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«Эволюция магнитного поля в квазидвумерном случайном потоке»

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MASTER’S THESIS
«Evolution of magnetic field in quasi-two-dimensional chaotic flow»

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1 Introduction

1.1 Relevance of the work

In the middle of the XX century it was found that the turbulent motion of a conducting liquid in a magnetic field can lead to spontaneous generation of sufficiently large magnetic fields (the so-called Dynamo effect)[1]. It is believed that this effect is responsible for the appearance of magnetic fields in a range of astronomical objects: planets, stars and galaxies [1], [2]. Indeed, let them have some initial field during their formation. Then, without the mechanism of maintaining the magnetic field due to various dissipations inherent in any conducting medium, it would fade with time. However, the amplitudes of the magnetic fields of astronomical objects slightly vary in foreseeable times, which is a good argument in favor of the dynamo effect.

The study of the behavior of the magnetic field of astronomical objects is not only of great interest from a fundamental point of view, but also has many important practical applications. In particular, understanding the processes taking place on the Sun can significantly improve the quality of predictions of various natural disasters. This will ensure the safety of many people, as well as avoid huge material losses, such as the destruction of buildings and other property, the failure of various communication and other systems, which often takes place during the natural disasters.

One of the theories explaining the maintenance of the amplitude of the magnetic field, can be the Dynamo theory [2]-[4]. In particular, a model qualitatively describing the observed pattern of the cyclic evolution of magnetic fields and sunspots on the surface of the Sun was proposed in [3]. The magnetic field is the most significant in the convective zone, where there is a strong shear flow. This zone is quite far from the center. It is a spherical layer in which strong quasi-two-dimensional flows of the medium are observed. The radial velocity component is much smaller than the shear one ($v_r \ll v_\tau$). In addition, there is a strong gradient of shear velocity along the radius due to the presence of finite viscosity. Thus, the flow is quasi-two-dimensional in this area. Along the surface of a constant radius (which is effectively plane at large radius values) the flow is two-dimensional with good accuracy, but the velocity field is strongly dependent on the radius. In this regard, it is of interest to study the intermediate case, which is not considered earlier – a quasi-two-dimensional flow,in which the velocity field depends on the vertical coordinate $\vec{v}(x, y, z, t)$, but has no vertical component ($v_z \equiv 0$).

In this paper, we consider a model in which the magnetic field obeys the following equation, that is well known in magnetohydrodynamics [5]:

$$\partial_t \mathbf{B} = (\mathbf{B}, \nabla)\mathbf{v} - (\mathbf{v}, \nabla)\mathbf{B} + \kappa \nabla^2 \mathbf{B} \tag{1}$$

It is obtained from Maxwell's equations under the locality assumptions of the Ohm's law $\mathbf{j} = \sigma \mathbf{E}$ and the unit of magnetic susceptibility ($\mu = 1$). However, these approximations are not always applicable to the description of the magnetic fields of astronomical objects, especially if the magnetic field is considered in the plasma. Particularly, it is assumed in [3], where the model qualitatively describing the evolution of the magnetic field on the Sun is considered, that the magnetic susceptibility may depend on the coordinates.

In addition to the description of the magnetic fields on the Sun, the study of the model under consideration is of considerable interest for the investigation of the classical formulation of the Dynamo effect in a quasi-two-dimensional flow. This type of flow is a certain "intermediate" mode in terms of dimension between two- and three-dimensional. For turbulent three-dimensional flows, the Dynamo effect was discovered and explained long ago. The situation with the Dynamo effect in two-dimensional flows is different. For a long time it was believed that it is not realized for all two-dimensional flows. So, in the works of Zeldovich [6], [7] was even formulated the so-called "anti-Dynamo theorem", which states the impossibility of generating strong local magnetic fields in a two-dimensional turbulent flow. However, the Dynamo effect was found in an efficient two-dimensional problem in [8], where the flow is assumed to be two-dimensional and the magnetic field to be three-dimensional.

1.2 The explanation of Dynamo effect

In this paper, we study the kinematic stage of the Dynamo effect, in which the magnetic field grows strong enough to influence the flow of the conductive liquid. First, we study in detail the case of infinitely large conductivity. Then there is no dissipation in the system, and the magnetic field will be "frozen" in the fluid flow:

$$\partial_t \mathbf{B} = (\mathbf{B}, \nabla)\mathbf{v} - (\mathbf{v}, \nabla)\mathbf{B} = \mathrm{rot}\,[\mathbf{v}, \mathbf{B}] \tag{2}$$

Large magnetic fields generation is observed not for all types fo flow. It should be a turbulent flow where there are singularities of the hyperbolic type, in the vicinity of which there is a compression of the flow in one direction and stretching in the other one. The velocity field has the following form in the vicinity of this point:

$$v_\alpha \approx V_\alpha^{(0)}(\mathbf{r}) + u_\alpha,\;\; u_\alpha = \sigma_{\alpha\beta} r_\beta,\;\; \begin{cases} u_x = \lambda x; \\ u_y = -\lambda y; \end{cases} \tag{3}$$

where λ is the Lyapunov exponent for a given flow. Every particle in the fluid velocity field $\mathbf{u}(\mathbf{r})$ will move along a hyperbolic path

$$\begin{cases} x = x_0 e^{\lambda t}; \\ y = y_0 e^{-\lambda t}; \end{cases} \qquad y = \frac{x_0 y_0}{x}. \tag{4}$$

The flow is compressed over y-axis and stretched over x-axis (Fig.3).

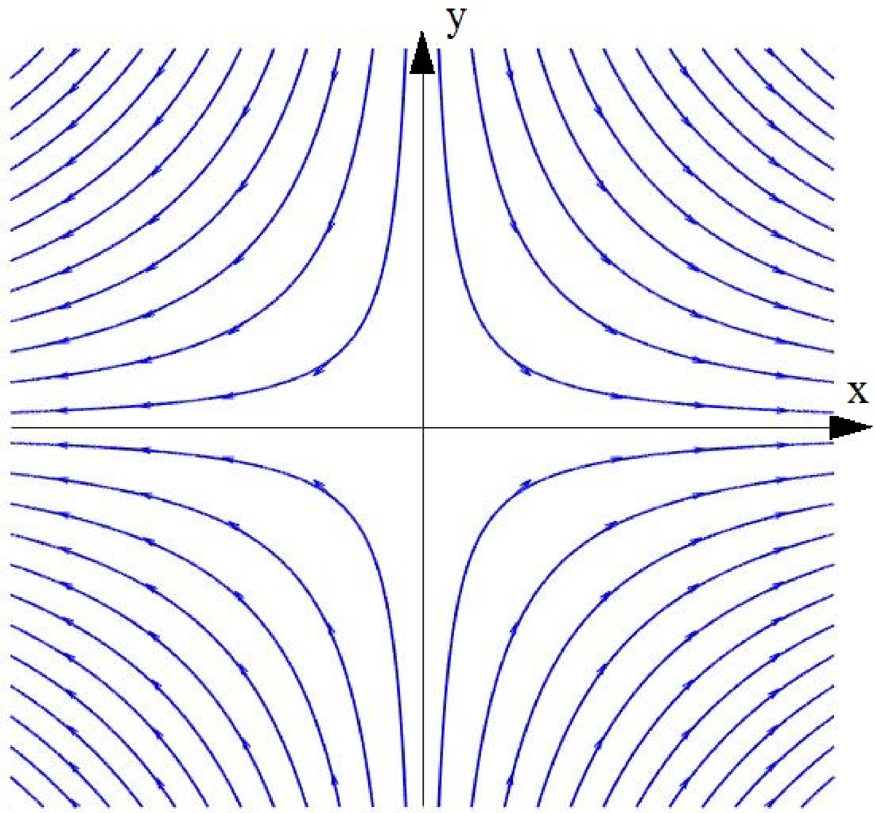

Рис. 1: Stream lines in the vicinity of singular point

Let's give a qualitative description of why one of the components of the field, namely B_x, will grows exponentially over time.

Suppose that at the initial moment of time the magnetic field is the most significant in the ball of radius l with the center at our singular point, and outside this ball it decreases very quickly with distance from the center. Since the magnetic field is "frozen" into the liquid, our ball will expand exponentially with time along the x-axis and shrink along the y-axis, turning into an ellipsoid. The field is large inside this ellipsoid and small enough outside. The value of the field will change in time. We will consider the area passing through a singular point and lying in the yz plane. The size of the site is much larger than l. The magnetic flux through this entire area will be determined by the flow through the cross-section of the area where the magnetic field is essentially by the Oyz plane. $\Phi = B_x(t)S_K(t) = \text{const}$. Since there is no movement along the axis z and the compression over y-axis occurs, the area of the described above section decreases exponentially with time: $S_{yz}(t) = S_{yz}(0)e^{-\lambda t}$. So, the x-component of the magnetic field will grow exponentially with time: $B_x \propto e^{\lambda t}$ (Fig. 2). Indeed, the magnetic field lines frozen into the flow will be compacted along one of the axes, along which the flow is compressed, and thus the value of the magnetic field induction will grow.

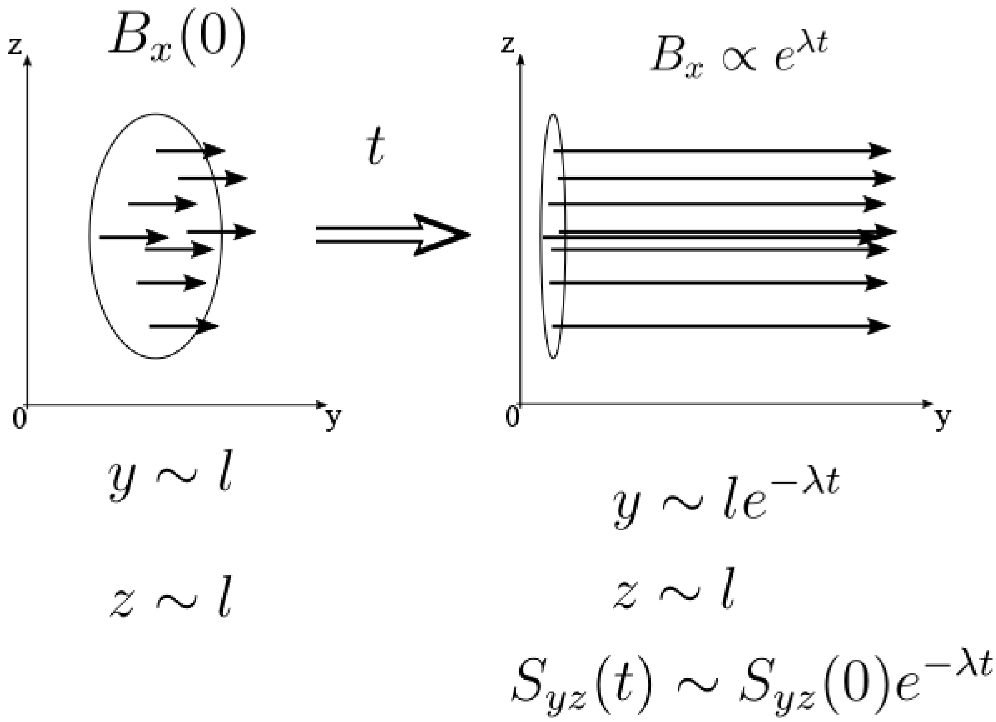

Рис. 2: Qualitative explanation of phenomenon: the growth of magnetic field

If the conductivity is finite, the dissipation will take place there:

$$\partial_t \mathbf{B} = (\mathbf{B}, \nabla)\mathbf{v} - (\mathbf{v}, \nabla)\mathbf{B} + \kappa \nabla^2 \mathbf{B}, \ \ \kappa = \frac{c^2}{4\pi\sigma} \tag{5}$$

Dissipation occurs at scales when the dissipative term is comparable to the term describing the temporal evolution of the field: $r_d = \sqrt{\frac{2\kappa}{\lambda}}$.

Again, we will hold arguments similar to those described above. The width of the area along the y-axis, where the magnetic field is significant, will decrease exponentially rapidly to the dissipation scale r_d and than stops. Up to this point, there will be an exponential growth of the field. Then the exponential growth of the magnetic field will stop.

This section describes a qualitative description of the Dynamo effect, showing the exponential growth of the magnetic field in a turbulent flow. The exact investigation of the magnetic field evolution for both the given and the random turbulent flow will be given below.

1.3 Formulation of the problem

Let us return to the equation (1) describing the evolution of the magnetic field. We will be interested in the kinematic stage, when the field is not so large as to influence the flow. From the equation (1) on the dynamics of the magnetic field it is seen that the energy is dissipated on the diffusion scale

$r_d = \sqrt{\frac{2\kappa}{\lambda}}$, where λ is the Lyapunov exponent of the considered flow.

We assume that fluctuations are created by volume forces, so the fluctuating component of the velocity does not depend on the vertical coordinate.

The correlations of the magnetic field are investigated at different times in this paper. В данной работе исследуются корреляции магнитного поля на различных временах.

First, the behavior of the correlator in the early times when the magnetic field is growing will be investigated. At this stage, the magnetic field blobs will be stretched into long filaments until the length of these filaments is compared with the characteristic correlation radius of the flow velocity field R. In this approximation, the scales of the coordinates on which the correlators will be studied are much smaller than the correlation radius of the velocity field, but many large than dissipation scales: $r_d \ll r \ll R$. The large parameter $\frac{R}{r_d} \gg 1$ is proportional to the square root of the Prandtl magnetic number Pr_m, that expresses the ratio of internal friction forces to magnetic force [2],[9],[10]. This parameter describes the fact that in our case the effect of the magnetic field on the fluid flow is small compared to the dissipation, and hence compared to other processes. The velocity field will depend linearly on the coordinates in the horizontal plane:

$$v_\alpha \approx V_\alpha^{(0)}(z) + \sigma_{\alpha\beta} r_\beta, \;\; \alpha, \beta = \overline{1,2}. \tag{6}$$

Then the question of the behavior of the correlators at later times will be investigated. At this stage the scales of the order velocity field correlation radius and larger one influence the evolution of the magnetic field.

We also assume that the characteristic width of the initial magnetic field distribution l is well divided between the diffusive length and the radius of the velocity correlations: $r_d \ll l \ll R$.

The behavior of the magnetic field is well described by the induction of the magnetic field in the formulation, when the velocity field does not change with time randomly. Otherwise, it is convenient to monitor not the induction of the magnetic field, which can strongly fluctuate, but their correlation functions $F_{mn}(\mathbf{r}, t) = \langle B_m(\mathbf{r}', t) B_n(\mathbf{r}' + \mathbf{r}, t)\rangle, \;\; m, n = \overline{1,3}$.

First, using the example of a simple model, it will be shown that the presence of a vertical gradient of the velocity field weakens the magnetic field. The effect will then be demonstrated on a model with a random linear velocity of the flow, where the behavior of the correlators will be found.

2 Two-dimensional model

Let's consider a simple model where the velocity field is constant in time.

2.1 Constant velocity

The easiest way to study first is a two-dimensional flow with a time-dependent and coordinate constant velocity:

$$v_\alpha(r,t) = V_\alpha^{(0)} + \sigma_{\alpha\beta} r_\beta \tag{7}$$

The matrix is traceless because of the incompressibility of the liquid. Let us consider for simplicity the case when it is diagonal:

$$\hat{\sigma} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \tag{8}$$

Let's write out componentwise the vector equation for the induction of a magnetic field evolving in a two-dimensional flow with a constant velocity:

$$\boxed{\begin{cases} \partial_t B_3 = -V_\beta^{(0)} \partial_\beta B_3 - \sigma_{\mu\nu} r_\nu \partial_\mu B_3 + \kappa \nabla^2 B_3; \\ \partial_t B_\alpha = B_\beta \sigma_{\alpha\beta} - V_\beta^{(0)} \partial_\beta B_\alpha - \sigma_{\beta\nu} r_\nu \partial_\beta B_\alpha + \kappa \nabla^2 B_\alpha; \end{cases}} \tag{9}$$

2.1.1 Equations in Fourier domain

It is convenient to solve the system in the momentum representation by the method of characteristics:

$$\boxed{\begin{cases} \partial_t b_k^3 = \sigma_{\mu\nu} k_\mu \partial_{k_\nu} b_k^3 - \kappa k^2 b_k^3 - i V_\beta^{(0)} k_\beta b_k^3, \qquad k^2 = k_\nu k_\nu + k_3^2; \\ \partial_t b_\alpha = \sigma_{\alpha\beta} b_\beta + \sigma_{\beta\nu} k_\beta \frac{\partial}{\partial k_\nu} b_\alpha - \kappa k^2 b_\alpha - i V_\beta^{(0)} k_\beta b_\alpha; \end{cases}} \tag{10}$$

Let's introduce matrix $\hat{W} = \mathrm{Texp}\left(\int\limits_0^t \mathrm{d}\tau \hat{\sigma}(\tau)\right) = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{pmatrix}$

It is convenient to make a momentum replacement connecting the momentum at the initial instant of time and into the current one:

$$\sigma_{\mu\nu} = \frac{dW_{\mu\beta}(t)}{dt} W_{\nu\beta}^{-1}(t),\ \ q_\alpha = W_{\nu\alpha}(t) k_\nu,\ \ k_\nu = W_{\mu\nu}^{-1} q_\mu,\ \ \frac{\partial}{\partial k_\nu} = \frac{dq_\rho}{dk_\nu}\frac{\partial}{\partial q_\rho} = W_{\nu\rho}\frac{\partial}{\partial q_\rho} \tag{11}$$

Let's firstly solve the equation on $b_3(k,t)$ by the method of characteristics:

$$b_3(k,t) = b_3(q,0) \exp\left[-\kappa \int\limits_0^t \mathrm{d}\tau q_\alpha(t) q_\nu(t) W_{\alpha\beta}^{-1}(\tau) W_{\nu\beta}^{-1}(\tau) - \kappa k_3^2 t - i V_\beta^{(0)} q_\alpha \int\limits_0^t \mathrm{d}\tau W_{\alpha\beta}^{-1}(\tau)\right] = \tag{12}$$

$$= b_3(q,0) \exp\left[-\kappa\left(\frac{q_1^2(t)}{2\lambda} + \frac{q_2^2(t) e^{2\lambda t}}{2\lambda} + k_3^2 t\right) - \frac{i}{\lambda}\left(V_1^{(0)} q_1 + V_2^{(0)} q_2 e^{\lambda t}\right)\right].$$

Vertical component of the field in the momentum representation:

$$\boxed{b_3(k,t) = b_3\left(\hat{W}^T(t) k, 0\right) \exp\left[-\kappa \int\limits_0^t \mathrm{d}\tau k \hat{W}(t,\tau) \hat{W}^T(t,\tau) k - \kappa k_3^2 t - i V_\beta^{(0)} k_\mu \int\limits_0^t \mathrm{d}\tau W_{\mu\beta}(t,\tau)\right]} \tag{13}$$

Introducing the new variable $c_\gamma = W^{-1}_{\gamma\alpha} b_\alpha$ and solving the remaining equations of the system by the same method of characteristics, we finally obtain the behavior of the components in the horizontal plane:

$$\boxed{b_\alpha(k,t) = W_{\alpha\gamma}(t) b_\gamma(\hat{W}^T k, 0) \exp\left[-\kappa k_\nu k_\rho \int\limits_0^t \mathrm{d}\tau W_{\nu\beta}(t,\tau) W_{\rho\beta}(t,\tau) - \kappa k_3^2 t - i V_\beta^{(0)} k_\nu \int\limits_0^t \mathrm{d}\tau W_{\nu\beta}(t,\tau)\right]} \tag{14}$$

We assume that at the initial moment of time the magnetic field has a Gaussian form and it is significant on scales $\sim l \gg r_d$

$$b_a(q,0) = i\epsilon_{abc} q_b A_c; \qquad A_c = \eta_c e^{-q^2 l^2}, \qquad \eta_c = (0, \eta, 0); \tag{15}$$

$$b_1(q,0) = -i\eta q_3 e^{-q^2 l^2}; \qquad b_2(q,0) = 0; \qquad b_3(q,0) = i\eta q_1 e^{-q^2 l^2} \tag{16}$$

With this initial magnetic field distribution, the induction y component remains zero: $B_2(r,t) = 0$. Further we make the inverse Fourier transform and find the expressions for the induction of the magnetic field.

Let's write expressions in terms of "center mass coordinates": $\tilde{x} = x - \frac{V_1^{(0)}}{\lambda} e^{\lambda t}$, $\tilde{y} = y - \frac{V_2^{(0)}}{\lambda}$.

$$\begin{cases} B_1(\tilde{r},t) = e^{\lambda t} \eta \frac{z}{2(l^2+\kappa t)} \frac{\exp\left[-\frac{(\tilde{x} e^{-\lambda t})^2}{4l^2} - \frac{e^{2\lambda t} \tilde{y}^2}{4(l^2 + r_d^2 e^{2\lambda t})} - \frac{z^2}{4(l^2+\kappa t)}\right]}{8\pi^{3/2} l \sqrt{l^2+\kappa t} \sqrt{l^2 + r_d^2 e^{2\lambda t}}} \\ B_3(\tilde{r},t) = -\eta \frac{\tilde{x} e^{-\lambda t}}{2l^2} \frac{\exp\left[-\frac{(\tilde{x} e^{-\lambda t})^2}{4l^2} - \frac{e^{2\lambda t} (\tilde{y})^2}{4(l^2 + r_d^2 e^{2\lambda t})} - \frac{z^2}{4(l^2+\kappa t)}\right]}{8\pi^{3/2} l \sqrt{l^2+\kappa t} \sqrt{l^2 + r_d^2 e^{2\lambda t}}} \end{cases} \tag{17}$$

Such a functional dependence of the magnetic field induction components on the coordinates and time makes it easy to understand how the field changes with time and on what spatial scales its magnitude is the most significant.

