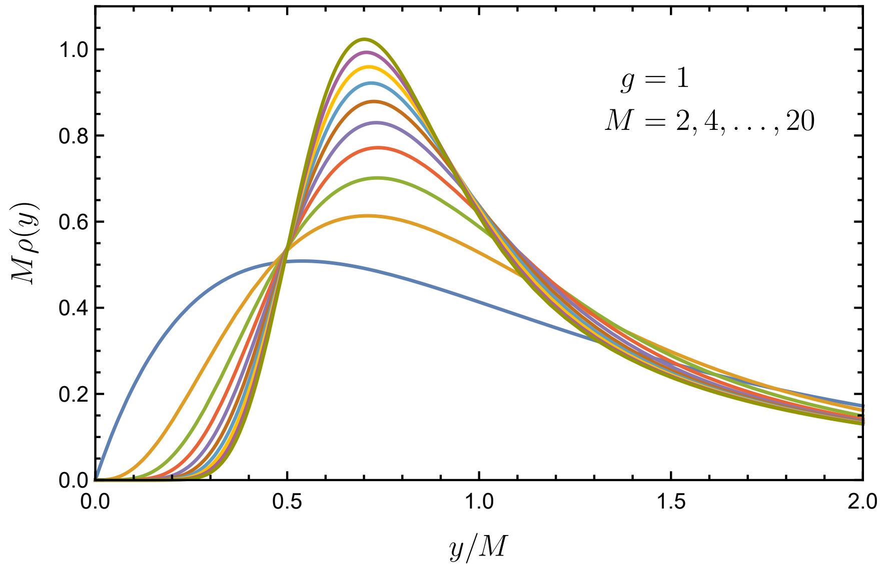

Figure 5: $\rho(y)$ in the symplectic class for a 0D system with M perfectly open channels.

The first ten functions are shown in Fig. 5. From formula (113) and (115) one can obtain the asymptotics for large y

$$\rho(y \to \infty) = \frac{M}{2y^2} + \frac{M}{2y^3} + \ldots \tag{119}$$

The first term is semiclassical, the next one is a quantum correction. Let's note that this correction has a different sign compared to the orthogonal class.

As in the orthogonal class, the asymptotics for small y was found from the explicit form of the distribution functions, for $M < 100$

$$\rho(y \to 0) = \frac{2^{M+1}M}{(M+2)!}\left(y^{M-1} - \frac{2M+2}{M+4}y^M + \ldots\right). \tag{120}$$

4.3.3 0D limit with two identical channels with arbitrary transmission coefficients

We can also calculate the resonance density for a zero-dimensional system attached to only two identical channels with arbitrary transmission coefficients. Let's introduce the generating function a little differently

$$\Gamma(\lambda, \lambda_1, \lambda_2, \alpha) = \sum_{n=1}^{\infty} \frac{\lambda_1^2 + \lambda_2^2 + g^2 + 2g\lambda_1\lambda_2 - 1}{(\lambda + g)^n}(-1)^n\alpha^n = \\ = -\frac{\alpha(\lambda_1^2 + \lambda_2^2 + g^2 + 2g\lambda_1\lambda_2 - 1)}{\lambda + g + \alpha}. \tag{121}$$

Performing analytical continuation, we obtain

$$\Gamma(-\lambda, -\lambda_1 + i0, \lambda_2) - \Gamma(-\lambda, -\lambda_1 - i0, \lambda_2) = \alpha(\lambda_1^2 + \lambda_2^2 + g^2 - 2g\lambda_1\lambda_2 - 1)\delta(\lambda - g - \alpha). \tag{122}$$

It allow us to take the integral over the bosonic eigenvalue

$$\rho(y) = -\frac{1}{8}\frac{\partial^2}{\partial\alpha^2}\frac{\partial^2}{\partial y^2}\int_{-1}^{1} d\lambda_1 \int_{-1}^{1} d\lambda_2\, J(g+\alpha, \lambda_1, \lambda_2)\Psi(g+\alpha, \lambda_1, \lambda_2). \tag{123}$$

