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MASTER'S THESIS

Current-voltage characteristics of an asymmetric Josephson junction

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Abstract

The Josephson effect is a well-known phenomenon of the small current flowing through a weak link between two superconductors. The current remains superconducting up to certain critical value (critical current) above which the voltage drop across the junction appears, and the current demonstrates dissipative component. Usually, the critical current is independent on the current direction. In this work, we consider the case when this dependence arises, and study possible consequences of direction-dependent critical current. In particular, we develop several perturbative approaches which allow to calculate corrections to the current-voltage characteristics (CVC) due to the asymmetry of the junction.

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Part I

Introduction

1 Motivation and definition

Definition: *Asymmetric Josephson junction* is a Josephson junction which has different critical currents (I_c^+ and I_c^-), depending on the direction of the current. The direction from left to the right is defined as positive (the corresponding critical current is denoted as I_c^+). These critical currents are supposed to be unequal: $I_c^+ \neq I_c^-$.

Simplest realization This property of the inequality of critical currents can be demonstrated by such simple systems as two-contact SQUID's (see fig. 1.1) whose shoulders are represented by two different Josephson junctions. "Different" means that the current-phase relations of these junctions (denoted as $I_1(\varphi)$ and $I_2(\varphi)$ on the fig. 1.1) are not proportionate to each other. These current-phase relations define the effective current-phase relation $I(\varphi) = I_1(\varphi + 2\pi\Phi/\Phi_0) + I_2(\varphi)$ of the whole SQUID which can be considered now as one Josephson junction. If the phase shift $2\pi\Phi/\Phi_0$, produced by magnetic flux Φ (related to the flux quantum $\Phi_0 = \pi\hbar/e$) put inside the SQUID's loop is non-integer times π, then we have:

$$\left|\max_{\varphi} I(\varphi)\right| \neq \left|\min_{\varphi} I(\varphi)\right|$$

which leads to $I_c^+ \neq I_c^-$ (Here and below, φ denotes the phase difference of the order parameters on the opposite sides of the junction).

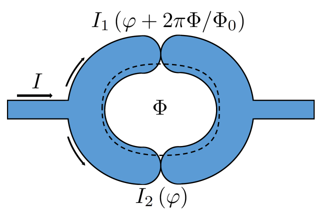


Fig. 1.1: Asymmetric Josephson junction based on two-contact SQUID. Asymmetry appear when $\Phi \neq n\Phi_0,\ (n+1/2)\,\Phi_0;\ n \in \mathbb{Z}$.

Example Let us to consider a minimal model, when one Josephson junction has sine-like current-phase relation and another has in addition a second harmonic:

$$I_1(\varphi) = I_c \sin\varphi \tag{1.1a}$$

$$I_2(\varphi) = I_c(\sin\varphi + A\sin 2\varphi) \tag{1.1b}$$

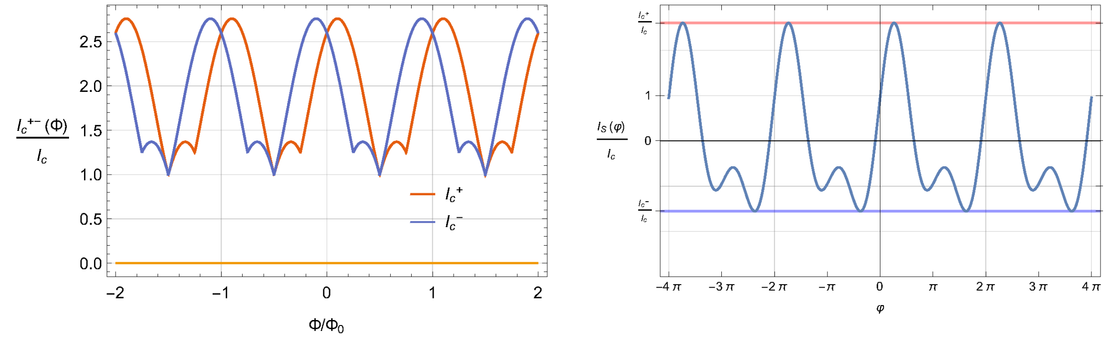
where A is a dimensionless parameter corresponding to the "asymmetry degree". Then the effective current-phase relation has a form:

$$I_S(\varphi) = I_c\left\{2\cos\left(\pi\frac{\Phi}{\Phi_0}\right)\sin(\varphi + \pi\Phi/\Phi_0) + A\sin 2\varphi\right\} \tag{1.2}$$

and maximum and minimum values I_c^+ and I_c^- in first order with respect to $A \ll 1$ demonstrate different values, when $\Phi \neq n\Phi_0,\ \left(n+\frac{1}{2}\right)\Phi_0$:

$$I_c^{\pm}/I_c \approx 2\cos\left(\pi\frac{\Phi}{\Phi_0}\right) \pm A\sin\left(2\pi\frac{\Phi}{\Phi_0}\right) \tag{1.3}$$

When A is not small the presence of the asymmetry can be clearly seen from the plot of $I(\varphi)$, and numeric calculation of the critical currents (see 1.2a and 1.2b). The sharp angles on the plot in 1.2a are due to the alternation of the local maxima and minima, which can arise in case of arbitrary (non-small) A.



(a) (b)

Fig. 1.2: (a) – critical current as function of the magnetic flux in case $A = 1$. (b) – Current-phase relation in case $A = 1$, $\Phi = 0.2\Phi_0$

This effect, despite its simplicity, as far as we know, has not been discussed previously in literature. However, when our work was already in process, the article [1], where a similar effect was found in a much more complex system, was published. In that work a similar effect was uncovered in a superconducting transition, through a Weyl semimetall, where the carrier transfer was carried out due to topologically protected edge states with different Fermi velocities.