2.1.2 The investigation of field behavior

In order to demonstrate clearly that in this case the dynamo effect actually takes place, we first consider the nondissipative case ($\kappa = 0$). Here, an unbounded exponential growth of the field component B_1 will be observed, and the value of B_3 will remain at the same level:

$$\begin{cases} B_1(\tilde{r},t) = e^{\lambda t} \eta \frac{z}{2l^2} \frac{\exp\left[-\frac{(\tilde{x} e^{-\lambda t})^2}{4l^2} - \frac{e^{2\lambda t} \tilde{y}^2}{4l^2} - \frac{z^2}{4l^2}\right]}{8\pi^{3/2} l^3} \\ B_3(\tilde{r},t) = -\eta \frac{\tilde{x} e^{-\lambda t}}{2l^2} \frac{\exp\left[-\frac{(\tilde{x} e^{-\lambda t})^2}{4l^2} - \frac{e^{2\lambda t} (\tilde{y})^2}{4l^2} - \frac{z^2}{4l^2}\right]}{8\pi^{3/2} l^3} \end{cases} \tag{18}$$

It worth mentioning that the components of magnetic field induction are essential at the following scales: $\delta\tilde{x} \sim l e^{\lambda t}$, $\delta\tilde{y} \sim l e^{-\lambda t}$, $\delta z \sim l$.

This behavior of the field components is in agreement with the magnetic flux conservation law. Indeed, consider the flux of the magnetic field through the plane yz. The main contribution to the flux through the plane is given by the flux of the component B_x through the region in the yz plane, where this field component is the most significant: $\Phi \sim B_x S_{yz}$. The width of this region along the z axis remains constant, and along the y axis exponentially decreases with time. This is the reason for the growth of B_x. For the same reason, B_z remains constant with time. The region in the xy plane, where this component is the most significant, does not change with time, because its width along one of the axes increases exponentially, and on the other one - decreases exponentially with the same rate.

Let's now consider the case of finite dissipation.

The components of magnetic field induction will behave differently at different scales of time. Let us analyze the behavior of the field in earlier and later stages.

$r_d e^{\lambda t} \ll l$:

$$\boxed{\begin{cases} B_3 \approx -\eta \frac{\tilde{x} e^{-\lambda t}}{2l^2} \frac{\exp\left[-\frac{(\tilde{x}e^{-\lambda t})^2}{4l^2} - \frac{e^{2\lambda t}(\tilde{y})^2}{4l^2} - \frac{z^2}{4l^2}\right]}{8\pi^{3/2} l^3}; \\ B_1 \approx e^{\lambda t} \eta \frac{z}{2l^2} \frac{\exp\left[-\frac{(\tilde{x}e^{-\lambda t})^2}{4l^2} - \frac{e^{2\lambda t}(\tilde{y})^2}{4l^2} - \frac{z^2}{4l^2}\right]}{8\pi^{3/2} l^3} \end{cases}}$$

At the initial stage, the exponential growth of the field B_1 is observed, the field B_3 decays exponentially as a result of dissipation. At this stage, the dynamo effect is manifested.

Dissipation will be significant at larger times and the growth of the field will cease:

$r_d e^{\lambda t} \gg l$:

$$\boxed{\begin{cases} B_3 \approx -\eta \frac{\tilde{x} e^{-2\lambda t}}{2l^2} \frac{\exp\left[-\frac{(\tilde{x}e^{-\lambda t})^2}{4l^2} - \frac{(\tilde{y})^2}{4r_d^2} - \frac{z^2}{4(l^2+\kappa t)}\right]}{8\pi^{3/2} l r_d \sqrt{l^2+\kappa t}} \\ B_1 \approx \eta \frac{z}{l^2+\kappa t} \frac{\exp\left[-\frac{(\tilde{x}e^{-\lambda t})^2}{4l^2} - \frac{(\tilde{y})^2}{4r_d^2} - \frac{z^2}{4(l^2+\kappa t)}\right]}{8\pi^{3/2} l r_d \sqrt{l^2+\kappa t}} \end{cases}}$$

Here the growth of the field B_1 terminates and the power decrease begins, in turn the field B_3 decreases exponentially.

It can be seen that at this stage, the characteristic scales on which the field is significant, $\delta\tilde{x} \sim l e^{\lambda t},\ \delta\tilde{y} \sim r_d,\ \delta z \sim \sqrt{l^2+\kappa t}.$

It is worth mentioning that the constant flow velocity in the two-dimensional problem can be eliminated by means of the Galileo transformation. For the two-dimensional problem with zero velocity, such analysis was made earlier in the work [11].

2.2 Quasi-two-dimensional flow

In real physical problems, where dynamo occurs, two-dimensional flows rarely occur. However, it is very common phenomenon when flows occur in which the vertical velocity component is suppressed, and the remaining components depend on the vertical coordinate.

We have considered the following flow in the previous section:

$$v_\alpha = V_\alpha^{(0)} + \sigma_{\alpha\beta} r_\beta \tag{19}$$

and we have assumed the term $V_\alpha^{(0)}$ to be constant. The incompressibility condition imposes no restrictions on the dependence of this velocity component on z. Flows with constant vertical gradient often take place in a range of astrophysical phenomena [2], [3]. Such a flow pattern is due to the presence of viscosity or other mechanisms. Such a flow with a strong vertical gradient will be considered in this section:

$$v_\alpha(r,t) = V_\alpha^{(0)}(z) + \sigma_{\alpha\beta} r_\beta,\;\; V_\alpha^{(0)} = \Sigma_\alpha z,\;\; \Sigma_1 = \Sigma,\;\; \Sigma_2 = 0,\;\; \Sigma \gg \lambda. \tag{20}$$

Let's write down the equations for all components of the field, taking into account that the two-dimensional flow can depend on the vertical coordinate:

$$\boxed{\begin{cases} \partial_t B_3 = -v_\beta \partial_\beta B_3 + \kappa \nabla^2 B_3; \\ \partial_t B_\alpha = B_\beta \partial_\beta v_\alpha - v_\beta \partial_\beta B_\alpha + B_3 \partial_3 v_\alpha + \kappa \nabla^2 B_\alpha; \end{cases}} \tag{21}$$

The equation for the vertical component of the magnetic field is homogeneous.

On the contrary, the equations for the components in the horizontal plane are inhomogeneous. Since $\hat{\sigma}$ does not depend on z, the contribution from the source has the form $B_3 \partial_3 v_\alpha = B_3 \Sigma_\alpha$.

$$\boxed{\begin{cases} \partial_t B_3 = -z\Sigma_\beta \partial_\beta B_3 - \sigma_{\mu\nu} r_\nu \partial_\mu B_3 + \kappa \nabla^2 B_3; \\ \partial_t B_\alpha = B_\beta \sigma_{\alpha\beta} - z\Sigma_\beta \partial_\beta B_\alpha - \sigma_{\beta\nu} r_\nu \partial_\beta B_\alpha + B_3 \Sigma_\alpha + \kappa \nabla^2 B_\alpha; \end{cases}}$$

2.2.1 Equations in Fourier domain

Similar to the previous section, it is convenient to solve the equation by the method of characteristics in momentum space:

$$\begin{cases} \partial_t b_3^k = \sigma_{\mu\nu} k_\mu \partial_{k_\nu} b_3^k + \Sigma_\beta k_\beta \partial_{k_3} b_3^k - \kappa k^2 b_3^k, \qquad k^2 = k_\nu k_\nu + k_3^2; \\ \partial_t b_\alpha^k = \sigma_{\alpha\beta} b_\beta^k + \sigma_{\mu\nu} k_\mu \frac{\partial}{\partial k_\nu} b_\alpha^k + \Sigma_\nu k_\nu \frac{\partial}{\partial k_3} b_\alpha^k - \kappa k^2 b_\alpha^k + b_3^k \Sigma_\alpha; \end{cases} \tag{22}$$

Here it is convenient to make the following change of impulses, connecting the pulse at the initial instant of time and into the current:

$$q_\alpha = W_{\nu\alpha}(t) k_\nu,\;\; k_3 = q_3 - q_\mu \Sigma_\nu \int\limits_0^t d\tau W_{\mu\nu}^{-1},\;\; k_\nu = W_{\mu\nu}^{-1} q_\mu,\;\; \frac{\partial}{\partial k_\nu} = \frac{dq_\rho}{dk_\nu} \frac{\partial}{\partial q_\rho} = W_{\nu\rho} \frac{\partial}{\partial q_\rho}$$

First we solve the equation by $b_3(k,t)$ by the method of characteristics:

$$b_3(k,t) = b_3\left(q,0\right) \exp\left[-\kappa \int\limits_0^t \mathrm{d}\tau \left\{ q_\nu(t) q_\gamma(t) W^{-1}_{\nu\mu}(\tau) W^{-1}_{\gamma\mu}(\tau) + \left(q_3 - q_\mu \Sigma_\nu \int\limits_0^\tau d\tau' W^{-1}_{\mu\nu}(\tau') \right)^2 \right\}\right] = \tag{23}$$

$$= b_3\left(q,0\right) \exp\left[-\frac{\kappa}{2\lambda}\left(q_1^2 + q_2^2 e^{2\lambda t}\right) - \kappa t \left(q_3 - \frac{\Sigma}{\lambda} q_1\right)^2\right].$$

For further convenience, we write out the vertical component of the field in the momentum representation in a general form for an arbitrary matrix $\hat{W}$:

$$b_3(\vec{k},t) = b_3\left(\hat{W}^T \vec{k}, k_3 + \int\limits_0^t \vec{k} d\tau \hat{W}(t,\tau)\vec{\Sigma}; 0\right) \tag{24}$$

$$\exp\left[-\kappa \int\limits_0^t \mathrm{d}\tau \left\{ \vec{k}\hat{W}(t,\tau)\hat{W}^T(t,\tau)\vec{k} + \left(k_3 + \vec{k}\int\limits_\tau^t d\tau' \hat{W}(t,\tau')\vec{\Sigma}\right)^2 \right\}\right]$$

The obtained expression can be rewritten as

$$b_3(\vec{k},t) = b_3(\vec{q},0)\exp\left\{-g(\vec{q},t)\right\} \tag{25}$$

Arguing analogously to the previous section, we find the horizontal components of the field:

$$b_\alpha(\vec{k},t) = W_{\alpha\gamma}(t)\left[b_\gamma(\vec{q},0) + b_3(\vec{q},0)\Sigma_\nu \int\limits_0^t \mathrm{d}\tau W^{-1}_{\gamma\nu}(\tau)\right]\exp\left\{-g(\vec{q},t)\right\} \tag{26}$$

Finally, we find the values of these fields.

We follow only one horizontal component of the field B_x, since the other one is small.

$$\boxed{\begin{cases} B_3 = -\eta \dfrac{\exp\left(-\frac{z^2}{4(l^2+\kappa t)} - \frac{y^2 e^{2\lambda t}}{4(L^2+\frac{\kappa}{2\lambda}e^{2\lambda t})} - \frac{(\tilde{x}e^{-\lambda t} - \frac{\Sigma}{\lambda}\frac{l^2}{\kappa t + l^2} z)^2}{4l^2(1+\frac{\Sigma^2}{\lambda^2}\frac{\kappa t}{\kappa t + l^2})}\right)}{16\pi^{3/2} l \sqrt{(l^2+\kappa t)(l^2+\frac{\kappa}{2\lambda}e^{2\lambda t})(1+\frac{\Sigma^2}{\lambda^2}\frac{\kappa t}{\kappa t+l^2})}} \dfrac{\tilde{x}e^{-\lambda t} - \frac{\Sigma}{\lambda} z \frac{l^2}{l^2+\kappa t}}{l^2(1+\frac{\Sigma^2}{\lambda^2}\frac{\kappa t}{\kappa t+l^2})} \\ B_1 = \eta e^{\lambda t} \dfrac{\exp\left(-\frac{z^2}{4(l^2+\kappa t)} - \frac{y^2 e^{2\lambda t}}{4(l^2+\frac{\kappa}{2\lambda}e^{2\lambda t})} - \frac{(\tilde{x}e^{-\lambda t} - \frac{\Sigma}{\lambda}\frac{l^2}{\kappa t + l^2} z)^2}{4l^2(1+\frac{\Sigma^2}{\lambda^2}\frac{\kappa t}{\kappa t + l^2})}\right)}{16\pi^{3/2} l \sqrt{(l^2+\kappa t)(l^2+\frac{\kappa}{2\lambda}e^{2\lambda t})(1+\frac{\Sigma^2}{\lambda^2}\frac{\kappa t}{\kappa t+l^2})}(l^2+\kappa t)} \cdot \left(z - \dfrac{\frac{\Sigma}{\lambda}}{1+(\frac{\Sigma}{\lambda})^2\frac{\kappa t}{\kappa t+l^2}}\left[\tilde{x}e^{-\lambda t} - \frac{\Sigma}{\lambda} z \frac{l^2}{l^2+\kappa t}\right]\right) \end{cases}}$$

The difference from the constant shear flow is the appearance of an additional factor $\frac{1}{\sqrt{1+\frac{\Sigma^2}{\lambda^2}\frac{\kappa t}{\kappa t+l^2}}}$ and the change in the scale along the x axis, on which the fields are essential. These changes are directly related to the presence of a non-zero vertical gradient, due to which the area within which the magnetic field is significantly stretched, and the module of the field becomes smaller.

It is worth mentioning that the additional contribution that comes from the source $B_3\partial_3 v_\alpha$ appears in B_1.

2.2.2 The investigation of field behavior

We study the behavior of field components at times larger than the inverse Lyapunov exponent: $\lambda t \geq 1$. During this time, the liquid particles that are initially close to each other diverge far enough.

To begin with, it is easier to investigate the nondissipative regime.

Then the presence of a finite dissipation will be taken into account. In contrast to the two-dimensional case, the dissipation is more complicated here. Depending on the relationships between the parameters of the problem, either the exponential growth of the field can be replaced by a power decrease, or the field can be small in the ratio parameter of the Lyapunov exponent to the vertical gradient.

2.2.3 Nondissipative case

First of all, let us consider the behavior of the x component of the magnetic field in the nondissipative limit $\kappa \to 0$:

$$B_1 \approx \eta e^{\lambda t}\left[z(1+\left(\frac{\Sigma}{\lambda}\right)^2)-\tilde{x}e^{-\lambda t}\frac{\Sigma}{\lambda}\right]\frac{\exp\left\{-\frac{\left(\tilde{x}e^{-\lambda t}-\frac{\Sigma}{\lambda}z\right)^2}{4l^2}-\frac{(y^2e^{2\lambda t}}{4l^2}-\frac{z^2}{4l^2}\right\}}{16\pi^{3/2}l^5} \propto \left[z\left(\frac{\Sigma}{\lambda}\right)^2-xe^{-\lambda t}\frac{\Sigma}{\lambda}\right]e^{\lambda t} \tag{27}$$

In this case, the field grows exponentially with time. In addition, the value of the field itself will grow with the parameter $\frac{\Sigma}{\lambda}$.

The value of the vertical component of the induction of the magnetic field, in turn, will remain at the same level:

$$B_3 \approx -\eta\left(xe^{-\lambda t}-\frac{\Sigma}{\lambda}z\right)\frac{\exp\left\{-\frac{\left(xe^{-\lambda t}-\frac{\Sigma}{\lambda}z\right)^2}{4l^2}-\frac{y^2e^{2\lambda t}}{4l^2}-\frac{z^2}{4l^2}\right\}}{16\pi^{3/2}l^5} \tag{28}$$

2.2.4 Finite dissipation

Now we assume the dissipation to be finite. Then there are two different time scales corresponding to two different stages of the inclusion of dissipation: $\begin{cases} r_d e^{\lambda t} \text{ vs } l; \\ \kappa t \text{ vs } \left(\frac{\lambda}{\Sigma}l\right)^2 \end{cases}$; $\begin{cases} \lambda t \text{ vs } \ln\left(\frac{l}{r_d}\right); \\ \lambda t \text{ vs } \left(\frac{\lambda}{\Sigma}\frac{l}{r_d}\right)^2 \end{cases}$; Depending on how these times relate to each other, different regimes are included in a different order.

Suppose first that $\left(\frac{\lambda}{\Sigma}\frac{l}{r_d}\right)^2 \ll \ln\left(\frac{l}{r_d}\right)$.

$\kappa t \ll \left(\frac{\lambda}{\Sigma}l\right)^2$:

$$\begin{cases} B_3 \approx -\eta\left[\tilde{x}e^{-\lambda t}-\frac{\Sigma}{\lambda}z\right]\frac{\exp\left\{-\frac{\left(\tilde{x}e^{-\lambda t}-\frac{\Sigma}{\lambda}z\right)^2}{4l^2}-\frac{\left(ye^{\lambda t}\right)^2}{4l^2}-\frac{(z)^2}{4l^2}\right\}}{16\pi^{3/2}l^5} \\ B_1 \approx \eta e^{\lambda t}\left[z(1+\left(\frac{\Sigma}{\lambda}\right)^2)-\tilde{x}e^{-\lambda t}\frac{\Sigma}{\lambda}\right]\frac{\exp\left\{-\frac{\left(\tilde{x}e^{-\lambda t}-\frac{\Sigma}{\lambda}z\right)^2}{4l^2}-\frac{\left(ye^{\lambda t}\right)^2}{4l^2}-\frac{(z)^2}{4l^2}\right\}}{16\pi^{3/2}l^5} \end{cases} \tag{29}$$

At these times, even nonzero dissipation is negligibly small and the x component of the magnetic field exponentially grows, the dynamo takes place. However, at a later stage, dissipation will play a significant role. Note that if the dissipation is sufficiently large in relation to the vertical gradient ($\left(\frac{\Sigma}{\lambda}\right)^2 \frac{\kappa}{\lambda l^2} \gg 1,\ \kappa \gg \lambda l^2 \left(\frac{\lambda}{\Sigma}\right)^2$), then such a scenario is not realized at any times.

Let's consider the limit $\left(\frac{\Sigma}{\lambda}\right)^2 \frac{\kappa t}{l^2} \gg 1,\ \kappa t \gg l^2 \left(\frac{\lambda}{\Sigma}\right)^2$. Then there will be a weakening of the magnetic field by the presence of a vertical gradient of the velocity field. In this limit the scale of the field width along the x axis gives a smallness in this parameter: $\frac{1}{\sqrt{1+\frac{\Sigma^2}{\lambda^2}\frac{\kappa t}{\kappa t+l^2}}} \sim \frac{\lambda}{\Sigma}\sqrt{\frac{\kappa t+l^2}{\kappa t}}$. The scale on the x-axis, on which the fields are significant, $\delta x \sim l\frac{\Sigma}{\lambda}\sqrt{\frac{\kappa t}{\kappa t+l^2}}$.

$r_d e^{\lambda t} \ll l \ll \frac{\Sigma}{\lambda}\sqrt{\kappa t}$:

$$\begin{cases} B_3 \approx -\eta\frac{\lambda}{\Sigma}\left(\tilde{x}e^{-\lambda t} - z\frac{\Sigma}{\lambda}\right)\dfrac{\exp\left[-\frac{(\tilde{x}e^{-\lambda t}-\frac{\Sigma}{\lambda}z)^2}{4(l^2+\frac{\Sigma^2}{\lambda^2}\kappa t)} - \frac{e^{2\lambda t}(y)^2}{4l^2} - \frac{z^2}{4l^2}\right]}{16\pi^{3/2}l^3\kappa t} \\ B_1 \approx \frac{\lambda}{\Sigma}e^{\lambda t}\eta\left(z\frac{l^2+\kappa t}{\kappa t} - \frac{\lambda}{\Sigma}\frac{l^2}{\kappa t}\tilde{x}e^{-\lambda t}\right)\dfrac{\exp\left[-\frac{(\tilde{x}e^{-\lambda t}-z\frac{\Sigma}{\lambda})^2}{4(l^2+\frac{\Sigma^2}{\lambda^2}\kappa t)} - \frac{e^{2\lambda t}(y)^2}{4l^2} - \frac{z^2}{4l^2}\right]}{16\pi^{3/2}l^3\kappa t} \end{cases} \tag{30}$$

Here there is an exponential growth of the field B_1, in which the dynamo effect manifests itself, the field B_3 decays exponentially as a result of dissipation. However, here we already have a weakening of the field due to a nonzero gradient in z.