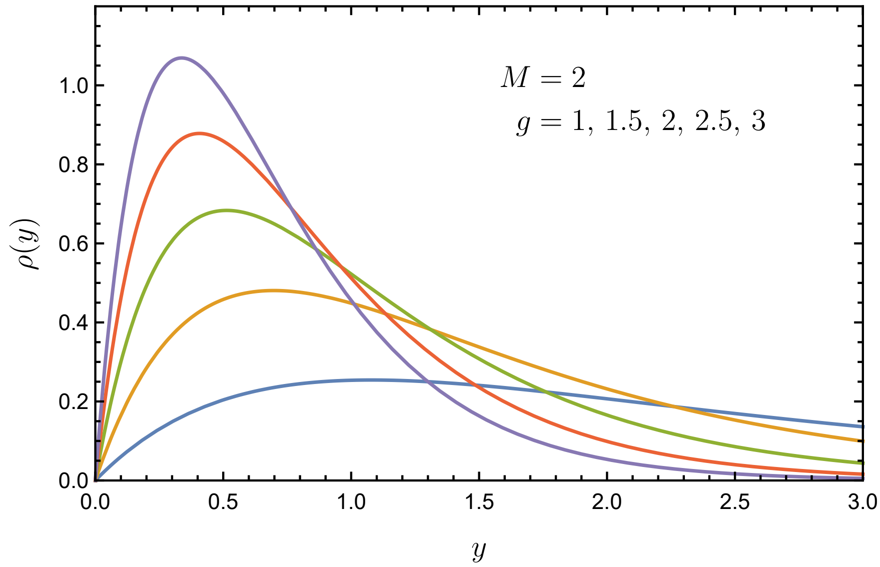

Figure 6: $\rho(y)$ in the symplectic class for a 0D system with two channels for different values of g.

Let's perform the inverse Laplace transform for $\rho(y)$. The exponential factor converts to a delta function which allows us to integrate over λ_2. After calculating the integral over λ_1 and taking the second derivative with respect to α, we obtain

$$\rho(t) = \begin{cases} \dfrac{t(2g-t)}{2(g-t)}, & g-1 < t < g+1, \\ 0, & \text{otherwise}. \end{cases} \tag{124}$$

The singularity at the point $t = g$ has the same nature as before. Applying the same argument we conclude that the distribution function is given by the principal value of the integral for the direct Laplace transform. Performing it, we get

$$\rho(y) = -\frac{\partial}{\partial y} e^{-gy} \left(\frac{\sinh y}{y} + g \operatorname{Shi} y \right). \tag{125}$$

In the case $g = 1$, the result coincides with the formula (116).

The resonance density for different values of g is shown in Fig. 6. Asymptotics for small and large y are

$$\rho(y) = \begin{cases} \left(g^2 - \dfrac{1}{3}\right)(y - gy^2 + \ldots), & y \to 0, \\ \dfrac{g^2-1}{2y} + \dfrac{g^2+1}{2y^2} + \ldots, & y \to \infty. \end{cases} \tag{126}$$

In the limit of small transmission coefficients (i.e., $g \gg 1$) we get the special case of χ^2-distribution with 4 degrees of freedom

$$\rho(y) \approx g^2 y e^{-gy}. \tag{127}$$

5 Summary and discussion

To summarize, we have developed a general approach to calculate the average distribution of scattering resonances. Our method is based on the supersymmetric nonlinear sigma model that describes both the studied disordered system and the attached ballistic wire. We derived general formulas (60), (81) and (106) for disordered systems of all three Wigner-Dyson symmetry classes. These formulas are applicable for an arbitrary set of transmission coefficients in the interface between the system and the wire.

We have specifically considered the zero-dimensional limit of the problem assuming identical transmission coefficients in all channels. In the unitary class, our formula (66) reproduces the result of [1]. In the orthogonal and symplectic classes we obtained formulas (91) and (115) which allow to easily compute the resonance distribution for any given number of channels with perfect transmission. Also, in the symplectic class with only one pair of channels, an exact formula (125) was obtained for any transmission coefficients.