In this paper, we propose to investigate how the asymmetry will affect the properties of the junction. In particular, it is proposed to investigate correction of current-voltage characteristic (CVC) and Shapiro steps. This is the subject of the first and second parts of this work, respectively.

2 Problem statement, definitions and structure of the thesis

Let us consider SQUID on the (1.1) with current-phase relations (1.1a) and (1.1b) in terms of the Resistive Shunted Junction (RSJ) model, in which the dissipative resistive part of the current is represented by the resistance R enabled in parallel with the Josephson junction, where R denotes the total resistance of the SQUID. We will neglect the capacitance of the junction and the inductance of the loop. In this approximation we will have the following equation for the phase $\varphi(t)$:

$$\frac{\hbar}{2eR}\frac{\partial\varphi}{\partial t} = I - I_S(\varphi) \tag{2.4}$$

where $I_S(\varphi)$ is defined in (1.2). Our task is to calculate the average voltage $\langle V(t)\rangle_t$, which is defined as $V(t) = \frac{\hbar}{2e}(\partial\varphi/\partial t)$.

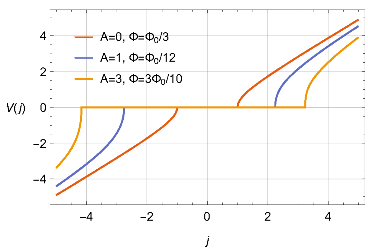


Fig. 2.1: Numeric result of the CVC in RSJ model under certain values of the parameters A and Φ

Notations Let us to introduce the convenient dimensionless notations. First, we see from (1.2) and (1.3) that in case $A = 0$ we have $I_c^+ = I_c^- = 2I_c|\cos\pi\Phi/\Phi_0|$, so it is natural to measure the current and other connected with current quantities in terms of $\tilde{I}_c(\Phi) \overset{\text{def}}{=} 2I_c\cos\pi\Phi/\Phi_0$. To be specific let us to introduce the following variables:

$$\begin{aligned} j &= \frac{I}{\tilde{I}_c(\Phi)} & j_S(\varphi) &= \frac{I_S(\varphi)}{\tilde{I}_c(\Phi)} \\ j_c^{\pm} &= \frac{I_c^{\pm}}{\left|\tilde{I}_c(\Phi)\right|} & \alpha &= A\frac{I_c}{\tilde{I}_c(\Phi)} \\ v &= \frac{V}{R\tilde{I}_c(\Phi)} \end{aligned} \tag{2.5}$$

Note that $\tilde{I}_c\left(\left(n+\frac{1}{2}\right)\Phi_0\right) = 0$ if $n \in \mathbb{Z}$, what means that critical current is zero in the case of purely sinusoidal current-phase $j_S(\varphi) = \sin\varphi$ (without second harmonic). We exclude the trivial case when $\Phi = \left(n+\frac{1}{2}\right)\Phi_0$ from our consideration. Next, we see that natural time scale for this problem is determined by an inverse frequency (ω_c^{-1}), where $\omega_c \overset{\text{def}}{=} \left(2eR\tilde{I}_c(\Phi)\right)/\hbar$. It is convenient to introduce dimensionless time:

$$\tau = \omega_c t \tag{2.6}$$

In these notations the equation (2.4) takes the form:

$$\dot{\varphi}(\tau) = j - j_S(\varphi) \tag{2.7}$$
$$j_S(\varphi) = \sin\varphi + \alpha \sin(2\varphi - 2\pi\Phi/\Phi_0) \tag{2.8}$$
$$\langle v \rangle = \langle \dot{\varphi} \rangle$$

where we also make the shift $\varphi \to \varphi - \pi\Phi/\Phi_0$ in formula (1.2), to relocate the flux dependence into the second-harmonic term. In this notation the average voltage identically coincides with the frequency of Josephson oscillations $\langle v \rangle = \omega$, so we will identify these values.

Solution in simplest case In the case of sine-like current-phase relation $j_S(\varphi) = \sin\varphi$, when $\alpha = 0$, equation (2.7) has an analytical solution:

$$\varphi_{\alpha 0}(\tau) = 2 \arctan\left[\frac{1}{j}\left(1 + \sqrt{j^2 - 1} \tan \frac{\sqrt{j^2-1}(\tau - \tau_0)}{2}\right)\right] \tag{2.9a}$$
$$\langle v \rangle = \operatorname{sgn}(j)\sqrt{j^2 - 1} \tag{2.9b}$$

In case $\alpha \neq 0$, the equation (2.7) allows only numerical solution without additional approximations. Case $j - j_c^{\pm} \ll j_c^{\pm}$ has been considered in [3]: in this case the CVC has a root-like behavior $\langle v \rangle \approx \pm\sqrt{2 j_S''(\varphi_{\pm}) \left|j - j_c^{\pm}\right|}$ regardless of the harmonic composition of the $j_S(\varphi)$.

Structure of the thesis The part II of the thesis is dedicated to calculation of the corrections to the CVC in different parametric regions. For the case $|j| \gg j_c^{\pm}$, (sec. (3)) we developed a perturbation theory which works for arbitrary $j_S(\varphi)$ and respectively α (look sec. (3.2)), and apply it for the case when j_S is defined by (2.8). After that we considered the case of arbitrary j but small $\alpha \ll 1$. We calculate the correction to the CVC up to $O(\alpha^5)$ terms when $\Phi = n\Phi_0$, and find the first asymmetric correction to the CVC in case of non-integer flux. The part III is about the non-stationary case, when the current steps (Shapiro steps) can be observed on the CVC. In this part the asymmetric correction for the width of the first Shapiro step was calculated in first order on the amplitude of irradiation, and in first order on the second harmonic amplitude.