$r_d e^{\lambda t} \gg l$:

$$\begin{cases} B_3 \approx -\eta e^{-\lambda t}\left(\frac{\lambda}{\Sigma}\right)^3 \dfrac{\exp\left(-\frac{z^2}{4(l^2+\kappa t)} - \frac{y^2}{4r_d^2} - \frac{(xe^{-\lambda t}-\frac{\Sigma}{\lambda}\frac{l^2}{\kappa t+l^2}z)^2}{4l^2\frac{\Sigma^2}{\lambda^2}\frac{\kappa t}{\kappa t+l^2}}\right)}{16\pi^{3/2}l^3(\kappa t)^{3/2}r_d}(\kappa t+l^2)\left(xe^{-\lambda t} - \frac{\Sigma}{\lambda}z\frac{l^2}{l^2+\kappa t}\right) \\ B_1 \approx \eta\frac{\lambda}{\Sigma}\dfrac{\exp\left(-\frac{z^2}{4(l^2+\kappa t)} - \frac{y^2}{4r_d^2} - \frac{(xe^{-\lambda t}-\frac{\Sigma}{\lambda}\frac{l^2}{\kappa t+l^2}z)^2}{4l^2\frac{\Sigma^2}{\lambda^2}\frac{\kappa t}{\kappa t+l^2}}\right)}{16\pi^{3/2}lr_d(\kappa t)^{3/2}}\left(z - \frac{\lambda}{\Sigma}xe^{-\lambda t}\right) \end{cases} \tag{31}$$

Here, the exponential growth of the field B_1 is replaced by a power decrease.

In this limit, in comparison with the purely two-dimensional situation, when the constant flow was not taken into account, the range along the x-axis at which the fields are significant increased in $\frac{\Sigma}{\lambda}$ times, but the field size decreased by the same factor .

After that we will consider the opposite case when $\left(\frac{\lambda}{\Sigma}\frac{l}{r_d}\right)^2 \gg \ln\left(\frac{l}{r_d}\right)$. At large times, the dynamics of the fields will be the same as in the opposite limit, but at intermediate times the behavior of the fields will be somewhat different.

At times $r_d e^{\lambda t} \ll l$ the behavior of the fields will be the same as in the nondissipative regime (29).

$\ln\left(\frac{l}{r_d}\right) \ll \lambda t \ll \left(\frac{\lambda}{\Sigma}\frac{l}{r_d}\right)^2$:

$$\begin{cases} B_3 \approx -\eta e^{-\lambda t} \frac{\exp\left\{-\frac{\left(xe^{-\lambda t}-\frac{\Sigma}{\lambda}z\right)^2}{4l^2}-\frac{y^2}{4r_d^2}-\frac{z^2}{4l^2}\right\}}{16\pi^{3/2}l^4 r_d}\left[xe^{-\lambda t}-\frac{\Sigma}{\lambda}z\right]; \\ B_1 \approx \eta \frac{\exp\left\{-\frac{\left(xe^{-\lambda t}-\frac{\Sigma}{\lambda}z\right)^2}{4l^2}-\frac{y^2}{4r_d^2}-\frac{z^2}{4l^2}\right\}}{16\pi^{3/2}l^4 r_d}\left[z-\frac{\Sigma}{\lambda}\left(xe^{-\lambda t}-\frac{\Sigma}{\lambda}z\right)\right]. \end{cases} \quad (32)$$

At this stage, the dynamo is already turned off due to dissipation, but the field is still growing with a gradient. At times $\lambda t \gg \left(\frac{\lambda}{\Sigma}\frac{l}{r_d}\right)^2$, the field values already decrease with increasing gradient (31).

2.3 Randomly located in space magnetic field blobs in the linear model

Suppose that such blobs of magnetic field, considered in the previous subsection, are accidentally located in space.

We denote $\rho^{(\vec{n})}$ as coordinate of the blob, $\eta^{(\vec{n})}$ – its amplitude, $\vec{n} = (n_1, n_2, n_3)$ – set of numbers, counting blobs. Later they will be summed up in all three directions.

We will consider the B_1 component and study what it will be equal to.

Instead of the variable η, an expression for this contribution will contain the expression $\Sigma_{\vec{n}}\eta^{(\vec{n})}e^{i(q,\rho^{(\vec{n})})}$. This multiplication can be effectively replaced by $\Sigma_{\vec{n}}\eta^{(\vec{n})}$ with the following coordinate change:

$$\begin{aligned} r_1 &\to r_1 + e^{\lambda t}\rho_1^{(\vec{n})}; \\ r_2 &\to r_2 + e^{-\lambda t}\rho_2^{(\vec{n})}; \\ r_3 &\to r_3 + \rho_3^{(\vec{n})}; \end{aligned} \quad (33)$$

Suppose that the nodes are located evenly with characteristic distances $R \geq l$.

We will be interested in two questions.

1). How does the magnetic field of such a system behave at arbitrary point?

2). What is the average energy density of such a system?

First we answer the first question.

We perform the substitution (33) in the expression for B_1 in this limit.

The coordinates of the blobs are given on the average as follows: $\rho_i^{n_i} \sim m_i R,\ i = \overline{1,3}$. The maximal value of $\rho_i^{n_i}$, that contribute to the total field are determined by the following inequalities:

$$\begin{cases} m_3 R \lesssim l; \\ m_2 R e^{-\lambda t} \lesssim r_d; \\ (m_1 - m_3\frac{\Sigma}{\lambda})R \lesssim \frac{\Sigma}{\lambda}\sqrt{\frac{\kappa t l^2}{\kappa t + l^2}} \end{cases} \qquad \begin{cases} m_3 = 0; \\ m_2 \lesssim \frac{r_d}{R}e^{\lambda t} = M_2; \\ m_1 = 0 \end{cases}$$

It can be seen that the amount is collected on fairly small numbers.

AFter that we will have the following expression:

$$B_1 \approx \sum_{\vec{n}} \eta^{(\vec{n})} \frac{\lambda}{\Sigma} \frac{\exp\left(-\frac{z^2}{4(l^2+\kappa t)} - \frac{\left(y+\rho_2^{(n_2)} e^{-\lambda t}\right)^2}{4r_d^2} - \frac{(\tilde{x}e^{-\lambda t} - \frac{\Sigma}{\lambda}\frac{l^2}{\kappa t + l^2} z)^2}{4l^2 \frac{\Sigma^2}{\lambda^2} \frac{\kappa t}{\kappa t + l^2}} \right)}{16\pi^{3/2} l r_d (\kappa t)^{3/2}} \left(z - \frac{\lambda}{\Sigma}\tilde{x} e^{-\lambda t} \right) \approx$$

$$\approx \sum_{n_2=1}^{M_2} \eta^{(n_2)} \frac{\lambda}{\Sigma} \frac{\exp\left(-\frac{z^2}{4(l^2+\kappa t)} - \frac{y^2}{4r_d^2} - \frac{(\tilde{x}e^{-\lambda t} - \frac{\Sigma}{\lambda}\frac{l^2}{\kappa t + l^2} z)^2}{4l^2 \frac{\Sigma^2}{\lambda^2} \frac{\kappa t}{\kappa t + l^2}} \right)}{16\pi^{3/2} l r_d (\kappa t)^{3/2}} \left(z - \frac{\lambda}{\Sigma}\tilde{x} e^{-\lambda t} \right) \qquad (34)$$

The sum of a large number of random variables $\eta^{(n_2)}$, each of which $\sim \eta$ is proportional to the square root of the number of summands, like the random walk problem:

$$\sum_{n_2=1}^{M_2} \eta^{(n_2)} \sim \eta\sqrt{M_2} = \eta\sqrt{\frac{r_d}{R}} e^{\lambda t/2} \qquad (35)$$

Thus, the resulting field of all blobs exponentially grows with time:

$$B_1(r,t) \approx \eta \frac{\lambda}{\Sigma} e^{\lambda t/2} \frac{\exp\left(-\frac{z^2}{4(l^2+\kappa t)} - \frac{y^2}{4r_d^2} - \frac{(\tilde{x}e^{-\lambda t} - \frac{\Sigma}{\lambda}\frac{l^2}{\kappa t + l^2} z)^2}{4l^2 \frac{\Sigma^2}{\lambda^2} \frac{\kappa t}{\kappa t + l^2}} \right)}{16\pi^{3/2} l \sqrt{r_d R} (\kappa t)^{3/2}} \left(z - \frac{\lambda}{\Sigma}\tilde{x} e^{-\lambda t} \right) \qquad (36)$$

The exponential growth of the field with time arises from the fact that the number of blobs that contribute to the total field exponentially grows with time. However, the exponent in this case is twice less than in the non-dissipative mode for one blob.

Now we will calculate the density of energy.

$$\epsilon = \frac{\int \mathrm{d}^3 r B_1^2}{V} \qquad (37)$$

We will integrate over the volume of the cube with the side $NR,\ N \gg 1$. Because the scale of the fields decrease is much smaller than the distance between neighboring clusters, then we can confine ourselves to considering a cube with one blob, i.e. $V = R^3$. The value of energy density comprises

$$\epsilon = \frac{\lambda}{\Sigma} \frac{1}{32(2\pi)^{3/2}} \frac{\eta^2 e^{2\lambda t}}{l R^4 (\kappa t)^{3/2}} \qquad (38)$$

Thus, at times $r_d e^{\lambda t} \gg l \gg \sqrt{\kappa t}$, the magnitude of the magnetic field induction and the energy density will increase exponentially with time and decrease along with the vertical gradient:

$$\boxed{B_x \propto \frac{\lambda}{\Sigma} e^{\lambda t/2}} \qquad (39)$$

$$\boxed{\epsilon \propto \frac{\lambda}{\Sigma} e^{2\lambda t}} \qquad (40)$$

Such an exponent of the energy density is given by the magnetic field exponentially growing with time and the field width along the axis x.

3 Chaotic flow

For most chaotic flows, the assumption that the velocity gradient matrix will remain diagonal and do not change with time will not work. For a better understanding of phenomena, it is meaningful to study the magnetic field in a chaotic flow with a random matrix of velocity gradients.

In the case when we are dealing with a random flow, it is convenient to monitor the behavior of the correlation functions, since the induction of the magnetic field fluctuates strongly.

For the two-dimensional case, the asymptotics of the correlation functions $F_{\alpha\beta}(\mathbf{r}, t)$ were first obtained in Ref. [12].

In contrast to the previous section, here we assume a random matrix $\hat{\sigma}$.

$$v_\alpha = V_\alpha^{(0)} + \sigma_{\alpha\beta}(t) r_\beta, \;\; V_\alpha^{(0)} = \Sigma_\alpha z, \;\; \Sigma \gg \lambda \tag{41}$$

3.1 Random gradient matrix

The statistics of the matrices $\hat{\sigma}$ should be isotropic, homogeneous in time, and it should take into account the incompressibility of the flow:

$$\langle \sigma_{\alpha\mu}(t) \sigma_{\beta\nu}(t') \rangle = 2\lambda \left(3\delta_{\alpha\beta}\delta_{\mu\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} - \delta_{\alpha\mu}\delta_{\beta\nu} \right) \delta(t - t') \tag{42}$$

When the correlator is being calculated, the averaging is made according to the statistics of the initial field distribution and the statistics of the velocity field. It turns out convenient [12] to solve the equations on the fields in the momentum domain and to average by the initial field distribution, moving to the coordinate space only after that and averaging over the flow statistics:

$$\begin{aligned} F_{mn}(r, t) = \langle B_m(r', t) B_n(r' + r, t) \rangle = \\ = \left\langle \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \int \frac{\mathrm{d}^3 q'}{(2\pi)^3} e^{ikr} \langle b_m(q', t) b_n(q, t) \rangle_{\text{field}} \right\rangle_{\text{flow}}, \;\; m, n = \overline{1,3} \end{aligned} \tag{43}$$

At the initial moment of time the field is assumed to be localized on the scale $\sim l$.

$$\langle b_m(q', 0) b_n(q, 0) \rangle_0 = l^2 \left(q^2 \delta_{mn} - q_m q_n \right) f(q^2 l^2) \delta(q + q'), \;\; m, n = \overline{1,3} \tag{44}$$

Such form of distribution is due to the solenoidal nature of the magnetic field.

Averaging over the flow statistics is conveniently represented in the form of a path integral over the velocity gradient matrices $\hat{\sigma}$

$$\langle \ldots \rangle_{\text{flow}} = \int \mathrm{D}\hat{\sigma} \ldots e^{-\frac{1}{32\lambda} \int\limits_0^t \mathrm{d}\tau \mathcal{L}\left[\sigma(\hat{\tau})\right]} \tag{45}$$

with a weight corresponding to its statistics:

$$\mathcal{L}\left[\hat{\sigma}(\tau)\right] = 3\mathrm{Tr}\left(\hat{\sigma}\hat{\sigma}^T\right) + \mathrm{Tr}\left(\hat{\sigma}^2\right) \tag{46}$$

3.2 Iwasawa's parametrization

To calculate the correlator, it is necessary to average over the ensemble of matrices $\hat{\sigma}$ an expression that depends explicitly on $\hat{W}$. Of course, we can represent the latter through the matrix of velocity gradients, but then we will have a time ordered exponent under the functional integral. This way seems rather cumbersome and thorny.

This difficulty can be avoided by introducing parametrization of the ensemble of matrices $\hat{W}$ and the subsequent averaging of the entire expression with respect to these parameters.

We will use in this case Iwasawa's parameterization [12], [13], which has the following form:

$$\hat{W} = \hat{O}\hat{D}\hat{T},\ \hat{O}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix},\ \hat{D}(\rho) = \begin{pmatrix} e^{\rho} & 0 \\ 0 & e^{-\rho} \end{pmatrix},\ \hat{T}(\chi) = \begin{pmatrix} 1 & \chi(t) \\ 0 & 1 \end{pmatrix} \tag{47}$$

The three time functions $\rho(t),\ \chi(t),\ \phi(t)$ with the initial conditions $\rho(0) = \chi(0) = \phi(0) = 0$ is sufficient for the parametrization of the set of all two-dimensional traceless matrices $\hat{\sigma}$:

$$\hat{\sigma} = \hat{O}(\phi) \begin{pmatrix} \dot{\rho} & \dot{\phi} + \dot{\chi} e^{2\rho} \\ -\dot{\phi} & -\dot{\rho} \end{pmatrix} \hat{O}^{-1}(\phi) = \hat{O}(\phi)\hat{X}\hat{O}^{-1}(\phi) \tag{48}$$

Next, a change of variables should be made. The we should calculate the Jacobian, rewrite the Lagrangian (46) according to (48) and the integrand in the new variables. We first deal with the Lagrangian. It can be rewritten in terms of the new parameterization:

$$\mathrm{Tr}\left(\hat{\sigma}^2\right) = \mathrm{Tr}\left(\hat{O}\hat{X}^2\hat{O}^{-1}\right) = 2\dot{\rho}^2 - 2\dot{\phi}^2 - 2\dot{\phi}\dot{\chi}e^{2\rho} \tag{49}$$

$$\mathrm{Tr}\left(\hat{\sigma}\hat{\sigma}^T\right) = \mathrm{Tr}\left(\hat{O}\hat{X}\hat{X}^T\hat{O}^{-1}\right) = 2\dot{\rho}^2 + \dot{\phi}^2 + \left(\dot{\phi} + \dot{\chi}e^{2\rho}\right)^2 \tag{50}$$

$$\begin{aligned} L\left[\hat{\sigma}(\tau)\right] &= 3\left(2\dot{\rho}^2 + \dot{\phi}^2 + \left(\dot{\phi} + \dot{\chi}e^{2\rho}\right)^2\right) + \left(2\dot{\rho}^2 - 2\dot{\phi}^2 - 2\dot{\phi}\dot{\chi}e^{2\rho}\right) = \\ &= 8\dot{\rho}^2 + 4\dot{\phi}^2 + 4\dot{\phi}\dot{\chi}e^{2\rho} + 3\dot{\chi}^2e^{4\rho} = 8\dot{\rho}^2 + 4(\dot{\phi} + \frac{\dot{\chi}}{2}e^{2\rho})^2 + 2\dot{\chi}^2e^{4\rho} \end{aligned} \tag{51}$$

The calculation of the path integral assumes the introduction of discretization. For example, a positive definite term containing the derivative $\hat{\sigma}$ (actually kinetic energy) can be introduce into the Lagrangian, for example, in the form $\tau_c^2 \mathrm{Tr}\left(\dot{\hat{\sigma}}^2\right)$, where τ_c is the noise correlation time in the problem, small in comparison with the inverse Lyapunov exponent λ. For this reason, noise can be assumed $\delta-$correlated.

However, [15], [16] regularization can be taken into account by introducing a counterterm into the Lagrangian, which can be obtained by a simple shift $\rho(t) \to \rho(t) + \epsilon\dot{\rho}(t),\ \epsilon \sim \tau_c$.

In the result the average (45) can be rewritten as [12]

$$\mathcal{N}\int \mathrm{D}\rho\mathrm{D}\phi\mathrm{D}\chi\left(\prod_{\tau} e^{2\rho(\tau)}\right)\exp\{-\frac{1}{4\lambda}L\left[\rho,\chi,\phi\right]\} \tag{52}$$

where the Lagrangian $L\left[\rho,\chi,\phi\right]$ has the following form

$$L\left[\rho,\chi,\phi\right]=(\dot{\rho}-2\lambda)^2+\frac{e^{4\rho}}{4}\dot{\chi}^2+\frac{1}{2}\left(\dot{\phi}+\dot{\chi}e^{2\rho}\right)^2 \tag{53}$$

We will consider large times $\lambda t\gg 1$. It can be shown [14] that for long times the average (45) is equivalent to the average with the following distribution function

$$P(\rho,\chi,\phi,t)\approx\mathcal{C}\frac{\rho}{(2\lambda t)^{3/2}}\left(1+\chi^2\right)^{-\frac{1}{2}-\frac{\rho}{4\lambda t}}\exp\left[-\frac{(\rho-2\lambda t)^2}{4\lambda t}\right],\ \ \lambda t\gg 1,\ \ \rho\gg 1,\ \ \mathcal{C}\sim 1 \tag{54}$$

In the vicinity of a singular point, the flow is arranged so that one of the axes is compressed. Integration with respect to the momentum corresponding to this direction leads to the appearance of a decreasing factor $e^{-\rho}$. The product of two matrices $\hat{W}$in both contributions is given by the factor $e^{2\rho}$. At large times $\lambda t\gg 1$ on the saddle trajectory, the functions χ and ϕ will be approximately constant, and ρ will grow linearly with time: $\rho=4\lambda t$. Thus, for large times the main contribution to the continual integral proportional to $e^{3\lambda t}$ will be given by a saddle trajectory on which the functions χ and ϕ are approximately constant, and ρ will grow linearly with time:$\rho=4\lambda t$.

Of course, it is possible to consider the behavior of magnetic fields in a plane where there is a very strong shear flow. We will further be interested in the behavior of the correlator in the vicinity of the zero flow velocity ($z=0$).

3.3 The results in quasi-two-dimensional case

As it was found in the model with diagonal matrix $\hat{\sigma}$, the highest values of fields are achieved for components in the plane. It is interesting to study its behavior.

In the foregoing the initial field statistics, the correlator consists of four contributions, the first of which corresponds to a homogeneous solution of the equations for the horizontal components of the field of the system (21), the fourth describes the presence of the ‘’source‘’ $B_3\partial_3 v_\alpha$ in the equations of the system (21) on the magnetic field, the second and third are the cross terms.

The resulting path integrals are calculated in the saddle-point approximation $\rho\approx 4\lambda t$ at times $\lambda t\gg 1$.

Let’s give more details of calculations.