Let us comment on the limits of applicability of our results. First, the nonlinear sigma model works for relatively low energies $\eta\tau \ll 1$. In terms of y

$$y \ll \frac{1}{\tau\Delta}. \tag{128}$$

Second, the boundary action of the form (32) implies that the supermatrix Q is constant along the interface. Let us denote the characteristic transverse size of the interface as d. The characteristic distance at which the supermatrix changes is $l_Q = \sqrt{D/\eta}$, so

$$y \ll \frac{D}{d^2\Delta}, \tag{129}$$

The condition for the applicability of the zero-dimensional limit is similar. The length l_Q must be greater than the typical grain length L

$$y \ll \frac{D}{L^2\Delta} = \frac{E_{\mathrm{Th}}}{\Delta}. \tag{130}$$

This is a more restrictive condition than (128) and (129).

Our study can be continued in several directions. First, the nonlinear supersymmetric sigma model formalism allows us to also study one-dimensional systems. We can thus extend the results of [1] to orthogonal and symplectic wires. Second, it is interesting to compare our results with numerical simulations. Third, our general formulas make it possible to compute the moments of the resonance widths distribution in a system with arbitrary transmission coefficients. In particular, it is interesting to derive the Moldauer–Simonius relation [26, 27] for orthogonal and symplectic classes from our results.

References

[1] *Fyodorov, Yan V.* Resonances in a single-lead reflection from a disordered medium: σ-model approach / Yan V. Fyodorov, Mikhail A. Skvortsov, Konstantin S. Tikhonov // *Annals Phys.* — 2024. — Vol. 460. — P. 169568.

[2] *Osman, Mohammed.* Chaotic scattering with localized losses: S-matrix zeros and reflection time difference for systems with broken time-reversal invariance / Mohammed Osman, Yan V. Fyodorov // *Phys. Rev. E.* — 2020. — Jul. — Vol. 102. — P. 012202.

[3] *Fyodorov, Yan V.* Distribution of zeros of the S-matrix of chaotic cavities with localized losses and coherent perfect absorption: non-perturbative results / Yan V Fyodorov, Suwun Suwunnarat, Tsampikos Kottos // *Journal of Physics A: Mathematical and Theoretical.* — 2017. — Jun. — Vol. 50, no. 30. — P. 30LT01.

[4] Random Matrix Theory Approach to Chaotic Coherent Perfect Absorbers / Huanan Li, Suwun Suwunnarat, Ragnar Fleischmann et al. // *Physical Review Letters.* — 2017. — Jan. — Vol. 118, no. 4.

[5] Predicting the statistics of wave transport through chaotic cavities by the random coupling model: A review and recent progress / Gabriele Gradoni, Jen-Hao Yeh, Bo Xiao et al. // *Wave Motion.* — 2014. — Vol. 51, no. 4. — Pp. 606–621. — Innovations in Wave Modelling.

[6] *Huang, Yiming.* Wave excitation and dynamics in non-Hermitian disordered systems / Yiming Huang, Yuhao Kang, Azriel Z. Genack // *Phys. Rev. Res.* — 2022. — Feb. — Vol. 4. — P. 013102.

[7] *Chen, Lei.* Statistics of Complex Wigner Time Delays as a Counter of S-Matrix Poles: Theory and Experiment / Lei Chen, Steven M. Anlage, Yan V. Fyodorov // *Phys. Rev. Lett.* — 2021. — Nov. — Vol. 127. — P. 204101.

[8] *Davy, Matthieu.* Probing nonorthogonality of eigenfunctions and its impact on transport through open systems / Matthieu Davy, Azriel Z. Genack // *Phys. Rev. Res.* — 2019. — Oct. — Vol. 1. — P. 033026.

[9] *Liu, C.* Dicke Phase Transition with Multiple Superradiant States in Quantum Chaotic Resonators / C. Liu, A. Di Falco, A. Fratalocchi // *Phys. Rev. X.* — 2014. — Jun. — Vol. 4. — P. 021048.