Part II
The Current-voltage characteristic (CVC)

3 High current mode $|j| \gg j_c^{\pm}$

3.1 Failure of the straightforward iterations

First of all we will consider the case $|j| \gg j_c^{\pm}$. A naive way to build perturbation theory is to use the straightforward iterations method; let us suppose the expansion:

$$\varphi(\tau) = \varphi_0(\tau) + \frac{1}{j}\varphi_1(\tau) + \frac{1}{j^2}\varphi_2(\tau) + \cdots \tag{3.10}$$

and after the substitution of this series into eq. (2.7), and following expansion we obtain the equation which defines the corrections:

$$\dot{\varphi}_0 = j \tag{3.11a}$$
$$\frac{1}{j}\dot{\varphi}_1 = -\sin\varphi_0 - \alpha \sin\left(2\varphi_0 - 2\pi\frac{\Phi}{\Phi_0}\right) \tag{3.11b}$$
$$\frac{1}{j^2}\dot{\varphi}_1 = -\frac{1}{j}\varphi_1 (\cos\varphi_0 + 2\alpha\cos(2\varphi_0 - 2\pi\Phi/\Phi_0)) \tag{3.11c}$$
$$\cdots\cdots$$

But if we start calculations we will encounter a problem which is not specific to this case $|j| \gg j_c^{\pm}$, but arises whenever the perturbation leads to the CVC changing, which makes the straightforward iteration method inapplicable.

Inapplicability of straightforward iteration Because of the connection between phase and voltage $\langle v\rangle = \langle\dot{\varphi}\rangle$, it is natural to suppose that the corrections to the unperturbed solution $\langle v_0\rangle$ are defined by the corrections for the phase $\left\langle \dot{\varphi}_j^{(k)} \right\rangle$ (here $\varphi_0(\tau) = j\tau$, as follows from (3.11a), and hence $\langle v_0\rangle = j$). But for the average $\left\langle \dot{\varphi}_j^{(k)} \right\rangle$ to be non-zero, the correction $\varphi_j^{(k)}(\tau)$ must grow infinitely with time, which makes iteration theory inapplicable to the corrections' calculation. We emphasize that this problem is not related to the specifics of the perturbation theory in the case $|j| \gg j_c^{\pm}$, but is common for any case where perturbation leads to the CVC changing.

Example The aforementioned problem arises for the second iteration in the case $|j| \gg j_c^{\pm}$. It arises regardless of the specific form of $j_S(\varphi)$, so for simplicity we take $\alpha = 0$. The solution which one can obtain from the equations (3.11b) and (3.11c) has the following form:

$$\frac{1}{j}\varphi_1 = \frac{1}{j}\cos j\tau \tag{3.12a}$$

$$\frac{1}{j^2}\varphi_2 = -\frac{1}{2j}\left(\tau + \frac{\sin 2j\tau}{2j}\right), \tag{3.12b}$$

where in 3.12b, we can see the term which grows with time. To manage this difficulty we should use a more sophisticated method.

3.2 Harmonic perturbation theory

Main idea We know that the frequency of the Josephson oscillations is proportional (in our notation just equal) to the average voltage. That means that the solution of the (2.7) has a following form:

$$\varphi(\tau) = \omega\tau + \sum_{n=0} A_n \cos n\omega\tau + B_n \sin n\omega\tau \tag{3.13}$$

An important feature of this form is the fact that the frequency of the periodic part of eq. (3.13) equals the coefficient of the linear term, which provides the needed property. Next, we suppose the coefficients A_n and B_n to be monotonously decreasing with the current's growth. This assumption is quite natural because in limit $j \to \infty$, we have only the linear term. Finally we suppose that harmonics come into play one by one, starting from the first. That allows us to calculate only the parameters in (3.13), which are not time-dependent. It can be done by expanding them into series. Actually the order of the leading term of the expansion of the coefficients A_n and B_n depends on the harmonics which constitute $j_S(\varphi)$ and in the general form expansions can be written as:

$$\omega = \omega_0 + \frac{1}{j}\omega_1 + \frac{1}{j^2}\omega_2 + \cdots \tag{3.14a}$$

$$A_n = \frac{1}{j^{n-k}}A_n^{(-k)} + \frac{1}{j^{n-k}}A_n^{(-k+1)} + \cdots + \frac{1}{j^n}A_n^{(0)} + \frac{1}{j^{n+1}}A_n^{(1)} + \cdots \tag{3.14b}$$

$$B_n = \frac{1}{j^{n-k}}B_n^{(-k)} + \frac{1}{j^{n-k}}B_n^{(-k+1)} + \cdots + \frac{1}{j^n}B_n^{(0)} + \frac{1}{j^{n+1}}B_n^{(1)} \cdots \tag{3.14c}$$

Where the parameters $A_n^{(m)}$, $B_n^{(m)}$ and ω_n do not contain smallness with respect to j.