3.3.1 Intermediate calculations

Taking into account the formulas(26),(43),(44) we will obtain:

$$F_{\alpha\beta}(r,t) = \left\langle \int d^3r \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} e^{ikr} \left\langle W_{\alpha\gamma}(t) \left[b_\gamma(q,0) + b_3(q,0)\Sigma_\nu \int_0^t \mathrm{d}\tau W^{-1}_{\gamma\nu}(\tau) \right] \right.\right.$$

$$\left.\left. W_{\beta\rho}(t) \left[b_\rho(q',0) + b_3(q',0)\Sigma_\mu \int_0^t \mathrm{d}\tau W^{-1}_{\rho\mu}(\tau) \right] \right\rangle_{\text{field}} \exp\left\{-g(q,t) - g(q',t)\right\} \right\rangle_{\text{flow}}$$

We expand the brackets and average each term in the field statistics:

$$F_{\alpha\beta}(r,t) = \left\langle \int d^3r \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} e^{ikr} \Big[W_{\alpha\gamma}(t)W_{\beta\rho}(t) \Big\langle b_\gamma(q,0)b_\rho(q',0) \Big\rangle_{\text{field}} + \right. \tag{55}$$

$$+\Big\langle b_\gamma(q,0)b_3(q',0)\Big\rangle_{\text{field}} W_{\alpha\gamma}(t)\Sigma_\mu \int_0^t \mathrm{d}\tau W_{\beta\mu}(t,\tau) + \Big\langle b_3(q,0)b_\rho(q',0)\Big\rangle_{\text{field}} W_{\beta\rho}(t)\Sigma_\mu \int_0^t \mathrm{d}\tau W_{\alpha\mu}(t,\tau) +$$

$$\left. +\Big\langle b_3(q,0)b_3(q',0)\Big\rangle_{\text{field}} \Sigma_\nu\Sigma_\gamma \int_0^t \mathrm{d}\tau W_{\alpha\nu}(t,\tau) \int_0^t \mathrm{d}\tau' W_{\beta\gamma}(t,\tau') \Big] \exp\left\{-g(q,t) - g(q',t)\right\} \right\rangle_{\text{flow}}$$

After averaging over the initial statistics of magnetic field induction, we have the following expression:

$$F_{\alpha\beta}(r,t) = \left\langle \int d^3r \int \frac{d^3q}{(2\pi)^3} e^{ikr} f(q^2l^2) l^2 \Big[\big(q^2 W_{\alpha\gamma}(t)W_{\beta\gamma}(t) - q_\gamma q_\rho W_{\alpha\gamma}(t)W_{\beta\rho}(t) \big) - \right. \tag{56}$$

$$-q_\gamma q_3 W_{\alpha\gamma}(t) \frac{\Sigma}{\lambda} \lambda \int_0^t \mathrm{d}\tau W_{\beta 1}(t,\tau) - q_3 q_\rho W_{\beta\rho}(t) \frac{\Sigma}{\lambda} \lambda \int_0^t \mathrm{d}\tau W_{\alpha 1}(t,\tau) +$$

$$\left. +(q_1^2 + q_2^2) \left(\frac{\Sigma}{\lambda}\right)^2 \lambda \int_0^t \mathrm{d}\tau W_{\alpha 1}(t,\tau) \lambda \int_0^t \mathrm{d}\tau' W_{\beta 1}(t,\tau') \Big] \exp\left\{-2g(q,t)\right\} \right\rangle_{\text{flow}}$$

We write out the explicit form of the matrices $\hat{W}$ in the Iwasawa parameterization and various combinations with them, which will often occur in further calculations:

$$\hat{W} = e^\rho \begin{pmatrix} \cos\phi & \chi\cos\phi \\ -\sin\phi & -\chi\sin\phi \end{pmatrix} + e^{-\rho} \begin{pmatrix} 0 & \sin\phi \\ 0 & \cos\phi \end{pmatrix} \tag{57}$$

$$\hat{W}^{-1} = e^\rho \begin{pmatrix} -\chi\sin\phi & -\chi\cos\phi \\ \sin\phi & \cos\phi \end{pmatrix} + e^{-\rho} \begin{pmatrix} \cos\phi & -\sin\phi \\ 0 & 0 \end{pmatrix} \tag{58}$$

$$\hat{W}\hat{W}^T_{\alpha\beta} \approx (1+\chi^2) e^{2\rho} u_\alpha u_\beta \tag{59}$$

$$\hat{W}(t_1)\hat{W}^{-1}(t_2)=\hat{W}(t_1,t_2)=e^{\rho_1+\rho_2}(\chi_1-\chi_2)\begin{pmatrix}-\cos\phi_1\sin\phi_2 & \cos\phi_1\cos\phi_2\\ \sin\phi_1\sin\phi_2 & -\sin\phi_1\cos\phi_2\end{pmatrix}+$$
$$+e^{\rho_1-\rho_2}\begin{pmatrix}\cos\phi_1\cos\phi_2 & \cos\phi_1\sin\phi_2\\ -\sin\phi_1\cos\phi_2 & -\sin\phi_1\sin\phi_2\end{pmatrix}+e^{\rho_2-\rho_1}\begin{pmatrix}-\sin\phi_1\sin\phi_2 & \sin\phi_1\cos\phi_2\\ -\cos\phi_1\sin\phi_2 & \cos\phi_1\cos\phi_2\end{pmatrix} \tag{60}$$

Let's introduce for convenience vector $u_\alpha=\begin{pmatrix}\cos\phi\\ -\sin\phi\end{pmatrix}_\alpha$. Then the construction $\int\limits_0^t \mathrm{d}\tau W_{\alpha 1}(t,\tau)$ that often occurs in a non-uniform contribution will be exponentially large for long times:

$$\lambda\int\limits_0^t \mathrm{d}\tau W_{\alpha 1}(t,\tau)\approx e^{\rho}co_1\cos\phi u_\alpha,\;\; co_1\cos\phi=\lambda\int\limits_0^t \mathrm{d}\tau e^{-\rho_\tau}\cos\phi_\tau \tag{61}$$

$$W_{\alpha\gamma}q_\gamma=\begin{pmatrix}e^{\rho}\cos\phi(q_1+\chi q_2)+e^{-\rho}q_2\sin\phi\\ -e^{\rho}\sin\phi(q_1+\chi q_2)+e^{-\rho}q_2\cos\phi\end{pmatrix}_\alpha\approx e^{\rho}(q_1+\chi q_2)u_\alpha \tag{62}$$

Note that we will study correlations in a horizontal plane in the vicinity of slow flow, i.e. at z=0. Taking into account the expressions for the matrix $\hat{W}$ we have an oscillating exponential, depending only on the momentum $q_2-\chi q_1$:

$$e^{ikr}=e^{ik_1x+ik_2y}\approx e^{ie^{\rho}(q_2-\chi q_1)(x\sin\phi+y\cos\phi)}=e^{ie^{\rho}(q_2-\chi q_1)r\sin(\phi+\phi_r)} \tag{63}$$

Taking into account the above calculations, the expression for the correlator is substantially simplified:

$$F_{\alpha\beta}=\Bigg\langle\int d^3r\int\frac{d^3q}{(2\pi)^3}\exp\left\{ie^{\rho}(q_2-\chi q_1)r\sin(\phi+\phi_r)\right\}f(q^2l^2)l^2e^{2\rho}u_\alpha u_\beta\exp\left\{-2g(q,t)\right\} \tag{64}$$
$$\left[(1+\chi^2)q^2-(q_1+\chi q_2)^2-2\frac{\Sigma}{\lambda}co_1\cos\phi(q_1+\chi q_2)q_3+(q_1^2+q_2^2)\left(\frac{\Sigma}{\lambda}\right)^2co_1^2\cos^2\phi\right]\Bigg\rangle_{\text{flow}}$$

All possible limits of dissipation will be considered and the correlators on all admissible scales will be calculated.

Note that the results strongly depend on the vertical gradient of the velocity field in a quasi-two-dimensional formulation.

3.3.2 Nondissipative limit

Assume that $\kappa\to 0$. Then $g(q,t)=0$.

Make the change of variables $Q_1=q_1,\;\; Q_2=q_2-\chi q_1,\;\; Q_3=q_3$ and rewrite correlator in terms of new variables:

$$F_{\alpha\beta} = \left\langle \int \frac{d^3Q}{(2\pi)^3} \exp\left\{ie^{\rho}Q_2 r\sin(\phi+\phi_r)\right\} f(l^2\left[Q_1^2+Q_3^2+(Q_2+\chi Q_1)^2\right]) l^2 e^{2\rho}u_\alpha u_\beta \right. \quad (65)$$

$$\left[(1+\chi^2)\left(Q_1^2+(Q_2+\chi Q_1)^2+Q_3^2\right)-(Q_1(1+\chi^2)+\chi Q_2)^2-\right.$$

$$\left.\left. -2\frac{\Sigma}{\lambda}co_1\cos\phi(Q_1(1+\chi^2)+\chi Q_2)Q_3+(Q_1^2+(\chi Q_1+Q_2)^2)\left(\frac{\Sigma}{\lambda}\right)^2 co_1^2\cos\phi^2\right]\right\rangle_{\text{flow}}$$

The characteristic different components of momentum have different scales. In particular, the scale of the momentum Q_2 will be determined by the oscillating exponent and the scale of the initial magnetic field induction distribution: $Q_2 \sim \min\left\{\frac{1}{re^{\rho}}, \frac{1}{l}\right\}$.

At the same time, the components of momentum in other directions are localized at the same scales as at the initial time: $Q_1 \sim Q_3 \sim \frac{1}{l}$.

We first consider the limit of large spatial scales of the correlator, when $r\left\langle e^{\rho}\right\rangle \gg l$, т.е. $re^{3\lambda t} \gg l$.

So, this is the case:

$re^{3\lambda t} \gg l$.

The greatest contribution is made by the term quadratic in the shear:

$$F_{\alpha\beta} \approx \left\langle \int \frac{dQ_1 dQ_3}{(2\pi)^2}\delta\left(e^{\rho}r\sin(\phi+\phi_r)\right) f(l^2\left[Q_1^2+Q_3^2+\chi^2Q_1^2\right]) l^2 e^{2\rho}u_\alpha u_\beta \right. \quad (66)$$

$$\left.\left[(1+\chi^2)(Q_1^2(1+\chi^2)+Q_3^2)-Q_1^2(1+\chi^2)^2\right]\left(\frac{\Sigma}{\lambda}\right)^2 co_1^2\cos^2\phi\right\rangle_{\text{flow}} \approx$$

$$\approx \left\langle \int \frac{dQ_1 dQ_3}{(2\pi)^2}\delta\left(e^{\rho}r\sin(\phi+\phi_r)\right) f(l^2\left[Q_1^2+Q_3^2+\chi^2Q_1^2\right]) l^2 e^{2\rho}u_\alpha u_\beta(1+\chi^2)Q_3^2\left(\frac{\Sigma}{\lambda}\right)^2 co_1^2\cos^2\phi\right\rangle_{\text{flow}}$$

In integrating over the angle, the integral will be typed only at the angles ϕ_r and $-\phi_r$, which determine the direction of the radius vector connecting the points at which the field:

$$F_{\alpha\beta} \approx \left\langle \frac{1}{r}\left[\delta\left(\phi+\phi_r\right)+\delta\left(\phi+\phi_r-\pi\right)\right]\frac{f_{00}}{l^2}e^{\rho}u_\alpha u_\beta\sqrt{1+\chi^2}\left(\frac{\Sigma}{\lambda}\right)^2 co_1^2\cos^2\phi\right\rangle_{\text{flow}} \sim \quad (67)$$

$$\sim \frac{f_{00}}{l^2 r}e^{3\lambda t}co_1^2\left(\frac{\Sigma}{\lambda}\right)^2\cos^2\phi_r\begin{pmatrix}\cos^2\phi_r & \sin\phi_r\cos\phi_r\\ \sin\phi_r\cos\phi_r & \sin^2\phi_r\end{pmatrix}_{\alpha\beta} \sim \frac{f_{00}}{l^2}e^{3\lambda t}co_1^2\left(\frac{\Sigma}{\lambda}\right)^2\cos^2\phi_r\frac{r_\alpha r_\beta}{r^3}$$

where the value f_{00} is defined by the initial distribution of magnetic field induction:

$$\int dQ_1 dQ_3 f(l^2\left[Q_1^2+Q_3^2+\chi^2Q_1^2\right])Q_3^2 = \frac{\int\limits_0^{2\pi}d\phi\int\limits_0^{\infty}drrr^2\sin^2\phi f(r^2)}{l^4\sqrt{1+\chi^2}} = \frac{\pi\int\limits_0^{\infty}dwwf(w)}{2l^4\sqrt{1+\chi^2}} = \frac{\pi f_{00}}{2l^4\sqrt{1+\chi^2}} \quad (68)$$

Note that the obtained correlator is large in terms of the square of the ratio of the shear to the characteristic value of the velocity gradient in the flow plane. In addition, it is anisotropic. For values of the angle $\phi = \pi/2,\ 3\pi/2$, it turns to zero.

Next we investigate the limit of the correlator of inductions taken at points sufficiently close to each other: $re^{3\lambda t} \ll l$. In this limit, it will not depend on the distance between two points. In fact, this is a single-point correlator.

Consideration of one-point correlator ($r \ll le^{-3\lambda t}$) is in that it itself corresponds to the energy density of the magnetic field:

$$F_{\alpha\beta}(r,t) = \Bigg\langle \int \frac{d^3q}{(2\pi)^3} f(q^2l^2) l^2 e^{2\rho} u_\alpha u_\beta \tag{69}$$

$$\left[(1+\chi^2)q^2 - (q_1+\chi q_2)^2 - 2\frac{\Sigma}{\lambda} co_1 \cos\phi (q_1+\chi q_2) q_3 + (q_1^2+q_2^2)\left(\frac{\Sigma}{\lambda}\right)^2 co_1^2 \cos^2\phi\right]\Bigg\rangle_{\text{flow}}$$

In this case, all the components of the momentum have the same characteristic scales as for the initial distribution, i.e. $q_i \sim \frac{1}{l}$. Therefore the main contribution will be given by the term that is proportional to the square of the shear:

$$F_{\alpha\beta}(r,t) \approx \Bigg\langle \int \frac{d^3q}{(2\pi)^3} f(q^2l^2) l^2 e^{2\rho} u_\alpha u_\beta (q_1^2+q_2^2)\left(\frac{\Sigma}{\lambda}\right)^2 co_1^2\cos^2\phi \Bigg\rangle_{\text{flow}} \sim \Bigg\langle co_1^2 \frac{f_{00}}{l} e^{2\rho} u_\alpha u_\beta \left(\frac{\Sigma}{\lambda}\right)^2 \cos^2\phi \Bigg\rangle_{\text{flow}}, \tag{70}$$

$f_{00} = \pi \int\limits_0^\infty dy y f(y)$. After the averaging of obtained expression with the distribution function (54) we will have the following expression:

$$F_{\alpha\beta}(r,t) \sim co_1^2 \left(\frac{\Sigma}{\lambda}\right)^2 \frac{f_{00}}{l^3} e^{8\lambda t} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\beta}, \quad r \ll le^{-3\lambda t} \tag{71}$$

As in the two-dimensional case, the single-point correlator has a diagonal structure and grows exponentially at a very high rate.

However, unlike the two-dimensional flow, it is anisotropic, one diagonal component is several times larger than the other. It is also proportional to the square of the shear and is large in this parameter in the absence of dissipation.

Thus, in a nondissipative mode, the correlators grow exponentially with the same Lyapunov exponents with time as in the two-dimensional case, depending on the circumstance, a single-point or a different-point correlator is considered. In addition, the correlators grow quadratically with the parameter $\frac{\Sigma}{\lambda}$.

$$\boxed{F_{\alpha\beta}(r,t) \sim \left(\frac{\Sigma}{\lambda}\right)^2 co_1^2 \frac{f_{00}}{l^3} e^{8\lambda t} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\beta}, \quad r \ll le^{-3\lambda t}} \tag{72}$$

$$\boxed{F_{\alpha\beta}(r,t) \sim \left(\frac{\Sigma}{\lambda}\right)^2 \frac{f_{00}}{l^2} e^{3\lambda t} c o_1^2 \cos^2\phi_r \frac{r_\alpha r_\beta}{r^3}, \;\; r \gg l e^{-3\lambda t}} \tag{73}$$

The main contribution to these relations is made by the inhomogeneous term. It should be noted that these relations are valid only in the limit $\frac{\Sigma}{\lambda} \gg 1$.

Just as in the two-dimensional case, the correlator at scales much less than the dissipation has a diagonal tensor structure, but it turns out to be anisotropic. On a large scale, the correlator is also anisotropic due to the dependence of the factor on the polar angle ϕ_r.

It is worth noting that when you compare the deterministic and random flows, there are some apparent inconsistencies in comparison. In particular, the single-point correlator and the energy density depend in different ways on the vertical gradient. However, in a quasi-two-dimensional problem, the dissipation mechanism is arranged quite difficult because of the presence of a shift and a vertical gradient. Therefore, in a deterministic and chaotic problem, the correlator and the energy density, respectively, do not have to depend on the same parameters in the same way. Thus, in a purely two-dimensional problem [11], [12], the energy density depends on the dissipation scale, and the single-point correlator does not.

3.3.3 Finite dissipation

Now the dissipation will be taken into account.

In the presence of finite dissipation, an additional factor $e^{-2g(q,t)}$, which will change the scales along the axes on which the fields are significant, arise. Let us find the value of them

$$g(q,t) = \kappa \int_0^t \mathrm{d}\tau \left\{k_1^2(\tau) + k_2^2(\tau) + k_3^2(\tau)\right\}$$

Each component can be expressed by all characteristics that determine it:

$$k_1(\tau) = W_{11}^{-1}(\tau) q_1 + W_{21}^{-1} q_2(\tau) = \sin\phi e^{\rho}(q_2 - \chi(\tau) q_1) - q_1 e^{-\rho}\cos\phi \tag{74}$$

$$k_2(\tau) = W_{12}^{-1}(\tau) q_1 + W_{22}^{-1} q_2(\tau) = \cos\phi e^{\rho}(q_2 - \chi(\tau) q_1) - q_1 e^{-\rho}\sin\phi \tag{75}$$

$$k_3(\tau) = q_3 - \Sigma\left(q_1 \int_0^\tau dt' W_{11}^{-1}(t') + q_2 \int_0^\tau dt' W_{21}^{-1}(t')\right) \approx$$
$$\approx q_3 - \frac{\Sigma}{\lambda} c_1 \left(e^{\rho}\sin\phi(q_2 - \chi q_1) - \cos\phi q_1\right) \tag{76}$$

$c_1 = \lambda \int\limits_0^\tau dt' e^{\rho(t')-\rho(\tau)} \approx \lambda \int\limits_0^t \mathrm{d}\tau e^{\rho(\tau)-\rho(t)}$. $c_2 = \lambda \int\limits_0^\tau dt' e^{-2\rho(t')} \approx \lambda \int\limits_0^t \mathrm{d}\tau e^{-2\rho(\tau)}$.

It is convenient to diagonalize the decreasing quadratic exponent that arises from dissipation:

$$g(q,t)=\kappa\int\limits_0^t \mathrm{d}\tau\left[\left(Q_2\sin\phi e^{\rho}-q_1\cos\phi e^{-\rho}\right)^2+\left(Q_2\cos\phi e^{\rho}+q_1\sin\phi e^{-\rho}\right)^2+k_3(\tau)^2\right]=$$

$$=\kappa\left[\frac{\alpha}{\lambda}e^{2\rho}\tilde{Q_2}^{\,2}+Q_3^2t+c_2\frac{Q_1^2}{\lambda}\right],$$

$$c=\lambda\int\limits_0^t \mathrm{d}\tau e^{2\rho(\tau)-2\rho(t)},\ \ \alpha=c\left(1+c_1^2\frac{\Sigma}{\lambda}^2\sin\phi^2\right)\qquad(77)$$

The following variables change has been made during the calculations

$$\begin{cases}Q_1=q_1;\\ \tilde{Q}_2=q_2-\chi q_1-Q_3e^{-\rho}\frac{\lambda}{\Sigma}\frac{c_1^2\sin\phi}{c\left\{1+\left(\frac{\Sigma}{\lambda}c_1\sin\phi\right)^2\right\}}=Q_2-Q_3e^{-\rho}\#;\\ Q_3=q_3+\frac{\Sigma}{\lambda}c_1\cos\phi q_1;\end{cases}$$

The correlator can be calculated using the construction K corresponding to the dissipation:

$$F_{\alpha\beta}=\left\langle\int\frac{d^3q}{(2\pi)^3}f(q^2l^2)l^2e^{2\rho}u_\alpha u_\beta\exp\left\{-2g(q,t)+ikr\right\}\right.\quad(78)$$

$$\left.\left[\left(1+\chi^2\right)\left(q^2-(q_1+\chi q_2)^2\right)\right)-2\frac{\Sigma}{\lambda}co_1\cos\phi(q_1+\chi q_2)q_3+(q_1^2+q_2^2)\left(\frac{\Sigma}{\lambda}\right)^2co_1^2\cos\phi^2\right]\right\rangle_{\text{flow}}=$$

$$=\left\langle Kl^2e^{2\rho}u_\alpha u_\beta\right\rangle_{\text{flow}}$$

First we integrate over the momenta, making a change in momentum, which diagonalizes the quadratic exponent:

$$K=\int(d^3q)f(q^2l^2)\exp\left\{-2g(q,t)+ikr\right\}(79)$$

$$\left[\left(1+\chi^2\right)\left(q^2-(q_1+\chi q_2)^2\right)\right)-2\frac{\Sigma}{\lambda}co_1\cos\phi(q_1+\chi q_2)q_3+(q_1^2+q_2^2)\left(\frac{\Sigma}{\lambda}\right)^2co_1^2\cos\phi^2\right]\approx$$

$$\approx\int(d^3Q)f\left(l^2\left(Q_1^2\left[1+\chi^2\right]\right)+(Q_3-\beta Q_1)^2\right)$$

$$\exp\left\{-2\kappa\frac{\alpha}{\lambda}\tilde{Q}_2^2e^{2\rho}+i\tilde{Q}_2e^{\rho}r\sin(\phi+\phi_r)+iQ_3\frac{\lambda}{\Sigma}r\sin(\phi+\phi_r)\frac{c_1\sin\phi}{c\left[1+\left(\frac{\Sigma}{\lambda}c_1\sin\phi\right)^2\right]}\right\}\exp\left\{-2\kappa Q_3^2t\right\}$$

$$\left[(1+\chi^2)Q_3^2-2\frac{\Sigma}{\lambda}co_1\cos\phi(Q_1+\chi Q_2+\chi^2Q_1)(Q_3-\beta Q_1)+(Q_1^2+(Q_2+\chi Q_1)^2)\left(\frac{\Sigma}{\lambda}\right)^2co_1^2\cos^2\phi\right];$$

The following equality was used during the calculations:

$$\exp\left(-l^2Q_1^2(1+\chi^2)\right)\exp\left(-l^2(Q_3-\beta Q_1)^2-2\kappa t\right)=$$

$$=\exp\left(-l^2Q_1^2(1+\chi^2+\beta^2\frac{2l^2+2\kappa t}{l^2+2\kappa t})\right)\exp\left[-(l^2+2\kappa t)\left(Q_3-\beta\frac{l^2}{l^2+2\kappa t}Q_1\right)^2\right]$$

We are interested in times when $\kappa t \gg l^2$.

The quadratic exponent depending on ϕ with the parameter $\frac{\lambda}{\Sigma}\frac{r}{r_d}$ arise from the integration over momentum $\tilde{Q}_2$. The quadratic exponent with a slightly smaller parameter $\frac{\lambda}{\Sigma}\frac{r}{\kappa t}$ occurs from the integration over Q_3 and it also depend on ϕ.