[10] *Gaspard, Pierre.* Semiclassical quantization of the scattering from a classically chaotic repellor / Pierre Gaspard, Stuart Rice // *The Journal of Chemical Physics.* — 1989. — 02. — Vol. 90. — Pp. 2242–2254.

[11] *Verbaarschot, J. J. M.* Grassmann Integration in Stochastic Quantum Physics: The Case of Compound Nucleus Scattering / J. J. M. Verbaarschot, H. A. Weidenmuller, M. R. Zirnbauer // *Phys. Rept.* — 1985. — Vol. 129. — Pp. 367–438.

[12] *Lewenkopf, C.H.* Stochastic versus semiclassical approach to quantum chaotic scattering / C.H Lewenkopf, H.A Weidenmüller // *Annals of Physics.* — 1991. — Vol. 212, no. 1. — Pp. 53–83.

[13] *Efetov, K.B.* Statistics of the levels in small metallic particles / K.B. Efetov // *Zh. Eksp. Teor. Fiz.* — 1982. — Aug. — Vol. 56, no. 2.

[14] *Fyodorov, Yan.* Statistics of resonance poles, phase shifts and time delays in quantum chaotic scattering: Random matrix approach for systems with broken time-reversal invariance / Yan Fyodorov, Hans-Jürgen Sommers // *J. Math. Phys.* — 1997. — 04. — Vol. 38. — Pp. 1918–1981.

[15] *Sommers, Hans-Jürgen.* S-matrix poles for chaotic quantum systems as eigenvalues of complex symmetric random matrices: from isolated to overlapping resonances / Hans-Jürgen Sommers, Yan V Fyodorov, Mikhail Titov // *Journal of Physics A: Mathematical and General.* — 1999. — feb. — Vol. 32, no. 5. — P. L77.

[16] *Kuhl, U.* Microwave experiments using open chaotic cavities in the realm of the effective Hamiltonian formalism / U. Kuhl, O. Legrand, F. Mortessagne // *Fortschritte der Physik.* — 2013. — Vol. 61, no. 2-3. — Pp. 404–419.

[17] *Kottos, Tsampikos.* Chaotic Scattering on Graphs / Tsampikos Kottos, Uzy Smilansky // *Phys. Rev. Lett.* — 2000. — Jul. — Vol. 85. — Pp. 968–971.

[18] Open system of interacting fermions: Statistical properties of cross sections and fluctuations / G. L. Celardo, F. M. Izrailev, V. G. Zelevinsky, G. P. Berman // *Physical Review E.* — 2007. — Sep. — Vol. 76, no. 3.

[19] *Efetov, Konstantin.* Supersymmetry in Disorder and Chaos / Konstantin Efetov. — Cambridge University Press, 1996.

[20] *Mirlin, A.D.* Conductance Fluctuations of Disordered Wires: Fourier Analysis on Supersymmetric Spaces / A.D. Mirlin, A. Mullergroeling, M.R. Zirnbauer // *Annals of Physics.* — 1994. — Vol. 236, no. 2. — Pp. 325–373.

[21] *Daniil, Antonenko.* Anderson localization at the boundary of a two-dimensional topological superconductor. — 2021. — 12.

[22] *Nazarov, Yu. V.* Limits of universality in disordered conductors / Yu. V. Nazarov // *Phys. Rev. Lett.* — 1994. — Jul. — Vol. 73. — Pp. 134–137.

[23] *Zirnbauer, Martin R.* Anderson localization and non-linear sigma model with graded symmetry / Martin R. Zirnbauer // *Nuclear Physics B.* — 1986. — Vol. 265, no. 2. — Pp. 375–408.

[24] *Mirlin, A. D.* Statistics of energy levels and eigenfunctions in disordered and chaotic systems: Supersymmetry approach / A. D. Mirlin // *Proc. Int. Sch. Phys. Fermi.* — 2000. — Vol. 143. — Pp. 223–298.

[25] *Porter, C.E.* Statistical Theories of Spectra: Fluctuations; a Collection of Reprints and Original Papers, with an Introductory Review / C.E. Porter. Perspectives in physics. — Academic Press, 1965.