Result: Behavior of the CVC at $|j| \gg |j_c^{\pm}|$ We have calculated these coefficients in the case where $j_S(\varphi)$ is given by (2.8), but this procedure remains the same in the case of any other j_S. In our case calculating up to the fourth order we find:

$$\varphi(\tau) = \omega\tau + f_1(\tau) + f_2(\tau) + f_3(\tau) + O\left(j^{-4}\right) \tag{3.15}$$

$$f_1 = \frac{1}{j}\cos\omega\tau + \frac{\alpha}{2j}\cos(2\omega\tau - 2\pi\phi) \tag{3.16a}$$

$$f_2 = -\frac{\alpha^2}{8j^2}\sin(4\omega\tau - 4\pi\phi) - \frac{5\alpha}{12j^2}\sin(3\omega\tau - 2\pi\phi) - \frac{5\alpha}{4j^2}\sin(\omega\tau - 2\pi\phi) - \frac{1}{4j^2}\sin 2\omega\tau \tag{3.16b}$$

$$\begin{aligned} f_3 = {} & \frac{61\alpha^2 + 12}{48j^3}\cos\omega\tau - \frac{\alpha^3}{24j^3}\cos(6\omega\tau - 6\pi\phi) + \frac{1}{j^3}\left(\frac{\alpha^3}{8} - \frac{2\alpha}{3}\right)\cos(2\omega\tau - 2\pi\phi) \\ & - \frac{19\alpha^2}{32j^3}\cos(3\omega\tau - 4\pi\phi) - \frac{13\alpha}{48j^3}\cos(4\omega\tau - 2\pi\phi) - \frac{97\alpha^2}{480j^3}\cos(5\omega\tau - 4\pi\phi) - \frac{1}{12j^3}\cos 3\omega\tau \end{aligned} \tag{3.16c}$$

Here we collect in f_n terms from (3.13) which have the same order:

$$f_n \stackrel{\text{def}}{=} \frac{1}{j^n} \sum_{m+k=n} A_m^{(k)} \cos m\omega\tau + B_m^{(k)} \sin m\omega\tau \tag{3.17}$$

The result for frequency:

$$\boxed{\omega = \langle v \rangle = j - \frac{\alpha^2+1}{2j} - \frac{3\alpha}{4j^2} \sin\left(2\pi\frac{\Phi}{\Phi_0}\right) - \frac{1+8\alpha^2+\alpha^4}{8j^3} + O\left(j^{-4}\right)} \tag{3.18}$$

Here we can see the flux-dependent term $\frac{3\alpha}{4j^2}\sin\left(2\pi\Phi/\Phi_0\right)$, which vanishes when $\Phi = n\Phi_0$, $\left(n+\frac{1}{2}\right)\Phi_0$, or $\alpha = 0$ and note that this term has an even power of j. That is the manifestation of the asymmetry in the CVC dependence. Also, one can see that this result is consistent with classical formula (2.9b) in case $\alpha = 0$:

$$\langle v \rangle = \operatorname{sgn}(j)\sqrt{j^2-1} \approx j - \frac{1}{2j} - \frac{1}{8j^3} + O\left(j^{-5}\right) \tag{3.19}$$

4 Arbitrary current. Small second harmonic $\alpha \ll 1$

For the intermediate region when $|j| > |j_c^{\pm}|$, but the condition is no longer satisfy, we should use another approximation $\alpha \ll 1$. For this case we will use the following notations for expansion of $\varphi(\tau)$:

$$\varphi(\tau) = \varphi_{\alpha 0}(\tau) + \alpha\varphi_{\alpha 1}(\tau) + \alpha^2\varphi_{\alpha 1}(\tau) + \cdots \tag{4.20}$$

Here, just like in a previous section the iteration method is inapplicable for calculation $\varphi_\alpha^{(k)}(\tau)$. However, there are quite intuitive method, which allow us to determine the functional form of the solution, and than use iterations to find the parameters of this form. But this approach allow us to find functional form in terms of the inverse function $\tau(\varphi)$ what is a disadvantage of this method. We perform calculation only for case of zero flux, when there is not any asymmetry, but still we have a correction due to second harmonic. Case of non-trivial flux will be considered further.

4.1 Direct integration

Main idea First we consider the case $\Phi = 0$, but this method can be generalize for arbitrary flux. Note that eq. (2.7) is an equation with separable variables. It is solution can be written as:

$$\begin{aligned} \tau(\varphi) &= \int^{\varphi} \frac{d\varphi'}{j - \sin\varphi' - \alpha\sin 2\varphi'} = \Bigg/ \begin{array}{ll} x = \tan\frac{\varphi'}{2} & \sin\varphi' = \frac{2x}{x^2+1} \\ d\varphi' = \frac{2dx}{x^2+1} & \end{array} \Bigg/ = \\ &= \frac{2}{j}\int \frac{x^2+1}{\left(x^2+1-2\frac{1}{j}x\right)\left(x^2+1\right) - \frac{\alpha}{j}4x\left(1-x^2\right)} dx \end{aligned} \tag{4.21}$$

Integral here is an integral of rational function with a polynomial of the fourth degree with real coefficient in denominator. Its roots are two pairs of complex numbers which we will denote as a and b. Integral in (4.21) can be taken:

$$\tau(\varphi) = \frac{2}{j}\int \frac{x^2+1}{(x-a)(x-a^*)(x-b)(x-b^*)} = \frac{2}{j}\left(\frac{1}{\operatorname{Im} a}\operatorname{Im}\left[\frac{\left(a^2+1\right)\log(x-a)}{(a-b)(a-b^*)}\right] + a \rightleftarrows b\right) \tag{4.22}$$

where, after expressing in terms of real functions and come back to the original variable φ, we obtain:

$$\begin{aligned} \tau - \tau_0 &= \left\{A_{a,b}\left(n\pi + \arctan\frac{\tan\frac{\varphi}{2} - \operatorname{Re} a}{\operatorname{Im} a}\right) + a \rightleftarrows b\right\} + B\log\frac{|x-a|^2}{|x-b|^2} \\ &\varphi \in (\pi(2n-1)\,;\,\pi(2n+1)] \end{aligned} \tag{4.23}$$

$$A_{a,b} = \frac{2}{j\operatorname{Im} a}\operatorname{Re}\left[\frac{a^2+1}{(a-b)(a-b^*)}\right] \tag{4.24}$$

$$B = \frac{1}{j\operatorname{Im} a}\operatorname{Im}\left[\frac{a^2+1}{(a-b)(a-b^*)}\right] \tag{4.25}$$