Parameter $\frac{\lambda}{\Sigma}\frac{r}{r_d}$ will be small or large at the different spatial scales. The correlator's behavior will be different depending on this.

First, the behavior of the correlator at large spatial scales will be considered $r \gg r_d\frac{\Sigma}{\lambda}$.

When integrating over the angle, the main contribution will be collected near the saddle values $\phi=-\phi_r,\ \ \phi=\pi-\phi_r$. The main contribution to the vicinity of these saddle points will come only from one of the exponents, namely $\exp\left\{-2\kappa\frac{\alpha}{\lambda}\tilde{Q}_2^2e^{2\rho}+i\tilde{Q}_2e^{\rho}r\sin(\phi+\phi_r)\right\}$. Thus, the construction of K depends on the parameters of the problem as follows:

$$K\approx\frac{f_{01}}{l^3\left[1+\chi^2+\beta^2\right]^{3/2}}\frac{\lambda}{\kappa}e^{-\rho}\frac{1}{\sqrt{\lambda t\alpha}}(1+\chi^2)\left(\frac{\Sigma}{\lambda}\right)^2\cos^2\phi\left[2co_1c_1\mathrm{sgn}(\cos\phi)+co_1^2\right]\exp\left(-\frac{\lambda r^2\sin^2(\phi+\phi_r)}{8\kappa\alpha}\right)\tag{80}$$

The value of the above construction, which determines the influence of dissipation, is substituted into the expression for the correlator:

$$F_{\alpha\beta}\approx\left\langle e^{\rho}\frac{f_{01}}{l(1+\chi^2+\beta^2)^{3/2}}\frac{\lambda}{\kappa}\frac{1+\chi^2}{\sqrt{\lambda t\alpha}}\left(\frac{\Sigma}{\lambda}\right)^2\cos^2\phi u_\alpha u_\beta\right.$$
$$\left.\left[2co_1c_1\mathrm{sgn}(\cos\phi)+co_1^2\right]\exp\left(-\frac{\lambda r^2\sin^2(\phi+\phi_r)}{8\kappa\alpha}\right)\right\rangle\tag{81}$$

Saddle points $\rho=4\lambda t,\ \ \phi=-\phi_r,\ \ \phi=\pi-\phi_r$ give following expressions:

$$F_{\alpha\beta}\approx f_{00}co_1^2\left(\frac{\Sigma}{\lambda}\right)^2e^{3\lambda t}\frac{r_\alpha r_\beta}{r^3}\cos^2\phi_r\int\limits_{-\infty}^{\infty}\frac{d\chi}{(\chi^2+1)^{1/2}(\chi^2+\beta_r^2+1)^{3/2}}=\tag{82}$$
$$=f_{00}co_1^2e^{3\lambda t}\frac{r_\alpha r_\beta}{r^3}\frac{\mathrm{EllipticK}\left[\frac{\beta_r^2}{\beta_r^2+1}\right]-\mathrm{EllipticE}\left[\frac{\beta_r^2}{\beta_r^2+1}\right]}{c_1^2\sqrt{\beta_r^2+1}},\ \ \beta_r=\frac{\Sigma}{\lambda}c_1\cos\phi_r$$

The correlator's behavior will be investigated in this limit as a function of the polar angle of the radius vector.

At moderate angles ($\cos\phi_r \gg \frac{\lambda}{\Sigma}\ \beta_r \gg 1$) shear weakens the value of the field:

$$F_{\alpha\beta} \approx \frac{\lambda}{\Sigma} \ln\left(\frac{\Sigma}{\lambda}\right) f_{00} e^{3\lambda t} \frac{co_1^2}{c_1^3 l\sqrt{\kappa t}\cos\phi_r} \frac{r_\alpha r_\beta}{r^3}, \ \ r \gg r_d \frac{\Sigma}{\lambda}, \ \ \cos\phi_r \gg \frac{\lambda}{\Sigma} \tag{83}$$

At the polar angles that are close to the $\pi/2$ ($\cos\phi_r \ll \frac{\lambda}{\Sigma}\ \beta_r \ll 1$) the correlator is essentially anisotropic:

$$F_{\alpha\beta} \approx f_{00} e^{3\lambda t} \frac{co_1^2}{c_1^2 l\sqrt{\kappa t}} \frac{r_\alpha r_\beta}{r^3} \beta_r^2, \ \ r \gg r_d \frac{\Sigma}{\lambda}, \ \ \cos\phi_r \ll \frac{\lambda}{\Sigma} \tag{84}$$

This behavior on close to $\pi/2$ angles is analogous to the non-dissipative case.

It will be shown further that correlator does not depend on the distance r at small scales $r \ll r_d \frac{\Sigma}{\lambda}$:

$$F_{\alpha\beta} = \left\langle \int \frac{d^3q}{(2\pi)^3} f(q^2l^2) l^2 e^{2\rho} u_\alpha u_\beta \exp\left\{-2g(q,t)\right\} \right. \tag{85}$$

$$\left. \left[(1+\chi^2)\left(q^2 - (q_1+\chi q_2)^2\right)\right) - 2\frac{\Sigma}{\lambda} co_1 \cos\phi (q_1+\chi q_2) q_3 + (q_1^2+q_2^2)\left(\frac{\Sigma}{\lambda}\right)^2 co_1^2 \cos\phi^2\right] \right\rangle_{\text{flow}} =$$

$$= \left\langle K l^2 e^{2\rho} u_\alpha u_\beta \right\rangle_{\text{flow}}$$

The integration over momentum should be considered separately:

$$K = \int (d^3q) f(q^2l^2) \exp\left\{-2g(q,t)\right\} \tag{86}$$

$$\left[(1+\chi^2)\left(q^2 - (q_1+\chi q_2)^2\right)\right) - 2\frac{\Sigma}{\lambda} co_1 \cos\phi (q_1+\chi q_2) q_3 + (q_1^2+q_2^2)\left(\frac{\Sigma}{\lambda}\right)^2 co_1^2 \cos\phi^2\right];$$

The scale at which the wave vectors in the exponent are essential are determined by the coefficients before the momenta in the quadratic part of the exponent. Thus, characteristic momenta are limited at this stage by the following values:

$$Q_1 \sim \frac{1}{r_d}, \ \ \tilde{Q}_2 \sim \frac{1}{r_d e^\rho}, \ \ Q_3 \sim \frac{1}{\sqrt{\kappa t}} \sim \frac{1}{r_d\sqrt{\lambda t}} \tag{87}$$

Let's investigate what will happened to the argument of function f.

$$\begin{cases} q_1 = Q_1; \\ q_2 = \tilde{Q}_2 + \chi Q_1 + Q_3 e^{-\rho}\#; \\ q_3 = Q_3 - \frac{\Sigma}{\lambda} c_1 \cos\phi Q_1 = Q_3 - \beta Q_1. \end{cases} \tag{88}$$

Taking into account the scale of integration limited only by the quadratic exponent, the expression for the argument of the function f will take the following form:

$$l^2\left[q_1^2+q_2^2+q_3^2\right]=l^2\left[Q_1^2+\left(\tilde{Q}_2+\chi Q_1+Q_3e^{-\rho}\#\right)^2+(Q_3-\beta Q_1)^2\right] \tag{89}$$

Further, several contributions are obtained, the behavior of which is to be established:

$$K\approx\int(d^3Q)f\left(l^2\left(Q_1^2\left[1+\chi^2\right]\right)+(Q_3-\beta Q_1)^2\right)\exp\left\{-2\kappa\frac{\alpha}{\lambda}\tilde{Q}_2^2e^{2\rho}\right\}\exp\left\{-2\kappa Q_3^2t\right\} \tag{90}$$
$$\left[(1+\chi^2)Q_3^2-2\frac{\Sigma}{\lambda}co_1\cos\phi(Q_1+\chi Q_2+\chi^2Q_1)(Q_3-\beta Q_1)+(Q_1^2+(Q_2+\chi Q_1)^2)\left(\frac{\Sigma}{\lambda}\right)^2co_1^2\cos^2\phi\right]\approx$$
$$\approx\int(dQ_1)(dQ_3)f\left(l^2\left(Q_1^2\left[1+\chi^2\right]\right)+(Q_3-\beta Q_1)^2\right)\sqrt{\frac{\lambda}{\kappa\alpha e^{2\rho}}}(1+\chi^2)$$
$$\left\{\left(\frac{\Sigma}{\lambda}\right)^2\cos\phi^2Q_1^2\left[2co_1c_1\text{sgn}(\cos\phi)+co_1^2\right]+Q_3^2-2\frac{\Sigma}{\lambda}co_1\cos\phi Q_1Q_3\right\}\exp\left[-2\kappa tQ_3^2\right]$$

It is convenient to integrate over momentum in parametrization in which the quadratic exponential is diagonal, and the function f depends only on the square of the momentum: $\tilde{Q}_3=Q_3-\beta\frac{l^2}{l^2+2\kappa t}Q_1\approx Q_3-\beta\frac{l^2}{2\kappa t}Q_1$:

$$K\approx\int(dQ_1)(d\tilde{Q}_3)f\left(l^2Q_1^2\left[1+\chi^2+\beta^2\right]\right)\sqrt{\frac{\lambda}{\kappa\alpha e^{2\rho}}}(1+\chi^2) \tag{91}$$
$$\left\{\left(\frac{\Sigma}{\lambda}\right)^2\cos\phi^2Q_1^2\left[2co_1c_1\text{sgn}(\cos\phi)+co_1^2+c_1^2\frac{l^4}{4(\kappa t)^2}-2co_1c_1\frac{l^2}{2\kappa t}\right]+\tilde{Q}_3^2\right\}\exp\left[-2\kappa t\tilde{Q}_3^2\right]\approx$$
$$\approx\frac{f_{00}}{l^3\left[1+\chi^2+\beta^2\right]^{3/2}}\frac{\lambda}{\kappa}e^{-\rho}\frac{1}{\sqrt{\lambda t\alpha}}(1+\chi^2)\left(\frac{\Sigma}{\lambda}\right)^2\cos^2\phi\left[2co_1c_1\text{sgn}(\cos\phi)+co_1^2\right]+$$
$$+\frac{f_0}{l\sqrt{1+\chi^2+\beta^2}(\kappa t)}\frac{\lambda}{\kappa}e^{-\rho}\frac{1}{\sqrt{\lambda t\alpha}}(1+\chi^2)$$

Previously the term containing Q_1^2 in the integrand will be considered.

In the saddle-point approximation, ρ will grow linearly with time, integration over χ will give some constant, but the integration with respect to the angle is of the greatest interest. We write The whole expression expression will be written and the area of angles where the integral is accumulated should be determined.

Thus, the field correlators in the horizontal plane will grow exponentially with time and for finite dissipation:

$$F_{\alpha\beta}^{(1)}\approx\left\langle\frac{f_{00}}{l\left[1+\chi^2+\beta^2\right]^{3/2}}\frac{\lambda}{\kappa}e^{\rho}\frac{1}{\sqrt{\lambda t\alpha}}(1+\chi^2)\left(\frac{\Sigma}{\lambda}\right)^2\cos^2\phi\left[2co_1c_1\text{sgn}(\cos\phi)+co_1^2\right]u_\alpha u_\beta\right\rangle_{\text{flow}}\sim \tag{92}$$
$$\sim\left(\frac{\Sigma}{\lambda}\right)^2e^{3\lambda t}\frac{f_{00}}{l\sqrt{\lambda t}}\frac{\lambda}{\kappa}\int\limits_{-\infty}^{\infty}d\chi\int\limits_0^{2\pi}d\phi\frac{\cos^2\phi u_\alpha u_\beta\left[2co_1c_1\text{sgn}(\cos\phi)+co_1^2\right]}{\sqrt{c(1+\chi^2)(1+\chi^2+(c_1\frac{\Sigma}{\lambda}\cos\phi)^2)^3\left(1+(c_1\frac{\Sigma}{\lambda}\sin\phi)^2\right)}}$$

$$\int d\phi \frac{\ln\left(\frac{\Sigma}{\lambda}\right)\cos^2\phi u_\alpha u_\beta}{(1+(c_1\frac{\Sigma}{\lambda}\cos\phi)^2)^{3/2}(1+(c_1\frac{\Sigma}{\lambda}\sin\phi)^2)^{1/2}} \tag{93}$$

The correlator of the field components in the plane is a second-rank tensor containing only 4 components. We integrate each of the components in the angles ϕ.

$$11: \left(\frac{\Sigma}{\lambda}\right)^2 \ln\left(\frac{\Sigma}{\lambda}\right) \int d\phi \frac{\cos^4\phi}{(c_1\frac{\Sigma}{\lambda})^3|\cos^3\phi|\sqrt{1+(c_1\frac{\Sigma}{\lambda}\sin\phi)^2}} = \frac{\lambda}{\Sigma}\ln(\frac{\Sigma}{\lambda})\frac{1}{c_1^3}\int d\phi \frac{|\cos\phi|}{\sqrt{1+(c_1\frac{\Sigma}{\lambda}\sin\phi)^2}} \approx (94)$$

$$\approx \frac{\lambda}{\Sigma}\ln(\frac{\Sigma}{\lambda})\frac{1}{c_1^3}\frac{4\ln\frac{\Sigma}{\lambda}}{c_1\frac{\Sigma}{\lambda}} = \left(\frac{\lambda}{\Sigma}\right)^2 \ln^2\left(\frac{\Sigma}{\lambda}\right)\frac{4}{c_1^4}$$

The cross components will be nullified:

$$12: \left(\frac{\Sigma}{\lambda}\right)^2 \ln\left(\frac{\Sigma}{\lambda}\right) \int d\phi \frac{\cos^3\phi\sin\phi}{(c_1\frac{\Sigma}{\lambda})^3|\cos^3\phi|\sqrt{1+(c_1\frac{\Sigma}{\lambda}\sin\phi)^2}} = \frac{\lambda}{\Sigma}\ln(\frac{\Sigma}{\lambda})\frac{1}{c_1^3}\int d\phi \frac{sign(\cos\phi)\sin\phi}{\sqrt{1+(c_1\frac{\Sigma}{\lambda}\sin\phi)^2}} = 0 (95)$$

The obtained expression equals to zero at large shear rates $\frac{\Sigma}{\lambda} \gg 1$. Thus, one-point correlator again turns out to be diagonal.

$$22: \left(\frac{\Sigma}{\lambda}\right)^2 \ln\left(\frac{\Sigma}{\lambda}\right) \int d\phi \frac{\cos^2\phi\sin^2\phi}{(c_1\frac{\Sigma}{\lambda})^3|\cos^3\phi|\sqrt{1+(c_1\frac{\Sigma}{\lambda}\sin\phi)^2}} \approx$$

$$\approx \frac{\lambda}{\Sigma}\ln(\frac{\Sigma}{\lambda})\frac{1}{2c_1^3}\int d\phi \frac{1+\cos(2\phi)}{\sqrt{(1+(\frac{c_1}{2}\frac{\Sigma}{\lambda}\sin(2\phi))^2)}} \approx \frac{\lambda}{\Sigma}\ln(\frac{\Sigma}{\lambda})\frac{1}{2c_1^3}4\frac{\ln(\frac{\Sigma}{\lambda})}{\frac{\Sigma}{\lambda}\frac{c_1}{2}} = \left(\frac{\lambda}{\Sigma}\right)^2 \ln^2\left(\frac{\Sigma}{\lambda}\right)\frac{4}{c_1^4} \tag{96}$$

Thus, the spatial tensor structure of the correlator will have the form $\delta_{\alpha\beta}$. The integrand multiplied by sgn$(\cos\phi)$ yields the identity zero in all three cases, so the answer must be multiplied only by co_1^2.

The following result can be obtained using similar arguments for another contribution:

$$F^{(2)}_{\alpha\beta} \sim \left(\frac{\lambda}{\Sigma}\right)^2 \ln^2\frac{\Sigma}{\lambda} e^{3\lambda t}\frac{f_0}{c_1^4\sqrt{c}}\sqrt{\frac{\lambda}{\kappa}}\frac{l}{(\kappa t)^{3/2}}\delta_{\alpha\beta} \tag{97}$$

The final answer for a single-point correlator when finite dissipation is taken into account has the following form:

$$F_{\alpha\beta} \sim \left(\frac{\lambda}{\Sigma}\right)^2 \ln^2\frac{\Sigma}{\lambda} e^{3\lambda t}\frac{f_{00}co_1^2\lambda}{c_1^4\sqrt{c}l\kappa\sqrt{\lambda t}}\delta_{\alpha\beta}, \ \ r \ll r_d\frac{\Sigma}{\lambda} \tag{98}$$

We note that a single-point correlator is isotropic in the case of finite dissipation and decreases with increasing shear.

3.4 Analysis of results and comparison to the two-dimensional problem

3.4.1 Nondissipative case

Here we write down the results for the two-dimensional case.

$\kappa = 0:$

$$F_{\alpha\beta}(r,t) \sim \frac{f_0}{l^2} e^{8\lambda t} \delta_{\alpha\beta}, \;\; r \ll l e^{-3\lambda t} \tag{99}$$

$$F_{\alpha\beta}(r,t) \sim \frac{f_0}{l} e^{3\lambda t} \frac{r_\alpha r_\beta}{r^3}, \;\; r \gg l e^{-3\lambda t} \tag{100}$$

Correlators grow exponentially with time, and the growth rates on small scales are much larger than at large ones. The correlators in the two-dimensional problem are isotropic.

In a quasi-two-dimensional problem, the correlators grow on large and small scales at the same rate, but they are anisotropic and depend on the shift:

$$\boxed{F_{\alpha\beta}(r,t) \sim \left(\frac{\Sigma}{\lambda}\right)^2 co_1^2 \frac{f_{00}}{l^3} e^{8\lambda t} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\beta}, \;\; r \ll l e^{-3\lambda t}} \tag{101}$$

$$\boxed{F_{\alpha\beta}(r,t) \sim \left(\frac{\Sigma}{\lambda}\right)^2 \frac{f_{00}}{l^2} e^{3\lambda t} co_1^2 \cos\phi_r^2 \frac{r_\alpha r_\beta}{r^3}, \;\; r \gg l e^{-3\lambda t}} \tag{102}$$

3.4.2 Dissipative case

In a two-dimensional problem with finite dissipation, the correlators behave with time as follows at different scales:

$$F_{\alpha\beta}(r,t) \sim \frac{f_0}{l r_d} e^{3\lambda t} \delta_{\alpha\beta}, \;\; r \ll r_d \tag{103}$$

$$F_{\alpha\beta}(r,t) \sim \frac{f_0}{l} e^{3\lambda t} \frac{r_\alpha r_\beta}{r^3}, \;\; r \gg r_d \tag{104}$$

In a quasi-two-dimensional random flow for finite dissipation, they also increase exponentially, but essentially depend on the shift:

$$\boxed{F_{\alpha\beta}(r) \sim \left(\frac{\lambda}{\Sigma}\right)^2 \ln^2 \frac{\Sigma}{\lambda} e^{3\lambda t} \frac{\lambda}{l\kappa\sqrt{\lambda t}} \frac{f_{00} co_1^2}{c_1^4 \sqrt{c}} \delta_{\alpha\beta}, \;\; r \ll r_d \frac{\Sigma}{\lambda}} \tag{105}$$

$$\boxed{F_{\alpha\beta}(r) \sim \frac{\lambda}{\Sigma} \ln\left(\frac{\Sigma}{\lambda}\right) e^{3\lambda t} \frac{f_{00} co_1^2}{c_1^3} \frac{1}{l\sqrt{\kappa t}\cos\phi_r} \frac{r_\alpha r_\beta}{r^3}, \;\; r \gg r_d \frac{\Sigma}{\lambda}, \;\; \cos\phi_r \gg \frac{\lambda}{\Sigma}} \tag{106}$$

$$\boxed{F_{\alpha\beta}(r) \sim \cos^2\phi_r \left(\frac{\Sigma}{\lambda}\right)^2 e^{3\lambda t}\frac{f_{00}co_1^2}{c_1^2}\frac{1}{l\sqrt{\kappa t}}\frac{r_\alpha r_\beta}{r^3},\ \ r \gg r_d\frac{\Sigma}{\lambda},\ \ \cos\phi_r \ll \frac{\lambda}{\Sigma}} \tag{107}$$

The behavior of correlators at large scales, as compared with dissipation, turns out to be anisotropic, in contrast to the two-dimensional problem.

Consider angles different from $\pi/2$ and $3\pi/2$. The cosine of these angles will be of the order of one: $\cos\phi_r \sim 1$. In this range of angles the correlation monotonously decreases on all scales. Indeed,

$$\frac{F_{\alpha\beta}(r \gg r_d\frac{\Sigma}{\lambda})}{F_{\alpha\beta}(r_1 \ll r_d\frac{\Sigma}{\lambda})} \sim \frac{F_{\alpha\beta}(r \gg r_d\frac{\Sigma}{\lambda})}{F_{\alpha\beta}(0)} \sim \frac{\frac{\lambda}{\Sigma}\frac{1}{\sqrt{\kappa t}}\frac{1}{r}}{\left(\frac{\lambda}{\Sigma}\right)^2\sqrt{\frac{\lambda}{\kappa}\frac{1}{\sqrt{\kappa t}}}} \sim \frac{r_d\frac{\Sigma}{\lambda}}{r} \ll 1 \tag{108}$$

At the large distances two-point correlator will tend to zero at $\phi_r = \pi/2,\ \ \phi_r = 3\pi/2$. One-point correlator will remains constant at the same angles.