[26] *Moldauer, P. A.* Unitary Models of Nuclear Resonance Reactions / P. A. Moldauer // *Phys. Rev.* — 1967. — May. — Vol. 157. — Pp. 907–921.

[27] *Simonius, M.* Unitarity and the energy averaged S-matrix / M. Simonius // *Physics Letters B.* — 1974. — Vol. 52, no. 3. — Pp. 279–281.

A Fourier analysis on the sigma-model manifold

In this Appendix we give the main properties and explicit forms for the eigenfunctions of the radial Laplace operators on the sigma-model manifolds.

In the unitary class the Laplace operator on the sigma model is defined as follows

$$\Delta = \frac{1}{J}\frac{\partial}{\partial\lambda_1}J(\lambda_1^2-1)\frac{\partial}{\partial\lambda_1} + \frac{1}{J}\frac{\partial}{\partial\lambda_2}J(1-\lambda_2^2)\frac{\partial}{\partial\lambda_2}, \tag{131}$$

$$J = \frac{1}{(\lambda_1-\lambda_2)^2}. \tag{132}$$

In the orthogonal and symplectic classes it has the form

$$\Delta_{\text{orth}} = -\Delta_{\text{sympl}} = \frac{1}{J}\frac{\partial}{\partial\lambda}J(1-\lambda^2)\frac{\partial}{\partial\lambda} + \frac{1}{J}\frac{\partial}{\partial\lambda_1}J(\lambda_1^2-1)\frac{\partial}{\partial\lambda_1} + \frac{1}{J}\frac{\partial}{\partial\lambda_2}J(\lambda_2^2-1)\frac{\partial}{\partial\lambda_2}, \tag{133}$$

$$J = \frac{|1-\lambda^2|}{(\lambda^2+\lambda_1^2+\lambda_2^2-2\lambda\lambda_1\lambda_2-1)^2}. \tag{134}$$

Eigenfunctions of the full Laplace operator transform according to irreducible representations of the group G under rotations of Q. Within each irreducible representation there is exactly one eigenfunction that depends on the λ variables only. Hence these eigenfunctions are also the eigenfunctions of the radial Laplace operators and form a complete basis for the functions of λ variables.

Expansion in this basis represents the Fourier transform

$$\Psi(\lambda) = \Psi_0 + \sum_{\boldsymbol{q}} \mu_{\boldsymbol{q}}\Psi_{\boldsymbol{q}}L_{\boldsymbol{q}}(\lambda), \tag{135}$$

where $\boldsymbol{q}$ is a set of indices (angular momenta) that enumerate irreducible representations of G

$$\boldsymbol{q} = \begin{cases} \{l,q\}, & \text{unitary}, \\ \{l,q_1,q_2\}, & \text{orthogonal}, \\ \{q,l_1,l_2\}, & \text{symplectic}. \end{cases} \tag{136}$$

Here q, q_1, and q_2 are continuous positive momenta in the bosonic sector and l, l_1, l_2 are integer non-negative momenta in the fermionic sector of the model.

Summation over continuous indices q implies the integral over them. In the orthogonal class, continuous momenta $q_{1,2}$ must be equal when $l=0$. In the symplectic class discrete momenta $l_{1,2}$ should have the same parity.

There is also a special eigenfunction of the Laplace operator wich is constant on the whole manifold. This eigenfunction does not correspond to any value of parameters $\boldsymbol{q}$ and should be added separately. The corresponding Fourier amplitude Ψ_0 of this constant eigenfunction is the first term in (135).