Here the additional πn in (4.23) included to choose the branch of arctan and obtain continuous function $\tau(\varphi)$. Now we have similar situation as in case of harmonic perturbation theory in previous sec. (3.2), where we have a knowledge about class of function where solution can be found. Here we obtain this solution by formal integration (4.21). Now we can calculate roots α and b using iterations.

The CVC in case $\alpha \ll 1$, $\Phi = 0$ using direct integration method By solving equation for the roots iteratively:

$$\left(x^2+1-2\frac{1}{j}x\right)\left(x^2+1\right)-\frac{\alpha}{j}4x\left(1-x^2\right)=0 \tag{4.26}$$

we obtain following expressions:

$$a=i+2i\alpha+2\alpha^2\left(i-2j\right)-4\alpha^3 i\left(4j^2-2ij-1\right)+2\alpha^4\left(40j^3-20ij^2-16j+3i\right)+O\left(\alpha^5\right) \tag{4.27a}$$

$$\begin{aligned} b=&\left(\frac{1}{j}+i\sqrt{1-\frac{1}{j^2}}\right)(1-2\alpha)+2\alpha^2 j\frac{2\cdot\operatorname{sgn}(j)\sqrt{j^2-1}-i}{\operatorname{sgn}(j)\sqrt{j^2-1}}+4\alpha^3 j\frac{4ij^2+2\cdot\operatorname{sgn}(j)\sqrt{j^2-1}-3i}{\operatorname{sgn}(j)\sqrt{j^2-1}}+\\ &+2\alpha^4 j\frac{20j^4\left(i-2\cdot\operatorname{sgn}(j)\sqrt{j^2-1}\right)+j^2\left(56\cdot\operatorname{sgn}(j)\sqrt{j^2-1}-33i\right)+4\left(3i-4\cdot\operatorname{sgn}(j)\sqrt{j^2-1}\right)}{\operatorname{sgn}(j)\left(j^2-1\right)^{3/2}}+O\left(\alpha^5\right)\end{aligned} \tag{4.27b}$$

and for coefficient $A_{a,b}$ and B:

$$A_{a,b}=24\alpha^2 j-80\alpha^4 j\left(14j^2-3\right)+O\left(\alpha^6\right) \tag{4.28a}$$

$$A_{b,a}=\frac{2\cdot\operatorname{sgn}(j)}{\sqrt{j^2-1}}\left(1-2\alpha^2\frac{6j^4-9j^2+2}{j^2-1}+2\alpha^4\frac{280j^8-760j^6+675j^4-200j^2+8}{\left(j^2-1\right)^2}+O\left(\alpha^6\right)\right) \tag{4.28b}$$

$$B=-2\alpha\left(1-4\alpha^2\left(10j^2-1\right)+O\left(\alpha^5\right)\right) \tag{4.28c}$$

Note, that, as should be, when $\alpha = 0$ expression (4.28b), (4.27a), (4.27b) and (4.23) give the inverse function of the unperturbed solution (2.9a):

$$\tau=\frac{2\cdot\operatorname{sgn}(j)}{\sqrt{j^2-1}}\arctan\frac{j\tan\frac{\varphi}{2}-1}{\operatorname{sgn}(j)\sqrt{j^2-1}} \tag{4.29}$$

Solution (4.23) allow us to extract the correction to voltage. We can write the average voltage in following form:

$$\langle v\rangle=\omega=\langle\dot\varphi\rangle=\lim_{\tau^*\to\infty}\frac{1}{\tau^*}\int_0^{\tau^*}\dot\varphi\left(\tau\right)d\tau=\lim_{\tau^*\to\infty}\frac{\varphi\left(\tau^*\right)}{\tau^*} \tag{4.30}$$

The procedure of matching of $\arctan\left(\cdots\right)$ in (4.23) which give us an additional πn mentioned above, give as value of $\tau\left(\varphi\right)$, at the matching point: $\tau\left(\pi\left(2n+1\right)\right)=\pi n\left(A_{a,b}+A_{b,a}\right)$. When $n\gg 1$, that equality and formula (4.30), give us the result:

$$\omega=\lim_{n\to\infty}\frac{2\pi n}{\pi n\left(A_{a,b}+A_{b,a}\right)}=\frac{2}{A_{a,b}+A_{b,a}} \tag{4.31}$$

$$\boxed{\begin{aligned}\omega=&\operatorname{sgn}(j)\sqrt{j^2-1}+2\alpha^2\left\{\frac{6j^4-9j^2+2}{\operatorname{sgn}(j)\sqrt{j^2-1}}-6j\left(j^2-1\right)\right\}+\\ &+\frac{2\alpha^4 j}{j^2-1}\left\{\frac{-136j^7+328j^5-249j^3+56j}{\operatorname{sgn}(j)\sqrt{j^2-1}}+136j^6-260j^4+136j^2-12\right\}+O\left(\alpha^5\right)\end{aligned}} \tag{4.32}$$

The high current expansion reproduce the result (3.18) of sec. (3.2):

$$\omega=j-\frac{\alpha^2+1}{2j}-\frac{\alpha^4+8\alpha^2+1}{8j^3}+O\left(j^{-5}\right) \tag{4.33}$$

Note that the result (4.32) is valid not only for $\Phi = 0$, but also for $n\Phi_0$. For the case $\Phi = n\Phi_0$ we have not any asymmetry and formula (4.32) agrees with this statement. Despite the direct integration method can be generalized for the case of non-trivial flux, we decide to use another more convenient method for calculation of asymmetric flux-dependent corrections. Thus, this section is of interest only in a methodological sense, providing us another way to calculate the CVC's corrections.