4 Finite radius of velocity correlation field

The reasoning of the previous section is applicable only when the magnetic field correlations are studied at scales small in comparison with the correlation radius of the velocity field R and at times until the length of the filament along which the magnetic field most significantly has grown to scales of the order of R. Components The velocities chosen at distances $r \gg R$ are uncorrelated with each other. Due to this, the magnetic field correlator will decrease with time.

The arguments of the previous section is applicable only in the case when we study the correlation of the magnetic field on scales small compared with the radius of correlation of the velocity field R, and the time until a length of filament along which the magnetic field is the most significant, has not grown to the scale of the order of R. Components of velocity selected at distances $r \gg R$, are uncorrelated between them. Due to this, the magnetic field correlator will decrease with time.

The qualitative picture can be described using magnetic field blobs picture. Suppose that there is arbitrary initial distribution of magnetic field which localized at the scale $\sim l$. The magnetic field lines begin to condense in the vicinity of a singular point of the flow, the magnetic field blobs are pulled into long filaments, the resulting field grows exponentially. This pattern will take place until the length of the filaments is compared with the correlation radius of the flow velocity field. Upon reaching this scale, the different ends of the thread will be uncorrelated with each other. The filaments will begin to bend and can overlap (Fig. 3).

Consider an arbitrary point in the neighborhood of the closely spaced ends of the filament. The resulting field at this point will be given by contributions from the ends of this filament, the induction vector of the magnetic field in which is directed in opposite directions. Therefore, the resulting field will be much weaker than an exponentially large field from each of the ends.

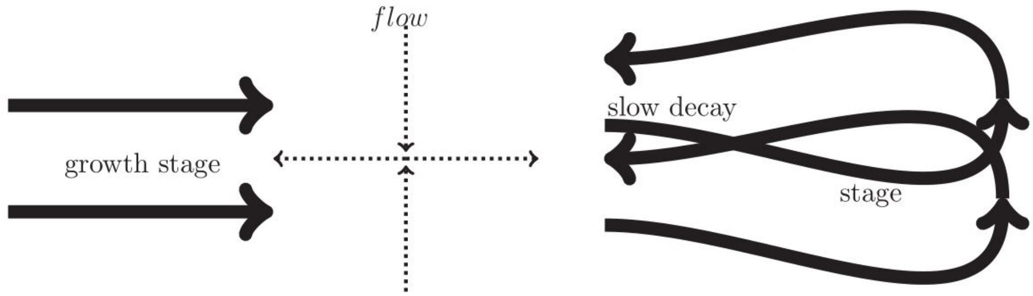

Рис. 3: Anticorrelation mechanism that terminates the dynamo [11]

It will be shown later that correlators will decrease in time in a power-law manner at times $\lambda t \gg \ln\left(\frac{R}{r_d}\right)$.

4.1 Kraichnan-Kazantsev model

We consider the situation when there is a constant flow with a vertical gradient of the velocity field onto which a two-dimensional turbulent flow is superimposed, independent of the vertical coordinate.

$$v_\alpha(r,z,t) = V_\alpha^{(0)}(z) + v_\alpha^{fl}(x,y,t) \tag{109}$$

In what follows, we will be interested in the behavior of the correlation functions in one horizontal plane $F_{mn}(r,z,t) = \langle B_m(r',z,t)B_n(r'+r,z,t)\rangle$. Further, in all notations of the flow velocity and magnetic induction, we omit the argument z, not forgetting the dependence of the fields on the vertical coordinate.

The fluctuating term of velocity is assumed to have correlation time that is small in comparison with the reciprocal Lyapunov exponent:

$$\left\langle v_\alpha^{fl}(r+r',t)v_\beta^{fl}(r',t')\right\rangle = C_{\alpha\beta}(r)\delta(t-t') \tag{110}$$

$C_{\alpha\beta}(r)$ – correlation function of velocity field that decreases at the scale $\sim R$. It is quadratic over coordinates at $r \ll R$ and it takes into acccount statistics of matrix $\hat{\sigma}$:

$$C_{\alpha\beta}(r) \approx \lambda\left(R^2\delta_{\alpha\beta} - 3r^2\delta_{\alpha\beta} + 2r_\alpha r_\beta\right) \tag{111}$$

It is worth noting that $C_{\alpha\beta}(r)\delta(t-t')$ - is the structural function of the Lagrangian velocities of the fluid flow. It is defined as follows:

$$S_L^{\alpha\beta}(r,t) = \left\langle v_L^\alpha(r,t'+t)v_L^\beta(r,t')\right\rangle, \tag{112}$$

v_L^α – component of the flow velocity in the Lagrangian reference frame. In this case it will be exactly the fluctuation part of the velocity field.

The correlator of the magnetic field's components will be considered at points taken in one horizontal plane.

On other scales, the correlator should be isotropic and satisfy the incompressibility of the flow velocity:

$$C_{\alpha\beta}(r) = C_1(r)\delta_{\alpha\beta} + C_2(r)\frac{r_\alpha r_\beta}{r^2}; \tag{113}$$

$$\frac{d}{dr}\left(C_1(r) + C_2(r)\right) = -\frac{C_2(r)}{r} \tag{114}$$

Functions $C_1(r)$ and $C_2(r)$ monotonically decrease and tend to zero at $r \gg R,\ C_2(0) = 0$.

Let us now study the behavior of the magnetic fields correlators in the plane. The derivation of the equation for field correlators resembles the derivation of the Focker-Planck equations.

It is convenient to write down the equation on magnetic field evolution:

$$\partial_t B_\alpha = B_\nu \partial_\nu v_\alpha - v_\nu \partial_\nu B_\alpha + \kappa \nabla^2 B_\alpha + B_3 \partial_3 v_\alpha; \tag{115}$$

$$\partial_t B_3 = -v_\nu \partial_\nu B_3 + \kappa \nabla^2 B_3 \tag{116}$$

and substrate fluctuation term in the velocity that we will further denote as v_α:

$$\partial_t B_\alpha = B_\nu \partial_\nu v_\alpha - v_\nu \partial_\nu B_\alpha + \kappa \nabla^2 B_\alpha + B_3 \partial_3 \Sigma_\alpha - z\Sigma_\nu \partial_\nu B_\alpha; \tag{117}$$

$$\partial_t B_3 = -v_\nu \partial_\nu B_3 + \kappa \nabla^2 B_3 - z\Sigma_\nu \partial_\nu B_3 \tag{118}$$

According to this equations the magnetic field induction can be changed during small amount due to the presence of velocity of the flow and dissipation:

$$\begin{aligned} B_\alpha(r', t+\delta t) &\approx B_\alpha(r', t) + \partial_t B_\alpha(r', t)\delta t \approx \\ &\approx B_\alpha(r', t) + \left(B_\nu \partial_\nu v_\alpha - v_\nu \partial_\nu B_\alpha + \kappa \nabla^2 B_\alpha + B_3 \Sigma_\alpha - z\Sigma_\nu \partial_\nu B_\alpha\right)\delta t \approx \\ \approx B_\alpha(r', t) + &\left(B_\nu \partial_\nu \int\limits_t^{t+\delta t} d\tau v_\alpha(\tau) - \int\limits_t^{t+\delta t} d\tau v_\nu(\tau)\partial_\nu B_\alpha\right) + \left(B_3\Sigma_\alpha - z\Sigma_\nu\partial_\nu B_\alpha + \kappa\nabla^2 B_\alpha\right)\delta t \approx \\ &\approx B_\alpha(r', t) + B_\nu \partial_\nu \int\limits_t^{t+\delta t} d\tau v_\alpha(\tau) - \int\limits_t^{t+\delta t} d\tau v_\nu(\tau)\partial_\nu B_\alpha \end{aligned} \tag{119}$$

It looks similarly for z-component of the field:

$$B_3(r', t+\delta t) \approx B_3(r', t) - \int\limits_t^{t+\delta t} d\tau v_\nu(\tau)\partial_\nu B_3 \tag{120}$$

The equation for the correlation function contains the first time derivative:

$$\partial_t F_{\alpha\beta}(r, t) = \langle B_\alpha(r', t)\partial_t B_\beta(r'+r, t)\rangle + \langle \partial_t B_\alpha(r', t) B_\beta(r'+r, t)\rangle \tag{121}$$

The direct substitution (115) give following result:

$$\begin{aligned}\partial_t F_{\alpha\beta}(r) = 2\kappa\nabla^2 F_{\alpha\beta}(r) + \Sigma_\alpha F_{3\beta}(r) + \Sigma_\beta F_{\alpha 3}(r) + \\ +\Big\langle B_\nu(r')\partial_\nu v_\alpha(r')B_\beta(r'+r) + B_\alpha(r')B_\nu(r'+r)\partial_\nu v_\beta(r'+r) - \\ -v_\nu(r')\partial_\nu B_\alpha(r')B_\beta(r'+r) - B_\alpha(r')\partial_\nu B_\beta(r'+r)v_\nu(r'+r)\Big\rangle\end{aligned} \tag{122}$$

The average in the right side contains derivatives of the fluctuating part of the velocity in the first degree. It is difficult to directly calculate the value of this average. The calculations will require a substitution of the magnetic field induction known at the time that occurred earlier at δt according to (119). Due to the short-correlated velocity field, we can choose the time interval δt much longer than the correlation time of the velocity field and much smaller than the inverse Lyapunov exponent.

There are several types of summands. There are summands linear in δt, and terms quadratic in velocity. In the limit $\delta t \to 0$, the terms linear by this time interval will be small, and the terms that are quadratic in the fluctuation velocity will remain moderate.

In the end, the following equation on correlator is obtained:

$$\partial_t F_{\alpha\beta} = \left[C_{\mu\nu}(0) - C_{\mu\nu}(r)\right]\partial_\mu\partial_\nu F_{\alpha\beta} + \partial_\mu C_{\nu\beta}(r)\partial_\nu F_{\alpha\mu} + \partial_\mu C_{\nu\alpha}(r)\partial_\nu F_{\mu\beta} - \tag{123}$$

$$-F_{\mu\nu}\partial_\mu\partial_\nu C_{\alpha\beta}(r) + \Sigma_\alpha F_{3\beta} + \Sigma_\beta F_{\alpha 3} + 2\kappa\nabla^2 F_{\alpha\beta} \tag{124}$$

It turns out to be hooked with cross correlators.

Equations for cross and z-z correlators are similarly derived:

$$\partial_t F_{3\alpha} = \left[C_{\mu\nu}(0) - C_{\mu\nu}(r)\right]\partial_\mu\partial_\nu F_{3\alpha} + \partial_\mu C_{\nu\alpha}(r)\partial_\nu F_{3\mu} + \Sigma_\alpha F_{33} + 2\kappa\nabla^2 F_{3\alpha} \tag{125}$$

$$\partial_t F_{33} = \left[C_{\mu\nu}(0) - C_{\mu\nu}(r)\right]\partial_\mu\partial_\nu F_{33} + 2\kappa\nabla^2 F_{33} \tag{126}$$

Note that in all equations, summands appearing in the derivation are proportional to $z\Sigma_\nu\partial_\nu$. The sum of all such summands in each equation yields zero when integrated by r' due to the presence of the total derivative in the integrand.

The behavior of autocorrelations of vertical components F_{33} and then cross correlators $F_{3\alpha}$ should be previously determined to find out the behavior of correlator's components in horizontal plane. Note that the main contribution in correlator at the early stage of the evolution will be given by nonhomogeneous term containing vertical component of magnetic field induction. It will be demonstrated later which term in correlator will be main and on what times.

Firstly, the method that allows to obtain correlator's evolution at large times will be illustrated at the case of two-dimensional flow. Then we will go back to the quasi-two-dimensional problem and determine the evolution of correlator in this case at large times.

4.2 The results of two-dimensional problem

The correlators of the magnetic field components in the plane at the initial stage grow exponentially with time. For a two-dimensional flow, it is possible to show [11] that this is due to the presence of a source in the system of equations for correlators. The equation for the correlators in the plane has the following form:

$$\partial_t F_{\alpha\beta} = \left[C_{\mu\nu}(0) - C_{\mu\nu}(r)\right] \partial_\mu \partial_\nu F_{\alpha\beta} + \partial_\mu C_{\nu\beta}(r) \partial_\nu F_{\alpha\mu} + \partial_\mu C_{\nu\alpha}(r) \partial_\nu F_{\mu\beta} - \\ - F_{\mu\nu} \partial_\mu \partial_\nu C_{\alpha\beta}(r) + 2\kappa \nabla^2 F_{\alpha\beta} \tag{127}$$

The two-dimensional problem is isotropic, the correlators do not depend on the polar angle. Therefore they are conveniently sought in the form $F_{\alpha\beta}(r) = \delta_{\alpha\beta} S(r) + \frac{r_\alpha r_\beta}{r^2} Y(r)$. Then such substitution in (127) will be made and all terms will be revealed. During calculations, many repetitive constructions arise, which can be conveniently calculated once (see Appendix), and then used repeatedly. After all the transformations in the equation, the corresponding coefficients for the tensors $\delta_{\alpha\beta}$ and $\frac{r_\alpha r_\beta}{r^2}$ will be separated and then the system of equations on them will be obtained (without writing out the dissipative contribution, taking it into account as a cut-off scale r_d):

$$\partial_t S = \left[C_1(0) - C_1(r) - C_2(r)\right] \partial_r^2 S + \left[C_1(0) - C_1(r) + 2C_2(r)\right] \frac{\partial_r S}{r} - \\ - \left[C_1''(r) + \frac{C_1'(r)}{r} + \frac{2C_2(r)}{r^2}\right] S(r) + \frac{2Y}{r^2} \left[C_1(0) - C_1(r) + \frac{C_1'(r)}{2r}\right]; \tag{128}$$

$$\partial_t Y = \left[C_1(0) - C_1(r) - C_2(r)\right] \partial_r^2 Y + \left[C_1(0) - C_1(r) + 2C_2(r)\right] \frac{\partial_r Y}{r} - \frac{4C_2(r)}{r} \partial_r \left(S + Y\right) \tag{129}$$

The following closed equation can be obtained on the function $\Phi = Y + r\partial_r \left(Y + S\right)$:

$$\partial_t \Phi = \left[C_1(0) - C_1(r) - C_2(r)\right] \left(\partial_r^2 \Phi - \frac{1}{r} \partial_r \Phi\right) \tag{130}$$

It is worth noting that all the scalar functions introduced above can be represented in the form of the action of some tensor operator on the correlator:

$$S = \left[\delta_{\alpha\beta} - \frac{r_\alpha r_\beta}{r^2}\right] F_{\alpha\beta}; \; Y = \left[\frac{2r_\alpha r_\beta}{r^2} - \delta_{\alpha\beta}\right] F_{\alpha\beta}; \; \Phi = r_\beta \partial_\alpha F_{\alpha\beta} \tag{131}$$

A closed system with two functions can be obtained by choosing Y as the second one that contains Φ in the inhomogeneous part:

$$\partial_t Y = \left[C_1(0) - C_1(r) - C_2(r)\right] \partial_r^2 Y + \left[C_1(0) - C_1(r) + 2C_2(r)\right] \frac{\partial_r Y}{r} + \frac{4C_2(r)}{r} Y - \frac{4C_2(r)}{r} \Phi \tag{132}$$

Φ here just acts as a source due to which the Y-component will give an exponentially growing contribution at the growth stage.

Let's look further, how the presence of a source provides growth on a certain range of times.

We write the system of equations on the scales of interest to us $r_d \ll r \ll R$:

$$\begin{cases} \lambda^{-1}\partial_t\Phi = \left(r^2\partial_r^2 - r\partial_r\right)\Phi + \mathcal{D}_\Phi(\Phi), \;\; r_d \ll r \ll R \\ \lambda^{-1}\partial_t Y = \left(r^2\partial_r^2 + 7r\partial_r + 8\right) Y - 8\Phi + \mathcal{D}_Y(Y), \;\; r_d \ll r \ll R. \end{cases} \tag{133}$$

We know the behavior of the functions $C_1(r)$ and $C_2(r)$ for $r \ll R$ and $r \gg R$. These functions determine the solution of the system (133). We do not know the exact behavior of these functions for $r \sim R$, so it is not enough to consider the behavior of correlators on scales $r \ll R$. It is necessary to find their behavior on $r \gg R$, and then match these two solutions on $r \sim R$.

$r \gg R$:

$$\begin{cases} \lambda^{-1}\partial_t\Phi = R^2\left(\partial_r^2 - \frac{1}{r}\partial_r\right)\Phi + \mathcal{D}_\Phi(\Phi), \;\; r \gg R \\ \lambda^{-1}\partial_t Y = R^2\left(\partial_r^2 + \frac{1}{r}\partial_r\right) Y - 8\Phi + \mathcal{D}_Y(Y), \;\; r \gg R. \end{cases} \tag{134}$$

The solution of the first equation should be found primarily. This is a partial differential equation, which is convenient to solve by making the Laplace transform in time:

$$\left(r^2\partial_r - r\partial_r - p\right)\phi_p = 0, \;\; r \ll R \tag{135}$$

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{p}{R^2}\right)\phi_p = 0, \;\; r \gg R \tag{136}$$

The first equation has a power-law solution, the solution of the second is expressed in terms of the modified Bessel functions:

$$\phi_p = r^{-1-\sqrt{p+1}}\left[1 + b_p\left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right], \;\; r \ll R \tag{137}$$

$$\phi_p \approx B_p R^{-\sqrt{p+1}} r K_1\left(\frac{\sqrt{p}r}{R}\right), \;\; r \gg R \tag{138}$$

Correlators of magnetic fields are defined throughout the space. They are smooth functions, therefore two different asymptotics should be matched on scales of the order of the correlation radius R. This system of equations can be written in the matrix form:

$$\left(b_p + 1, b_p + 1 + \sqrt{p+1}(b_p - 1)\right)\hat{g} = B_p\left(K_1(\sqrt{p}, -\sqrt{p}K_0(\sqrt{p}))\right) \tag{139}$$

It is interesting to study the behavior of the coefficients at small p, since they correspond to large times, the behavior of the system on which we are interested. From the fact that the elements of the matrix $\hat{g}$ are finite, we can establish the behavior of the asymptotic coefficients on small scales $b_p \sim b_0 + b_1 p \ln p, \;\; b_0, b_1 \sim 1, \;\; p \to 0$ and on large scales $B_p \sim p, \;\; \phi_p \sim p\frac{r}{R}K_1(\frac{r}{R}\sqrt{p}), \;\; p \to 0.$

The Green function is obtained from solution of (135):

$$G_\Phi(p|r, r') = \frac{(r')^{-2+\sqrt{p+1}}}{2\sqrt{p+1}} r^{-1-\sqrt{p+1}}\left[1 + b_p\left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right], \;\; r > r' \tag{140}$$

$$G_\Phi(p|r,r') = \frac{(r')^{-2-\sqrt{p+1}}}{2\sqrt{p+1}} r^{-1+\sqrt{p+1}} \left[1 + b_p \left(\frac{r'}{R}\right)^{2\sqrt{p+1}}\right], \;\; r < r' \tag{141}$$

In the Laplace domain, the function is determined by the convolution of the Green's function with the initial field distribution, which is localized on the scales $\sim l \sim r_d$ (this assumption is used here and below to simplify calculations and avoid a range of intermediate asymptotics).

$$\Phi_p(r) = \int\limits_0^\infty dr' \Phi^{(0)}(r') G_\Phi(p|r,r') \approx f_0 \int\limits_{r_d}^{l} dr' G_\Phi(p|r,r') \sim$$

$$\sim f_0 \frac{1}{2\sqrt{p+1}(1-\sqrt{p+1})} \left(\frac{r}{r_d}\right)^{-1-\sqrt{p+1}} \left[1 + b_p \left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right] \tag{142}$$

The evolution of function $\Phi(r,t)$ can be found by its image $\Phi_p(r)$:

$$\Phi(r,t) = \int\limits_{c-i\infty}^{c+i\infty} \frac{dp}{2\pi i} e^{p\lambda t} \Phi_p(r) = f_0 \int\limits_{c-i\infty}^{c+i\infty} \frac{dp}{2\pi i} e^{p\lambda t} \frac{1}{2\sqrt{p+1}(1-\sqrt{p+1})} \left(\frac{r}{r_d}\right)^{-1-\sqrt{p+1}} \left[1 + b_p \left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right] \tag{143}$$

The saddle-point approximation allows to calculate this integral at large times $\lambda t \gg 1$. Function will primarily grow in a complex manner with time [11]:

$$\Phi(r,t) \propto \frac{f_0}{\sqrt{\lambda t}} \frac{r}{r_d} \exp\left(-\lambda t - \frac{1}{4\lambda t} \ln^2 \frac{r}{r_d}\right), \;\; 2\lambda t < \ln\left(\frac{R}{r_d}\right) \tag{144}$$

It is worth noting noted that it is the most significant along the trajectory in (r, t) the space $r_m(t) = r_d e^{2\lambda t}$. When $r_m(t)$ reaches the correlation scale of the flow velocity field, this approximation will not be applicable.