The weight $\mu_{\boldsymbol{q}}$ in the Fourier transform (135) can be found from the normalization integrals

$$\int d\lambda\, J(\lambda)L_{\boldsymbol{q}}(\lambda)L_{\boldsymbol{q}'}(\lambda) = \frac{1}{\mu_{\boldsymbol{q}}}\delta_{\boldsymbol{q},\boldsymbol{q}'}, \tag{137}$$

$$\sum_{\boldsymbol{q}} \mu_{\boldsymbol{q}}L_{\boldsymbol{q}}(\lambda)L_{\boldsymbol{q}}(\lambda') = \frac{1}{J(\lambda)}\delta(\lambda-\lambda'). \tag{138}$$

It is convenient to choose the normalization of the eigenfunctions and hence the weight $\mu_{\boldsymbol{q}}$ such that all Fourier components of the delta function on the sigma-model manifold are equal to unity

$$\delta(Q - \Lambda) = 1 + \sum_{\boldsymbol{q}} \mu_{\boldsymbol{q}} L_{\boldsymbol{q}}(\lambda). \tag{139}$$

The eigenfunctions $L_{\boldsymbol{q}}(\lambda)$ vanish at $\lambda = 1$, so we can find the amplitude of the constant mode as the value of the function $\Psi(\lambda)$ at this point

$$\Psi_0 = \Psi(\lambda = 1). \tag{140}$$

Using relation (137) one can express all other Fourier components as

$$\Psi_{\boldsymbol{q}} = \int d\lambda\, J(\lambda)(\Psi(\lambda) - \Psi_0) L_{\boldsymbol{q}}(\lambda). \tag{141}$$

Eigenfunctions of the Laplace operator on the sigma-model manifold can be explicitly calculated using Iwasawa decomposition of the group G. These functions are constructed as plane waves in the corresponding Iwasawa coordinates averaged over rotations by the group K, see [20] and [21].

In the unitary class the eigenfunctions are expressed through Legendre polynomials

$$L_{l,q}(\lambda_1, \lambda_2) = -\frac{1}{2}((l+1/2)^2 + q^2)(\lambda_1 - \lambda_2) P_{-1/2+iq}(\lambda_1) P_l(\lambda_2), \tag{142}$$

$$\Delta L_{l,q}(\lambda_1, \lambda_2) = -((l+1/2)^2 + q^2) L_{l,q}(\lambda_1, \lambda_2). \tag{143}$$

The coefficient $\mu_{l,q}$ is

$$\mu_{l,q} = \frac{4(l+1/2) q \tanh(\pi q)}{((l+1/2)^2 + q^2)^2}. \tag{144}$$

This corresponde to the proper normalization such that (139) holds.

In the orthogonal and symplectic classes the eigenfunctions contain products of three Legendre functions and do not factorize. We will use the following notation:

$$R_\nu^\mu(x) = \begin{cases} (x^2 - 1)^{\mu/2} P_\nu^\mu(x), & x > 1, \\ (1 - x^2)^{\mu/2} \mathrm{P}_\nu^\mu(x), & -1 < x < 1, \end{cases} \tag{145}$$

where $P_\nu^\mu(x)$ is the standard associated Legendre function, and $\mathrm{P}_\nu^\mu(x)$ is the Ferrers function.

In the orthogonal class the eigenfunctions of the Laplace operator are

$$\begin{aligned} L_{l,q_1,q_2}(\lambda, \lambda_1, \lambda_2) &= R_l(\lambda) R^1_{-1/2+iq_1}(\lambda_1) R^1_{-1/2+iq_2}(\lambda_2) \\ &+ \frac{1}{2}\left(l^2 + l + q_1^2 + q_2^2 + \frac{1}{2}\right)(\lambda - \lambda_1\lambda_2) R_l(\lambda) R_{-1/2+iq_1}(\lambda_1) R_{-1/2+iq_2}(\lambda_2) \\ &+ \frac{1}{2}\left(l^2 + l + q_1^2 - q_2^2\right)(\lambda_1 - \lambda\lambda_2) R_l^{-1}(\lambda) R^1_{-1/2+iq_1}(\lambda_1) R_{-1/2+iq_2}(\lambda_2) \\ &+ \frac{1}{2}\left(l^2 + l - q_1^2 + q_2^2\right)(\lambda_2 - \lambda\lambda_1) R_l^{-1}(\lambda) R_{-1/2+iq_1}(\lambda_1) R^1_{-1/2+iq_2}(\lambda_2) \\ &+ \left(\frac{1}{4}\left(l^2 + l + q_1^2 + q_2^2 + \frac{1}{2}\right)^2 - \left(q_1^2 + \frac{1}{4}\right)\left(q_2^2 + \frac{1}{4}\right)\right) \\ &\times (\lambda^2 + \lambda_1^2 + \lambda_2^2 - 2\lambda\lambda_1\lambda_2 - 1) R_l^{-1}(\lambda) R_{-1/2+iq_1}(\lambda_1) R_{-1/2+iq_2}(\lambda_2), \end{aligned} \tag{146}$$