4.2 Thompson's method

The method which allows us to calculate the corrections in case of non-trivial $\left(\Phi\neq n\Phi_0;\ \left(n+\frac{1}{2}\right)\Phi_0\right)$ was clearly presented by Thompson [4] (other explanation can be found in [2]) for the different problem: finding corrections in case when ordinary Josephson junction irradiated by small electrical field, where in the CVC the current step called Shapiro steps can be observed. We find that this method is applicable also for autonomous problem of calculation the correction due to second harmonics.

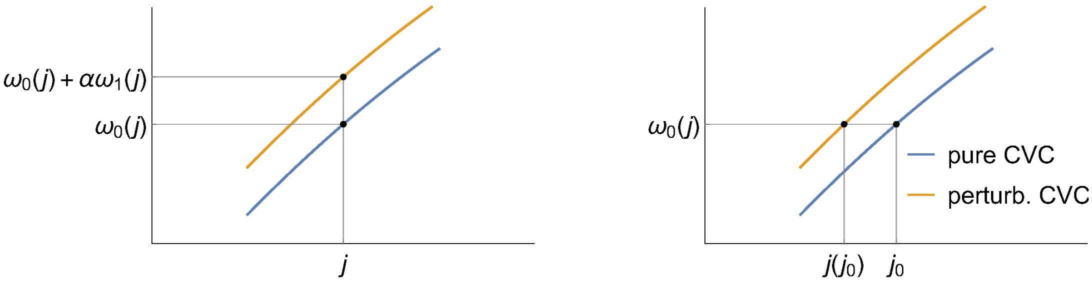
Main idea An attempts to find the correction on voltage as it was mentioned at 3.1 lead to inapplicability of the iteration method. But calculation of corrections to voltage (look fig. 4.1a) is not unique way to find the corrections in to the CVC. Another way is to look for the current corrections, that correspond to the same voltage value as in unperturbed case (look fig. 4.1b). In that picture the CVC is represented by implicit function of the auxiliary parameter which we denote j_0, which has a clear physical meaning: if $\omega\left(j\right)$ – is the CVC of our asymmetric Josephson junction, j_0 is a current that would correspond to the voltage ω in unperturbed symmetric case. Between j and j_0 there is a functional dependence, which we can calculate using an expansion $j\left(j_0\right)$ into power series with respect to α:

$$\begin{cases} \omega\left(j_0\right) = \langle\dot{\varphi}_{\alpha 0}\rangle + \cancelto{0}{\alpha\langle\dot{\varphi}_{\alpha 1}\rangle} + \cancelto{0}{\alpha^2\langle\dot{\varphi}_{\alpha 2}\rangle} + \cancelto{0}{\cdots} \\ j\left(j_0\right) = j_0 + \alpha j_1\left(j_0\right) + \alpha^2 j_2\left(j_0\right) + \cdots \end{cases} \tag{4.34}$$

where the corrections $j_k\left(j_0\right)$ is chosen in such a way to compensate the correction $\omega_k = \langle\dot{\varphi}_{\alpha k}\rangle$ to the ω ($\varphi_{\alpha k}\left(\tau\right)$ was defined in (4.20)), so $\omega\left(j_0\right)$ has the following form:

$$\omega\left(j_0\right) = \omega_0\left(j_0\right) = \operatorname{sgn}\left(j_0\right)\sqrt{j_0^2 - 1} \tag{4.35}$$

In other words, j_k compensate the growth terms in $\varphi_{\alpha k}\left(\tau\right)$. This compensation allow us to use iterations to find corrections $\varphi_{\alpha k}$ This procedure in details described at the next section.



(a) Direct determination of the perturbed CVC (b) The implicit determination of the perturbed CVC

Fig. 4.1

The CVC calculation at $\alpha \ll 1$ using Thompson's method After substitution the expansion for j (4.34) into (2.7) together with the expansion for φ:

$$\varphi\left(\tau\right) = \varphi_{\alpha 0}\left(\tau\right) + \alpha\varphi_{\alpha 1}\left(\tau\right) + \alpha^2\varphi_{\alpha 2}\left(\tau\right) + \cdots \tag{4.36}$$

we obtain an ordinary equation for Josephson junction with sine-like $j_S\left(\varphi\right)$ for zero order, when $\alpha = 0$ (as it should be):

$$\dot{\varphi}_{\alpha 0} + \sin\varphi_{\alpha 0} = j_0 \tag{4.37}$$

and a linear equation for $\varphi_{\alpha 1}$ at the first order:

$$\dot{\varphi}_{\alpha 1} + \varphi_{\alpha 1}\cos\varphi_{\alpha 0} = j_1 - \sin\left(2\varphi_{\alpha 0} - 2\pi\frac{\Phi}{\Phi_0}\right) \tag{4.38}$$

where solution of (4.37) $\varphi_{\alpha 0}$ has been already written in (2.9a). This is linear equation which has simple solution:

$$\varphi_{\alpha 1} = \frac{1}{K_0\left(\tau\right)}\int^{\tau} K_0\left(\tau'\right)\left\{j_1\left(j_0\right) - \sin\left(2\varphi_{\alpha 0} - 2\pi\frac{\Phi}{\Phi_0}\right)\right\} d\tau' \tag{4.39}$$

where integrating factor $K_0\left(\tau\right)$ is a solution of the autonomous equation:

$$K_0\left(\tau\right) = \exp\left\{-\int^{\tau}\cos\varphi_{\alpha 0}\, d\tau'\right\} \tag{4.40}$$