The growth of the function is replaced by a decrease having a power-law manner [11] at times $\lambda t \gg \ln\left(\frac{R}{r}\right)$:

$$\Phi(r,t) \propto f_0 \frac{r^2}{R^2} \frac{1}{(\lambda t)^2}, \;\; \lambda t \gg \ln\left(\frac{R}{r}\right). \tag{145}$$

The Green function for Y(r,t) is found in a similar way[11]:

$$G_Y(p|r,r') = r^{-3+\sqrt{p+1}} \frac{(r')^{2-\sqrt{p+1}}}{2\sqrt{p+1}} \left[1 + a_p \left(\frac{r'}{R}\right)^{2\sqrt{p+1}}\right], \;\; r < r' \tag{146}$$

$$G_Y(p|r,r') = r^{-3-\sqrt{p+1}} \frac{(r')^{2+\sqrt{p+1}}}{2\sqrt{p+1}} \left[1 + a_p \left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right], \;\; r > r' \tag{147}$$

The expression for the solution is following:

$$y_p(r) = \int\limits_0^\infty dr' G_Y(p|r,r') \left(y^{(0)}(r') - 8\Phi_p(r')\right) \tag{148}$$

The term $\Phi_p(r)$, will prevail over other non-homogeneous one, since unlike $y^{(0)}(r)$ that is localized on the scale of the initial distribution, it is essential on a larger spatial scale. This contribution increases with increasing r, like the Green's function, which ensures exponential growth at one of the stages.

The solution can be separated by two parts:

$$y_p(r) = y_p^{(1)}(r) + y_p^{(2)}(r); \tag{149}$$

$$y_p^{(1)}(r) \sim \frac{f_0 r^{1-\sqrt{p+1}}}{(p+1)(2-\sqrt{p+1})} \left(\frac{R}{r}\right)^{4-2\sqrt{p+1}} \left[1 - \left(\frac{r}{R}\right)^{4-2\sqrt{p+1}}\right]; \tag{150}$$

$$y_p^{(2)}(r) \sim \frac{f_0 r^{1-\sqrt{p+1}}}{(p+1)(2+\sqrt{p+1})} \left(\frac{R}{r}\right)^{4-2\sqrt{p+1}} \left[a_p + b_p + \frac{2a_p b_p}{2+\sqrt{p+1}}\right] \tag{151}$$

The first contribution contains a singular point $p = 3$, so we will consider the contours of integration lying in the space $Re(p) < 3$.

In the saddle-point approximation the asymptotics can be determined at different time intervals [11]

$$Y(r,t) \propto f_0 \frac{l}{r} e^{3\lambda t}, \ \ \frac{1}{4\ln}\frac{r}{r_d} \ll \lambda t \leq \frac{1}{4}\ln\frac{R^2}{lr}; \tag{152}$$

After that, the field continues to grow in a somewhat more complicated way, slower than exponentially, and the manner of this growth depends on the radius of correlation of the velocity field R:

$$Y(r,t) \propto \frac{f_0}{\sqrt{\lambda t}} \frac{R^4}{r^3 l} \exp\left\{-\lambda t - \frac{1}{4\lambda t}\ln^2\frac{R^2}{rl}\right\}, \ \ \frac{1}{4}\ln\frac{R^2}{lr} \leq \lambda t \leq \frac{1}{2}\ln\frac{R^2}{lr}; \tag{153}$$

Finally, the value of the function reaches a maximum value proportional to the parametrically large value $\frac{R^2}{r^2}$, then such growth is changed by a power decrease in time:

$$Y(r,t) \propto f_0 \frac{R^2}{r^2}\frac{1}{(\lambda t)^2}, \ \ \lambda t \gg \ln\frac{R^2}{lr} \tag{154}$$

The last asymptotics is obtained near the singular point $p = 0$, which describes the behavior of the correlator at large times.

Thus, the correlation functions of the magnetic field components in the horizontal plane at scales much smaller than the correlation radius of the velocity field $r \ll R$ decay in the limit of the large times following a power-law:

$$F_{\alpha\beta}(r) \propto \frac{f_0}{(\lambda t)^2}\left(-\delta_{\alpha\beta} + 2\frac{r_\alpha r_\beta}{r^2}\right)\frac{R^2}{r^2}, \ \ r_d \ll r \ll R \tag{155}$$

$$F_{\alpha\beta}(r) \propto \frac{f_0}{(\lambda t)^2}\frac{R^2}{r^2}\delta_{\alpha\beta}, \ \ r \lesssim r_d \tag{156}$$

Thus, a single-point correlator has a diagonal form, and a point-to-point correlator is determined by both contributions $\delta_{\alpha\beta}$ и $\frac{r_\alpha r_\beta}{r^2}$.

It is worth noting that the tensor of a different-point correlator is traceless.

4.3 The results of quasi-two-dimensional flow

4.3.1 Correlation function F_{33}

The equation for this correlation function is following:

$$\partial_t F_{33} = \left[C_{\mu\nu}(0) - C_{\mu\nu}(r)\right]\partial_\mu\partial_\nu F_{33}(r) + 2\kappa\nabla^2 F_{33} \tag{157}$$

Let's reveal $C_{\mu\nu}$ using relation (113) and substituting:

$$\partial_t F_{33} = \left[C_1(0) - C_1(r)\right]\partial_\mu^2 F_{33}(r) + 2\kappa\nabla^2 F_{33} + C_2(r)\frac{r_\mu r_\nu}{r^2}\partial_\mu\partial_\nu F_{33}; \tag{158}$$

Note that $r_\mu r_\nu \partial_\mu\partial_\nu = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 - \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = (r\partial_r)^2 - r\partial_r = r^2\partial_r^2$.

The correlator of vertical fields is invariant with respect to rotation in the horizontal plane, therefore it will not depend on the polar angle:

$\partial_\mu^2 = \partial_r^2 + \frac{1}{r}\partial_r$.

With this in mind, we have:

$$\partial_t F_{33} = \left[C_1(0) - C_1(r) - C_2(r)\right]\partial_r^2 F_{33} + \left[C_1(0) - C_1(r)\right]\frac{1}{r}\partial_r F_{33} + 2\kappa\nabla^2 F_{33}; \tag{159}$$

Now we will deal with the dissipative term. It should be noted that the scale of dissipation due to the strong shift changes and becomes $\tilde{r}_d = r_d\frac{\Sigma}{\lambda}$. The dissipative summand on the scales of interest is small in comparison with the others, but it can not be neglected. It can be taken into account in a cut-off dissipation scale. It is denoted as $\mathcal{D}\left(F\right)$.

The remaining terms contain only derivatives with respect to coordinates in the plane, and for small z the correlator depends little on the vertical coordinate. Thus, the equation can immediately be reduced to a two-dimensional:

The equation on F is following:

$$\partial_t F(r) = \left[C_1(0) - C_1(r) - C_2(r)\right]\partial_r^2 F(r) + \left[C_1(0) - C_1(r)\right]\frac{1}{r}\partial_r F(r) + \mathcal{D}\left(F\right) \tag{160}$$

Let's write down one more time the correlators of velocity field at $r \ll R$:

$$C_1(r) = \lambda(R^2 - 3r^2);\ \ C_2(r) = 2\lambda r^2;\ \ C_1(0) - C_1(r) = 3\lambda r^2;\ \ C_1(0) - C_1(r) - C_2(r) = \lambda r^2 \tag{161}$$

The dissipative term can be omitted in the region of interest, taking into account dissipation only in the form of a scaling scale.

The equation (159) will have the following form in two different limits:

$$\begin{aligned}\lambda^{-1}\partial_t F &= r^2\partial_r^2 F + 3r\partial_r F; \qquad \tilde{r}_d \ll r \ll R \\ \lambda^{-1}\partial_t F &= R^2\left(\partial_r^2 F + \frac{1}{r}\partial_r F\right); \qquad r \gg R\end{aligned} \tag{162}$$

Those equations can be solved using Laplace transformation $F \to F_p$, $\lambda^{-1}\partial_t \to p$. Then we match two different solutions at $r \ll R$ and $r \gg R$ at the boundary $r \sim R$:

$$\left(r^2\partial_r^2 + 3r\partial_r - p\right) F_p = 0; \qquad \tilde{r}_d \ll r \ll R \tag{163}$$
$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{p}{R^2}\right) F_p = 0; \qquad r \gg R$$

General solution of the first one:

$$F_p = r^{-1-\sqrt{1+p}} + c_p R^{-2\sqrt{p+1}} r^{-1+\sqrt{p+1}} \tag{164}$$

General solution of the second one:

$$F_p = C_p R^{-1-\sqrt{p+1}} K_0\left(\frac{r}{R}\sqrt{p}\right) \tag{165}$$

Green function at $\tilde{r}_d \ll r \ll R$:

$$G_F(p|r,r') = \frac{1}{(r')^2 W(r')} u(r)v(r'), \qquad r > r' \tag{166}$$

$$G_F(p|r,r') = \frac{1}{(r')^2 W(r')} u(r')v(r), \qquad r' > r; \tag{167}$$

where $u(r) = r^{-1+\sqrt{p+1}}$, $v(r) = r^{-1-\sqrt{p+1}}\left[1 + c_p\left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right]$.

$$G_F(p|r,r') = \frac{(r')^{\sqrt{p+1}}}{2\sqrt{p+1}} r^{-1+\sqrt{1+p}}\left[1 + c_p\left(\frac{r'}{R}\right)^{2\sqrt{p+1}}\right], r < r' \tag{168}$$

$$G_F(p|r,r') = \frac{(r')^{\sqrt{p+1}}}{2\sqrt{p+1}} r^{-1-\sqrt{1+p}}\left[1 + c_p\left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right], r > r' \tag{169}$$

Matching:

$$\left(F_p(R-0), F_p'(R-0)\right)\hat{g} = C_p\left(F_p(R+0), F_p'(R+0)\right) \tag{170}$$

$$\left(c_p + 1, -(c_p+1) + \sqrt{p+1}(c_p - 1)\right)\hat{g} = C_p\left(K_0(\sqrt{p}), -\sqrt{p}K_1(\sqrt{p})\right) \tag{171}$$

gives the following result at small p:

$$c_p \sim c_0 + c_1 \ln^{-1}(p), \;\; C_p \sim \ln^{-1}(p) \tag{172}$$

In the end, the evolution of the correlators of the components of the magnetic field in the plane should be found. For this purpose it will be enough to find the Laplace transform of functions F_{33} and $F_{3\alpha}$.

$$F_p(r) = \int\limits_0^\infty \mathrm{d}r' G_F(p|r,r') F_p^{(0)}(r') = f_0 \int\limits_{\tilde{r}_d}^{l} \mathrm{d}r' (r')^{\sqrt{p+1}} \frac{r^{-1-\sqrt{p+1}}}{2\sqrt{p+1}} \left(1 + c_p \left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right) \approx \tag{173}$$

$$\approx f_0 \left(\frac{r}{\tilde{r}_d}\right)^{-1-\sqrt{p+1}} \frac{1}{2\sqrt{p+1}(\sqrt{p+1}+1)} \left(1 + c_p \left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right);$$

Here and further the scale of dissipation and the scale of initial field distribution is assumed to be of one order to avoid consideration of a large number of intermediate asymptotics: $l \sim \tilde{r}_d$.

We will primarily study correlator's behavior at not very large times t.

$$F(r,t) = \int\limits_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}p}{2\pi i} e^{p\lambda t} f_0 \left(\frac{r}{\tilde{r}_d}\right)^{-1-\sqrt{p+1}} \frac{1}{2\sqrt{p+1}(\sqrt{p+1}+1)} \left(1 + c_p \left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right) = \tag{174}$$

$$= f_0 \frac{\tilde{r}_d}{r} \int\limits_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}p}{2\pi i} e^{p\lambda t - \sqrt{p+1}\ln\left(\frac{r}{\tilde{r}_d}\right)} \frac{1}{2\sqrt{p+1}(\sqrt{p+1}+1)} \left(1 + c_p \left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right);$$

The saddle-point is determined by the maximum of the last exponent at times $2\lambda t < \ln\left(\frac{r}{\tilde{r}_d}\right)$. We can't just find a minimum of written earlier exponent at times smaller than the latest one because in this case we will have a saddle-point $p < 0$ but $p = 0$ – is a singular point. Here the second term is not taken into account. So, the value of the exponent at the saddle-point is $\lambda t - \frac{1}{2\sqrt{p+1}} \ln\left(\frac{r}{\tilde{r}_d}\right) = 0; \qquad p = -1 + \left(\frac{\ln\left(\frac{r}{\tilde{r}_d}\right)}{2\lambda t}\right)^2$.

We will get:

$$F(r,t) \propto f_0 \frac{\tilde{r}_d}{r} \frac{1}{\sqrt{\lambda t}} \frac{1}{1 + \frac{\ln\left(\frac{r}{\tilde{r}_d}\right)}{2\lambda t}} e^{-\lambda t - \frac{\ln^2\left(\frac{r}{\tilde{r}_d}\right)}{4\lambda t}}, \qquad 1 \ll 2\lambda t < \ln\left(\frac{r}{\tilde{r}_d}\right) \tag{175}$$

The calculations are similar to the case described below, there it is painted a little more.

$$F(r,t) = f_0 \frac{\tilde{r}_d}{r} \int\limits_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}p}{2\pi i} e^{p\lambda t - \sqrt{p+1}\ln\left(\frac{r}{\tilde{r}_d}\right) + 2\sqrt{p+1}\ln\left(\frac{r}{R}\right)} \frac{1}{2\sqrt{p+1}(\sqrt{p+1}+1)} c_p = \tag{176}$$

$$= f_0 \frac{\tilde{r}_d}{r} \int\limits_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}p}{2\pi i} e^{p\lambda t - \sqrt{p+1}\ln\left(\frac{R^2}{r\tilde{r}_d}\right)} \frac{1}{2\sqrt{p+1}(\sqrt{p+1}+1)} c_p$$

Let's find saddle point at larger times:

$$\lambda t - \frac{\ln\left(\frac{R^2}{r\tilde{r}_d}\right)}{2\sqrt{p+1}} = 0; \qquad p_1 = -1 + \left(\frac{\ln\left(\frac{R^2}{r\tilde{r}_d}\right)}{2\lambda t}\right)^2.$$

$$F(r,t) \propto X e^{-\lambda t - \frac{1}{4\lambda t}\ln^2\left(\frac{R^2}{r\tilde{r}_d}\right)}; \qquad \frac{1}{2}\ln\left(\frac{r}{\tilde{r}_d}\right) < \lambda t < \frac{1}{2}\ln\left(\frac{R^2}{r\tilde{r}_d}\right) \tag{177}$$

$$X \sim f_0 \frac{\tilde{r}_d}{r} \int \frac{\mathrm{d}p}{2\pi i} \frac{c_p}{2\sqrt{p+1}(\sqrt{p+1}+1)} e^{Z''(p_1)(p-p_1)^2/2} = /Z''(p_1) = 2\frac{(\lambda t)^3}{\ln^2\left(\frac{R^2}{r\tilde{r}_d}\right)}/ = f_0 c_0 \frac{\tilde{r}_d}{r} \frac{1}{\sqrt{\lambda t}} \frac{1}{1 + \frac{\ln(R^2/(r\tilde{r}_d))}{2\lambda t}}.$$

There are no singularity at $p=0$. Therefore we will have the following expression at $\lambda t \gg \ln^2\left(\frac{R^2}{r\tilde{r}_d}\right)$

$$F(r,t) \propto f_0 c_0 \frac{\tilde{r}_d}{r}\frac{1}{\sqrt{\lambda t}} e^{-\lambda t} \tag{178}$$

So we obtain exponentially decreasing with time solution in the absence of source.

4.3.2 Correlation function $F_{3\alpha}$

The equation for this correlation function is following:

$$\partial_t F_{3\alpha} = \left[C_{\mu\nu}(0) - C_{\mu\nu}(r)\right]\partial_\mu\partial_\nu F_{3\alpha}(r,\phi) + \partial_\mu C_{\nu\alpha}(r)\partial_\nu F_{3\mu} + 2\kappa\nabla^2 F_{3\alpha} + \Sigma_\alpha F_{33}(r) \tag{179}$$

The correlator's dependence on vertical coordinate is still weak. However, in contrast to the correlator of vertical components, the cross correlator depend on the polar angle. We write the equation by a scalar quantity $P(r,\phi) = r_\alpha F_{3\alpha}(r,\phi)$:

$$\begin{aligned}\partial_t P = \left\{\left[C_1(0) - C_1(r)\right] r_\alpha \partial_\nu^2 F_{3\alpha} - C_2(r)\frac{r_\mu r_\nu}{r^2} r_\alpha \partial_\mu\partial_\nu F_{3\alpha}\right\} + r_\alpha \partial_\mu C_1(r)\partial_\alpha F_{3\mu} + \\ + r_\alpha\partial_\mu\left[C_2(r)\frac{r_\alpha r_\nu}{r^2}\right]\partial_\nu F_{3\mu} + \Sigma r\cos\phi F + \mathcal{D}\left(F_{3\alpha}\right) r_\alpha\end{aligned} \tag{180}$$

Let us analyze what happens to each term separately, multiplying the entire equation by r_α :

$$\partial_\nu\partial_\nu\left(r_\alpha F_{3\alpha}\right) = r_\alpha\partial_\nu\partial_\nu F_{3\alpha} + 2\partial_\nu F_{3\nu} \tag{181}$$

$$\partial_\mu\partial_\nu P = \partial_\mu\partial_\nu\left(r_\alpha F_{3\alpha}\right) = r_\alpha\partial_\mu\partial_\nu F_{3\alpha} + \partial_\nu F_{3\mu} + \partial_\mu F_{3\nu} \tag{182}$$

$$\frac{1}{r}\partial_r P = \frac{r_\nu}{r^2}\partial_\nu\left(F_{3\mu} r_\mu\right) = \frac{r_\mu r_\nu}{r^2}\partial_\mu F_{3\nu} + \frac{r_\mu}{r^2} F_{3\mu} \tag{183}$$

$$\frac{r_\mu r_\nu}{r^2} r_\alpha\partial_\mu\partial_\nu F_{3\alpha} = \partial_r^2 P - 2\left[\frac{1}{r}\partial_r P - \frac{P}{r^2}\right] \tag{184}$$

$$\begin{aligned}\left[C_{\mu\nu}(0) - C_{\mu\nu}(r)\right] r_\alpha\partial_\mu\partial_\nu F_{3\alpha} = \\ = \left[C_1(0) - C_1(r)\right]\left(\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2\right)P(r,\phi) - 2\partial_\nu f_{3\nu}\right) - C_2(r)\left(\partial_r^2 - 2\frac{\partial_r}{r} + \frac{2}{r^2}\right)P(r,\phi)\end{aligned} \tag{185}$$

Due to the solenoidal nature of the magnetic field, the convolution is expressed in terms of the correlator of vertical fields F_{33}:

$$\partial_\nu F_{3\nu} = \left(\partial_1\left\langle B_3(r')B_1(r'+r)\right\rangle + \partial_2\left\langle B_3(r')B_2(r'+r)\right\rangle\right) = \tag{186}$$

$$= \left\langle B_3(r')\left[\partial_1 B_1(r'+r) + \partial_2 B_2(r'+r)\right]\right\rangle = -\partial_3 F_{33} \tag{187}$$

In the spatial region of interest to us, this term turns out to be negligibly small in comparison with another term in the inhomogeneous part of the equation.

$$r_\alpha \partial_\nu C_1(r) \partial_\alpha F_{3\nu} = \frac{r_\alpha}{r} C_1'(r) r_\nu \partial_\alpha F_{3\nu} = \frac{C_1'(r)}{r} \left[r\partial_r P - P \right] \tag{188}$$

$$r_\alpha \partial_\mu \left(C_2(r) \frac{r_\alpha r_\nu}{r^2} \right) \partial_\nu F_{3\mu} = \frac{C_2'(r)}{r} \left[r\partial_r P - P \right] + r_\alpha C_2(r) \partial_\mu \left[\frac{r_\alpha r_\nu}{r^2} \right] \partial_\nu F_{3\mu} \approx \approx \frac{C_2'(r)}{r} \left[r\partial_r P - P \right] \tag{189}$$

From the condition for the incompressibility of a liquid, the connection between the functions can be obtained C_1 и C_2: $C_1'(r) + C_2'(r) + \frac{C_2(r)}{r} = 0$. Then

$$\partial_\nu C_{\alpha\mu}(r) r_\alpha \partial_\mu F_{3\nu}(r,\phi) \approx -C_2(r) \left[\frac{2}{r} \partial_r P - \frac{2P}{r^2} \right] \tag{190}$$

Then we collect all founded contributions in one equation, omitting the diffuse term, since the scale of the dissipation will be taken into account in the future as a cut-off of the integral on the dissipation scale:

$$\partial_t P = \left[C_1(0) - C_1(r) \right] \left(\partial_r^2 P + \frac{1}{r} \partial_r P + \frac{\partial_\phi^2 P}{r^2} \right) - C_2(r) \partial_r^2 P + +\Sigma r \cos\phi F; \tag{191}$$

The non-homogeneous term, proportional to z, is negligible compared to another non-homogeneous term in the range of interest.

$$\partial_t P = \left[C_1(0) - C_1(r) - C_2(r) \right] \partial_r^2 P + \left[C_1(0) - C_1(r) \right] \left(\frac{1}{r} \partial_r P + \frac{\partial_\phi^2 P}{r^2} \right) + \Sigma r \cos\phi F; \tag{192}$$

Then the variables in the equation are separated:

$$P(r,\phi) = \Pi(r) \cos\phi; \tag{193}$$

$$\partial_t \Pi = \left[C_1(0) - C_1(r) - C_2(r) \right] \partial_r^2 \Pi + \left[C_1(0) - C_1(r) \right] \left(\frac{1}{r} \partial_r \Pi - \frac{\Pi}{r^2} \right) + \Sigma r F \tag{194}$$