$$\Delta L_{l,q_1,q_2}(\lambda, \lambda_1, \lambda_2) = -\left(l^2 + l + q_1^2 + q_2^2 + \frac{1}{2}\right) L_{l,q_1,q_2}(\lambda, \lambda_1, \lambda_2). \tag{147}$$

The normalization coefficient μ_{l,q_1,q_2} is

$$\mu_{l,q_1,q_2} = \frac{16l(l+1)(l+1/2)q_1q_2\tanh(\pi q_1)\tanh(\pi q_2)}{\left(l^2+(q_1+q_2)^2\right)\left(l^2+(q_1-q_2)^2\right)\left((l+1)^2+(q_1+q_2)^2\right)\left((l+1)^2+(q_1-q_2)^2\right)}. \tag{148}$$

In the symplectic class the eigenfunctions have the same structure up to interchanging compact and noncompact moment

$$\begin{aligned} L_{q,l_1,l_2}(\lambda,\lambda_1,\lambda_2) &= R_{-1/2+iq}(\lambda)R^1_{l_1}(\lambda_1)R^1_{l_2}(\lambda_2) \\ &- \frac{1}{2}\left(\frac{1}{4}+q^2+l_1^2+l_1+l_2^2+l_2\right)(\lambda-\lambda_1\lambda_2)R_{-1/2+iq}(\lambda)R_{l_1}(\lambda_1)R_{l_2}(\lambda_2) \\ &- \frac{1}{2}\left(\frac{1}{4}+q^2+l_1^2+l_1-l_2^2-l_2\right)(\lambda_1-\lambda\lambda_2)R^{-1}_{-1/2+iq}(\lambda)R^1_{l_1}(\lambda_1)R_{l_2}(\lambda_2) \\ &- \frac{1}{2}\left(\frac{1}{4}+q^2-l_1^2-l_1+l_2^2+l_2\right)(\lambda_2-\lambda\lambda_1)R^{-1}_{-1/2+iq}(\lambda)R_{l_1}(\lambda_1)R^1_{l_2}(\lambda_2) \\ &+ \left[\frac{1}{4}\left(\frac{1}{4}+q^2+l_1^2+l_1+l_2^2+l_2\right)^2-(l_1^2+l_1)(l_2^2+l_2)\right] \\ &\times(\lambda^2+\lambda_1^2+\lambda_2^2-2\lambda\lambda_1\lambda_2-1)R^{-1}_{-1/2+iq}(\lambda)R_{l_1}(\lambda_1)R_{l_2}(\lambda_2), \end{aligned} \tag{149}$$

$$\Delta L_{l,q_1,q_2}(\lambda,\lambda_1,\lambda_2) = -\left(\frac{1}{4}+q^2+l_1^2+l_1+l_2^2+l_2\right)L_{l,q_1,q_2}(\lambda,\lambda_1,\lambda_2). \tag{150}$$

The normalization factor μ_{q,l_1,l_2} is

$$\mu_{q,l_1,l_2} = \frac{32(l_1+1/2)(l_2+1/2)(q^2+1/4)q\tanh(\pi q)}{\left((l_1+l_2+3/2)^2+q^2\right)\left((l_1+l_2+1/2)^2+q^2\right)\left((l_1-l_2+1/2)^2+q^2\right)\left((l_1-l_2-1/2)^2+q^2\right)}. \tag{151}$$

These functions in both classes are properly normalized according to (139), see [21].