In order to calculate this integral we do not need to substitute φ_0 inside cosine function in (4.40), it is enough do differentiate 4.37 to find:

$$\ddot{\varphi}_{\alpha 0} = -\dot{\varphi}_{\alpha 0}\cdot\cos\varphi_{\alpha 0} \tag{4.41}$$

from where we get after integration:

$$K_0(\tau) = \exp\left\{\int^{\tau} \frac{\ddot{\varphi}_{\alpha 0}}{\dot{\varphi}_{\alpha 0}} d\tau'\right\} = \frac{C}{\dot{\varphi}_{\alpha 0}} \tag{4.42}$$

Now, we should calculate $j_1(j_0)$. Condition which determine $j_1(j_0)$ is vanishing the corresponding correction $\omega_1 = 0 \Leftrightarrow \langle\dot{\varphi}_{\alpha 1}\rangle = 0$. This condition is met when the correction $\varphi_{\alpha 1}$ does not contain growing with time terms. In other word we should demand the time-average of integrand in (4.39) to be zero:

$$\left\langle \frac{j_1(j_0) - \sin(2\varphi_{\alpha 0} - 2\pi\Phi/\Phi_0)}{\dot{\varphi}_{\alpha 0}} \right\rangle = 0 \tag{4.43}$$

After necessary calculation we see that average $\left\langle \frac{\sin 2\varphi_{\alpha 0}}{\dot{\varphi}_{\alpha 0}} \right\rangle = 0$, as should be, because first asymmetric correction should vanished when flux $\Phi = n\Phi_0$, $\left(n + \frac{1}{2}\right)\Phi_0$, and the flux-dependent part of the correction must contain $\# \sin 2\pi\Phi/\Phi_0$. For $j_1(j_0)$,we obtain a result:

$$j_1(j_0) = -\sin\left(2\pi\frac{\Phi}{\Phi_0}\right) \frac{\left\langle \dot{\varphi}_{\alpha 0}^{-1} \cos 2\varphi_0 \right\rangle}{\left\langle \dot{\varphi}_{\alpha 0}^{-1} \right\rangle} = -\sin\left(2\pi\frac{\Phi}{\Phi_0}\right) \frac{2j_0^3 - 3j_0 - 2\cdot \operatorname{sgn}(j_0)\left(j_0^2 - 1\right)^{3/2}}{j_0} \tag{4.44}$$

For calculation correction to the voltage, we should keep in mind that in this picture the CVC has a form of implicit function (4.34):

$$\begin{cases} \omega(j_0) = \operatorname{sgn}(j_0)\sqrt{j_0^2 - 1} \\ j(j_0) = j_0 - \alpha \sin\left(2\pi\frac{\Phi}{\Phi_0}\right) \frac{2j_0^3 - 3j_0 - 2\cdot\operatorname{sgn}(j_0)\left(j_0^2-1\right)^{3/2}}{j_0} \end{cases} \tag{4.45}$$

At the first order with respect to α we will have:

$$j_0(j) = j + \alpha \sin\left(2\pi\frac{\Phi}{\Phi_0}\right) \frac{2j^3 - 3j - 2\cdot\operatorname{sgn}(j)\left(j^2 - 1\right)^{3/2}}{j} \tag{4.46}$$

and after substitution it into $\omega(j_0)$ in (4.45) we finally get the result:

$$\boxed{\omega(j) = \operatorname{sgn}(j)\left(\sqrt{j^2 - 1} + \alpha \sin\left(2\pi\frac{\Phi}{\Phi_0}\right) \frac{2j^3 - 3j - 2\cdot\operatorname{sgn}(j)\left(j^2-1\right)^{3/2}}{\sqrt{j^2 - 1}}\right) + O\left(\alpha^2\right)} \tag{4.47}$$

At the limit $j \gg 1$, we obtain the result which coincide with the result (3.18) of harmonic perturbation theory in sec. (3.2).