We obtain equation on Π using (161):

$$\begin{aligned} \lambda^{-1} \partial_t \Pi &= \left[r^2 \partial_r^2 + 3r\partial_r - 3 \right] \Pi + \frac{\Sigma}{\lambda} r F; \qquad r \ll R \\ \lambda^{-1} \partial_t \Pi &= R^2 \left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \right) \Pi + \frac{\Sigma}{\lambda} r F; \qquad r \gg R \end{aligned} \tag{195}$$

Laplace transformation: $\qquad \Phi \to \Phi_p, \qquad \lambda^{-1}\partial_t \to p.$

$$\begin{aligned} \left(r^2 \partial_r^2 + 3r\partial_r - 3 - p \right) \Phi_p = 0; &\qquad \tilde{r}_d \ll r \ll R \\ \left(\partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} - \frac{p}{R^2} \right) \Phi_p = 0; &\qquad r \gg R \end{aligned} \tag{196}$$

Solutions:

$$\Pi_p \sim r^{-1-\sqrt{p+4}} \left(1 + b_p \left(\frac{r}{R}\right)^{2\sqrt{p+4}}\right), \ r \ll R \tag{197}$$

$$\Pi_p \sim B_p R^{-1-\sqrt{p+4}} K_1 \left(\frac{r\sqrt{p}}{R}\right), \ r \gg R \tag{198}$$

Green function at $\tilde{r}_d \ll r \ll R$:

$$G_\Pi(p|r,r') = \frac{(r')^{-\sqrt{p+4}}}{2\sqrt{p+4}} \left[1 + b_p \left(\frac{r'}{R}\right)^{2\sqrt{p+4}}\right] r^{-1+\sqrt{p+4}}, \ \ r < r' \tag{199}$$

$$G_\Pi(p|r,r') = \frac{(r')^{\sqrt{p+4}}}{2\sqrt{p+4}} \left[1 + b_p \left(\frac{r}{R}\right)^{2\sqrt{p+4}}\right] r^{-1-\sqrt{p+4}}, \ \ r > r' \tag{200}$$

Matchinf of two solutions:

$$\left(b_p + 1, \ -b_p - 1 + \sqrt{p+4}(b_p+1)\right) \hat{g} = B_p \left(K_1(\sqrt{p}), \ -\frac{1}{2}\left(K_0(\sqrt{p}) + K_2(\sqrt{p})\right)\right); \tag{201}$$

$$b_p \sim b_0 + b_1 p, \ \ B_p \sim p, \ \ p \to 0 \tag{202}$$

$$\Pi(r,t) = \int\limits_{c-i\infty}^{c+i\infty} \frac{dp}{2\pi i} e^{p\lambda t} \int\limits_0^R dr' G_\Pi(p|r,r') \left(\Pi_p^{(0)}(r') + \frac{\Sigma}{\lambda} r' F_p(r')\right) \tag{203}$$

For further calculations namely to study the behavior of the correlator of the magnetic field components in the plane it is sufficient to know only the image of this function:

$$\Pi_p(r) \approx \frac{f_0}{2\sqrt{p+4}(\sqrt{p+4}+1)} \left(\frac{\tilde{r}_d}{r}\right)^{2\sqrt{p+4}} \left[1 + b_p \left(\frac{r}{R}\right)^{2\sqrt{p+4}}\right] +$$

$$+ \int\limits_0^R dr' G_\Pi(p|r,r') \frac{\Sigma}{\lambda} r' f_0 \left(\frac{r'}{\tilde{r}_d}\right)^{-1-\sqrt{p+1}} \frac{1}{2\sqrt{p+1}(\sqrt{p+1}+1)} \left(1 + c_p^{33} \left(\frac{r'}{R}\right)^{2\sqrt{p+1}}\right) =$$

$$= \Pi_p^{hom}(r) + \Pi_p^{(1)}(r) + \Pi_p^{(2)}(r); \tag{204}$$

which is formed from a homogeneous and two non-homogeneous parts. The contribution from the homogeneous part will be significantly less than from the non-homogeneous part, since it will be typed on the scale of the initial distribution l, which is much smaller than the characteristic distances on which the behavior of the different-point correlators is studied. The first inhomogeneous part is determined by the Green function at $r' < r$:

$$\Pi_p^{(1)}(r) = \frac{\Sigma}{\lambda} \frac{f_0 r^{-1-\sqrt{p+4}}}{4\sqrt{p+1}\sqrt{p+4}(1+\sqrt{p+1})} \left[1 + b_p \left(\frac{r}{R}\right)^{2\sqrt{p+4}}\right] \cdot$$

$$\cdot \int\limits_0^r dr' (r')^{1+\sqrt{p+4}} \left(\frac{r'}{\tilde{r}_d}\right)^{-1-\sqrt{p+1}} \left[1 + c_p^{33} \left(\frac{r'}{R}\right)^{2\sqrt{p+1}}\right] = \frac{\Sigma}{\lambda} \frac{f_0 r}{4\sqrt{p+1}\sqrt{p+4}(1+\sqrt{p+1})} \left(\frac{r}{\tilde{r}_d}\right)^{-1-\sqrt{p+1}} \tag{205}$$

$$\cdot \left[\frac{1}{1+\sqrt{p+4}-\sqrt{p+1}} + \frac{c_p^{33}}{1+\sqrt{p+1}+\sqrt{p+4}} \left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right] \left[1 + b_p \left(\frac{r}{R}\right)^{2\sqrt{p+4}}\right]$$

and the second one forms at the scale $r' > r$:

$$\Pi_p^{(2)}(r) = \frac{\Sigma}{\lambda} \frac{f_0}{4\sqrt{p+1}\sqrt{p+4}(1+\sqrt{p+1})\tilde{r}_d} \left(\frac{r}{R}\right)^{-1+\sqrt{p+4}} \left(\frac{R}{\tilde{r}_d}\right)^{-\sqrt{p+1}} \tag{206}$$

$$\left[\frac{1}{1-\sqrt{p+1}-\sqrt{p+4}} + \frac{b_p}{1-\sqrt{p+1}+\sqrt{p+4}} + \frac{c_p^{33}}{1+\sqrt{p+1}-\sqrt{p+4}} + \frac{c_p^{33} b_p}{1+\sqrt{p+1}+\sqrt{p+4}}\right]$$

The ultimate goal is to study the behavior of the correlations of the magnetic field components in the plane, for which it is sufficient to know the behavior of the cross correlator $F_{3\alpha}$.

4.3.3 Correlation function $F_{\alpha\beta}$

We will further investigate the correlations of the components of the magnetic field in the horizontal plane.

The equations for the correlators have the following form:

$$\partial_t F_{\alpha\beta} = \hat{L}_{\alpha\beta}^{\mu\nu} F_{\mu\nu} + (\Sigma_\alpha r_\beta + \Sigma_\beta r_\alpha) \frac{\Pi(r)\cos\phi}{r^2}, \tag{207}$$

where $\hat{L}_{\alpha\beta}^{\mu\nu}$ is he same operator as in two-dimensional problem.

The quasi-two-dimensional situation differs from the two-dimensional one by the presence of an inhomogeneous term, which depends on the vertical component of the magnetic field. As for the homogeneous part some quantities such as the effective dissipation scale and the amplitude f_0 should be redefined. Both these circumstances are caused by the presence of a shear flow.

Let us find out how much will be the contribution to the correlator produced by inhomogeneous terms. The behavior of magnetic fields at moderate polar angles $\sin\phi_r \sim 1,\ \cos\phi_r \sim 1$ will be considered. Then the general structure of the tensor will be the same as in the two-dimensional case, but the evolution of the scalar functions Φ and Y, which determine the behavior of correlators, can change due to the presence of an inhomogeneous contribution. Further it is necessary to understand, how much this evolution will change.

First we investigate the closed equation on Φ, which was previously defined as $\Phi = r_\beta \partial_\alpha F_{\alpha\beta}$:

$$\partial_t \Phi = [C_1(0) - C_1(r) - C_2(r)] \left(\partial_r^2 - \frac{1}{r}\partial_r\right) \Phi + c_1 \Sigma \frac{\Pi(r)}{r} + c_2 \Sigma \Pi'(r) \tag{208}$$

As will be shown later, the contributions from the inhomogeneous part turn out to be small, so we are not interested in the exact values of the numerical coefficients c_1 and c_2.

The solution will have the following form n the image space:

$$\Phi_p(r) = \Phi_p^{2D}(r) + \Sigma \int_0^R dr' G_\Phi(p|r,r') \left[c_1 \frac{d}{dr} \Pi_p(r') + c_2 \frac{\Pi_p(r')}{r'} \right] \tag{209}$$

We first find a derivative of the first part of $\Pi_p^{(1)}(r)$ of the inhomogeneous term arising due to the shear flow:

$$\frac{d}{dr}\Pi_p^{(1)}(r) = \frac{\Sigma}{\lambda} \frac{f_0}{4\sqrt{p+1}\sqrt{p+4}(1+\sqrt{p+1})} \left(\frac{r}{\tilde{r}_d}\right)^{-1-\sqrt{p+1}}$$
$$\left[-\frac{\sqrt{p+1}}{1+\sqrt{p+4}-\sqrt{p+1}} + \frac{c_p^{33}\sqrt{p+1}}{1+\sqrt{p+1}+\sqrt{p+4}} \left(\frac{r}{R}\right)^{2\sqrt{p+1}} + \right. \tag{210}$$
$$\left. + \frac{b_p(2\sqrt{p+4}-\sqrt{p+1})}{1+\sqrt{p+1}+\sqrt{p+4}} \left(\frac{r}{R}\right)^{2\sqrt{p+4}} + \frac{c_p^{33} b_p(2\sqrt{p+4}+\sqrt{p+1})}{1+\sqrt{p+1}+\sqrt{p+4}} \left(\frac{r}{R}\right)^{2\sqrt{p+4}+2\sqrt{p+1}} \right]$$

and than the second one $\Pi_p^{(2)}(r)$:

$$\frac{d}{dr}\Pi_p^{(2)}(r) = \frac{\Sigma}{\lambda} \frac{f_0(\sqrt{p+4}-1)}{4\sqrt{p+1}\sqrt{p+4}(1+\sqrt{p+1})\tilde{r}_d R} \left(\frac{r}{R}\right)^{-2+\sqrt{p+4}} \left(\frac{R}{\tilde{r}_d}\right)^{-\sqrt{p+1}} \tag{211}$$
$$\left[\frac{1}{1-\sqrt{p+1}-\sqrt{p+4}} + \frac{b_p}{1-\sqrt{p+1}+\sqrt{p+4}} + \frac{c_p^{33}}{1+\sqrt{p+1}-\sqrt{p+4}} + \frac{c_p^{33} b_p}{1+\sqrt{p+1}+\sqrt{p+4}} \right]$$

A contribution containing $\frac{\Pi_p(r)}{r}$ will have the same powers of r as the above, but slightly different coefficients depending on p.

The term $\Pi_p^{(2)}(r)$ gives an expression, part of the terms in which has a singularity at the point $p = 0$:

$$\int_0^R dr' G_\Phi(p|r,r') \frac{\Sigma}{\lambda} \frac{d}{dr'} \Pi^{(2)}(r') = \frac{f_0 \frac{\Sigma}{\lambda}}{8\sqrt{p+4}(p+1)(1+\sqrt{p+1}) R \tilde{r}_d} \cdot$$
$$\cdot \left[\int_0^r dr' \left(\frac{r'}{R}\right)^{\sqrt{p+4}-2} \left(\frac{R}{\tilde{r}_d}\right)^{-\sqrt{p+1}} (r')^{\sqrt{p+1}-2} r^{-1-\sqrt{p+1}} \left[1 + b_p \left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right] + \right. \tag{212}$$
$$\left. + \int_r^R dr' \left(\frac{r'}{R}\right)^{\sqrt{p+4}-2} \left(\frac{R}{\tilde{r}_d}\right)^{-\sqrt{p+1}} (r')^{-\sqrt{p+1}-2} r^{-1+\sqrt{p+1}} \left[1 + b_p \left(\frac{r'}{R}\right)^{2\sqrt{p+1}}\right] \right] \cdot$$
$$\cdot \left[\frac{1}{1-\sqrt{p+1}-\sqrt{p+4}} + \frac{b_p}{1-\sqrt{p+1}+\sqrt{p+4}} + \frac{c_p^{33}}{1+\sqrt{p+1}-\sqrt{p+4}} + \frac{c_p^{33} b_p}{1+\sqrt{p+1}+\sqrt{p+4}} \right] =$$

After integration, there will be several contributions, some of which will be singular at $p \to 0$:

$$= \frac{f_0 \frac{\Sigma}{\lambda}}{8\sqrt{p+4}(p+1)(1+\sqrt{p+1})R\tilde{r}_d}$$

$$\cdot \left[\left(\frac{r}{R}\right)^{\sqrt{p+4}-2} \left(\frac{R}{\tilde{r}_d}\right)^{-\sqrt{p+1}} \frac{1}{\left(\sqrt{p+1}+\sqrt{p+4}-3\right) r^2} \left[1 + b_p \left(\frac{r}{R}\right)^{2\sqrt{p+1}}\right] + \right.$$

$$\left. + \left(\frac{r}{R}\right)^{\sqrt{p+4}-2} \left(\frac{R}{\tilde{r}_d}\right)^{-\sqrt{p+1}} \frac{1}{r^2\left(3+\sqrt{p+1}-\sqrt{p+4}\right)} + b_p \frac{1}{R^2\left(\sqrt{p+4}+\sqrt{p+1}-3\right)} \left(\frac{r}{R}\right)^{\sqrt{p+1}-1} \left(\frac{\tilde{r}_d}{R}\right)^{\sqrt{p+1}} \right]$$

$$\cdot \left[\frac{1}{1-\sqrt{p+1}-\sqrt{p+4}} + \frac{b_p}{1-\sqrt{p+1}+\sqrt{p+4}} + \frac{c_p^{33}}{1+\sqrt{p+1}-\sqrt{p+4}} + \frac{c_p^{33} b_p}{1+\sqrt{p+1}+\sqrt{p+4}}\right]$$

We will consider the behavior of the fields at the longest times $\lambda t \gg \ln \frac{R}{r}$. We are looking for a maximum of exponent at large times:

$$\lambda t - \frac{1}{2\sqrt{p+4}} \ln \frac{R}{r} - \frac{1}{2\sqrt{p+1}} \ln \frac{R}{\tilde{r}_d} - \frac{1}{\sqrt{p+4}+\sqrt{p+1}-3} \left(\frac{1}{2\sqrt{p+1}} + \frac{1}{2\sqrt{p+4}}\right) = 0 \quad (214)$$

The saddle point will approach zero at long times:

$$\lambda t - \frac{1}{3p_0/4} \left(\frac{1}{2} + \frac{1}{4}\right) = 0; \quad (215)$$

$$p_0 = \frac{1}{\lambda t} \to 0 \quad (216)$$

Integration in the vicinity of this saddle point yields the same power law $\propto \frac{1}{(\lambda t)^2}$, as in the two-dimensional case.

Similar results are obtained for the remaining terms for other contributions to Φ, as well as for the equation for the function Y, which has the following form:

$$\partial_t Y = \hat{L}_Y^{2D} Y + \hat{L}_\Phi^{2D} \Phi + c_3 \Sigma \frac{\Pi(r)}{r}, \;\; c_3 \sim 1 \quad (217)$$

$\hat{L}_Y^{2D}, \;\; \hat{L}_\Phi^{2D}$ – are the same operators as in the two-dimensional case.

Thus, due to the shear flow, the dissipation scale changes, increasing by $\frac{\Sigma}{\lambda}$ times, as well as the initial amplitude f_0. The new amplitude has the form $\frac{f_0}{\sqrt{\tilde{\kappa} t}} \sim \frac{1}{\tilde{r}_d \sqrt{\lambda t}} \sim \frac{\lambda}{\Sigma} \frac{1}{\sqrt{\kappa t}}$.

Thus, for large times, single-point and point-to-point correlators decrease in the same power manner with time, however, they depend on the shear in different ways:

$$\boxed{F_{\alpha\beta} \propto \frac{\lambda}{\Sigma} \frac{f_0}{\sqrt{\kappa t}(\lambda t)^2} \left(-\frac{2 r_\alpha r_\beta}{r^2} + \delta_{\alpha\beta}\right) \frac{R^2}{r^2}, \;\; \lambda t \gg \ln \frac{R}{r}, \;\; r \gg r_d \frac{\Sigma}{\lambda}} \quad (218)$$

$$\boxed{F_{\alpha\beta} \propto \left(\frac{\lambda}{\Sigma}\right)^3 \frac{f_0}{\sqrt{\kappa t}(\lambda t)^2} \frac{R^2}{r_d^2} \delta_{\alpha\beta}, \;\; \lambda t \gg \ln \frac{\lambda R}{\Sigma r_d}, \;\; r \ll r_d \frac{\Sigma}{\lambda}} \tag{219}$$

The above results are valid for moderate polar angles sin ϕ_r, cos $\phi_r \sim 1$. At angles close to 0 or $\pi/2$ due to the anisotropy of the problem dips can occur when the polar angle is approached to these values.

A single-point correlator decreases faster with increasing shear than a point-to-point correlation function. We also note that the correlators are monotonically decreasing as a function of distance.

5 Conclusion

Thus, in this paper the behavior of the magnetic field in a chaotic quasi-two-dimensional flow with a strong shift at times much higher than the inverse Lyapunov exponent is considered. In the Lagrangian frame of reference, the situation is similar to the two-dimensional problem, in which there is a singular point and there is no additional flow, and differs from it only by the appearance of a strong shear.

In a two-dimensional problem, the fields grow exponentially with time, then the correlations weaken, and the fields begin to decrease with time following a power-law. The nature of the change of the magnetic field with time in this formulation is the same as in the two-dimensional case, up to a multiplier that occurs due to the diffusion along the vertical coordinate z. Thus, the regularities obtained in the work confirm the possibility of the appearance of magnetic fields in astrophysical objects due to the Dynamo effect. The structure of the correlators is very similar to the two-dimensional case. However, the presence of the strong shear effect on the amplitude of the fields. In particular, the values of the fields in the non-dissipative case significantly exceed the values of the fields for the two-dimensional case and are large in the vertical gradient parameter $\frac{\Sigma}{\lambda} \gg 1$. In the presence of finite dissipation, the opposite effect occurs. The main difference from the two-dimensional case arises only in the fact that there is a change in the effective scale of dissipation r_d, which increases in proportion to the magnitude of the shear in $\frac{\Sigma}{\lambda}$ times. Thus, with the final dissipation at sufficiently large times, the value of the field begins to decrease with the increase in the shear. Since in real flows there is always a final dissipation, the presence of a shear in the flows leads to a weakening of the Dynamo effect.

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6 Appendix: The differentiation of tensors

Formulas for the differentiation of certain tensors that depend on polar coordinates are presented in this section. They were often encountered in finding the behavior of correlators over long times:

$$\partial_\nu \left(\frac{r_\alpha r_\beta}{r^2}\right) = \frac{\delta_{\alpha\nu} r_\beta + \delta_{\beta\nu} r_\alpha}{r^2} - 2\frac{r_\alpha r_\beta r_\nu}{r^4}; \tag{220}$$

$$\partial_\alpha \left(\frac{r_\alpha r_\beta}{r^2} \right) = \frac{r_\beta}{r^2}; \tag{221}$$

$$\partial_\mu \frac{\delta_{\beta\nu} r_\alpha}{r^2} = \delta_{\beta\nu} \frac{r^2 \delta_{\alpha\mu} - 2 r_\alpha r_\mu}{r^4}; \tag{222}$$

$$\partial_\mu \frac{r_\alpha r_\beta r_\nu}{r^4} = \frac{r^4 (\delta_{\beta\mu} r_\alpha r_\nu + \delta_{\alpha\mu} r_\beta r_\nu + \delta_{\nu\mu} r_\alpha r_\beta) - 4 r^2 r_\alpha r_\beta r_\nu r_\mu}{r^8}; \tag{223}$$

$$\begin{aligned} \partial_\nu \partial_\mu \left(\frac{r_\alpha r_\beta}{r^2} \right) = \frac{\delta_{\alpha\nu}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\nu}}{r^2} + 8 \frac{r_\alpha r_\beta r_\nu r_\mu}{r^6} - \\ -2 \frac{\delta_{\beta\mu} r_\alpha r_\nu + \delta_{\nu\mu} r_\alpha r_\beta + \delta_{\alpha\mu} r_\beta r_\nu + \delta_{\beta\nu} r_\alpha r_\mu + \delta_{\alpha\nu} r_\beta r_\mu}{r^4}; \end{aligned} \tag{224}$$

$$\partial_\nu \partial_\nu \left(\frac{r_\alpha r_\beta}{r^2} \right) = -\frac{2}{r^2} \left(\frac{2 r_\alpha r_\beta}{r^2} - \delta_{\alpha\beta} \right); \tag{225}$$

$$r_\mu \partial_\mu \left(\frac{r_\alpha r_\beta}{r^2} \right) = 0; \tag{226}$$

$$r_\alpha \partial_\nu \left(\frac{r_\alpha r_\beta}{r^2} \right) = \delta_{\beta\nu} - \frac{r_\beta r_\nu}{r^2}. \tag{227}$$