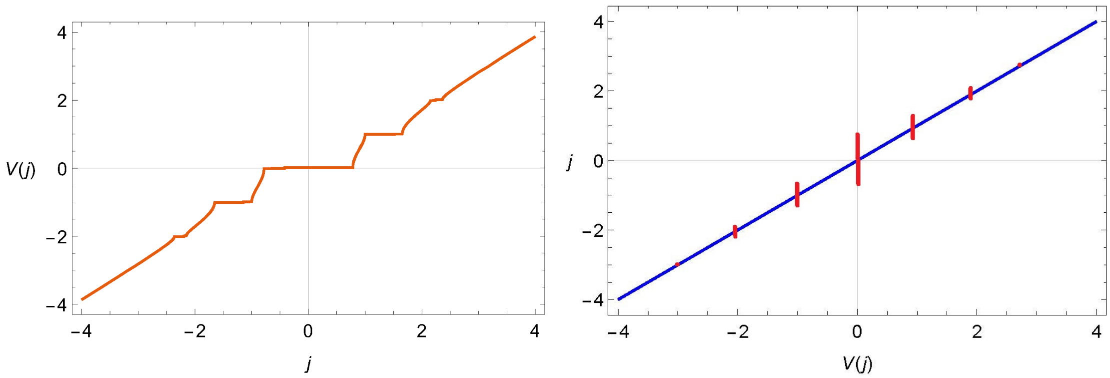
Part III
Shapiro steps

In that part we interest in asymmetric consequences on the Shapiro steps. Shapiro steps is the phenomena of arising constant current steps on the CVC in case when junction is irradiated by electric field. This effect can be easily explained if we suppose that voltage on the junction is installed, and modulated by electrical field as time-periodic function. In this problem statement the voltage-time dependence determine $\varphi(\tau)$, and allow us to find the current. This simple analysis had been done before [3]. It was demonstrated, that in resonant cases, when the frequency of the incident irradiation is equal to the frequency of Josephson oscillations or divides it completely at the CVC voltage spikes are arised (so called Shapiro spikes, look fig. 4.2b).

More physical problem statement, is a case when the incident irradiation defined the current-time dependence. In that case we also can observe step-like features on the CVC (so-called Shapiro steps) and these features also arise when the frequency of the incident irradiation is equal to the frequency of Josephson oscillations or divides it completely. In order to calculate voltage we should to solve a non-autonomous differential equation (2.7) with $j = j(\tau) = j_{dc} + j_{ac}\sin(\Omega\tau + \delta)$:

$$\dot{\varphi} + j_S(\varphi) = j_{dc} + j_{ac}\sin(\Omega\tau + \delta) \tag{4.48}$$

This equation cannot be solved analytically already in the sinusoidal case $j_S(\varphi) = \sin\varphi$. The numerical solution shows arising of the current steps (look fig. 4.2a). Approximate method stated in 4.2, was used



(a) (b)

Fig. 4.2: Shapiro steps a); and Shapiro spikes b)

by Thompson in [4] to find the correction to the φ and then to the CVC; in particular he calculate the width of the steps under assumption of smallness irradiation's amplitude $j_{ac} \ll j_{dc}$. Here we generalize this method for the case of arbitrary $j_S(\varphi)$ in first order with respect to j_{ac}, and express the width of the step in terms of autonomous solution φ_0 eq. (4.48) with $j_{ac} = 0$. For the our minimal model case (when j_S defined in (2.8)) the equation (4.48) with $j_{ac} = 0$ is equivalent to case which we had considered in part II, so further we will partially use and partially extend the result of this part.

5 Shapiro-steps problem in first order with respect to j_{ac}

In this section we will develop perturbation theory using $j_{ac} \ll 1$ as a small parameter. Despite the fact that eq. (4.48) is totally different than (2.7), Thompson's method turns out to be quite convenient. Also shape of the current step's on the fig. 4.2a tell us that the representation of CVC as an implicit function (schematically shown at the fig. 4.1b) seem to be natural for this problem.

5.1 Expression in terms of autonomous problem

According to the Thompson's method we should perform the expansion of φ in (4.48):

$$\varphi(\tau) = \varphi_0(\tau) + j_{ac}\varphi_1(\tau) + j_{ac}^2\varphi_1(\tau) + \cdots \tag{5.49}$$

and consider the correction to the current with respect to j_{ac}. The CVC dependence now (as in 4.2) is represented as an implicit function:

$$\begin{cases} \omega(j_0) = \langle\dot{\varphi}_0\rangle + \cancelto{0}{j_{ac}\langle\dot{\varphi}_1\rangle} + \cancelto{0}{j_{ac}^2\langle\dot{\varphi}_2\rangle} + \cancelto{0}{\cdots} \\ j_{dc}(j_0) = j_0 + j_{ac}j_1(j_0) + j_{ac}^2 j_2(j_0) + \cdots \end{cases} \tag{5.50}$$

The zero-order solution here φ_0 satisfies unperturbed equation:

$$\dot{\varphi}_0 = j - j_S(\varphi_0) \tag{5.51}$$

For the first order correction $\varphi_1(\tau)$ we will obtain linear equation again:

$$\dot{\varphi}_1 + \varphi_1 \cos\varphi_0 = j_1 + \sin(\Omega\tau + \delta) \tag{5.52}$$

with the solution:

$$\varphi_1(\tau) = \frac{1}{K(\tau)} \int^{\tau} K(\tau')\left(j_1 + \sin(\Omega\tau' + \delta)\right) d\tau' \tag{5.53}$$

Now, $K(\tau)$ is defined by whole current-phase relation $j_S(\varphi)$:

$$K(\tau) = \exp\left\{ -\int^{\tau} \left.\frac{\partial j_S(\varphi)}{\partial\varphi}\right|_{\varphi=\varphi_0} d\tau' \right\} \tag{5.54}$$

using the same trick as in 4.2 with differentiation of the unperturbed equation (5.51) (at that time) we can calculate $K(\tau)$ for the arbitrary $j_S(\varphi)$. Differentiation of (5.51) give us:

$$\ddot{\varphi}_0 = -\dot{\varphi}_0 \left.\frac{\partial j_S(\varphi)}{\partial\varphi}\right|_{\varphi=\varphi_0} \tag{5.55}$$

and after substitution to the (5.54), and performing integration we can get the result $K(\tau) = C\dot{\varphi}_0^{-1}$. The correction j_1 should be chosen to compensate the growing terms in (5.53). This condition can be written as equality to zero of integrand's average in (5.53):

$$\left\langle \frac{j_1 + \sin(\Omega\tau + \delta)}{\dot{\varphi}_0} \right\rangle = 0 \tag{5.56}$$

here, δ – is the time shift between phases of $\varphi(\tau)$ oscillations and incident irradiation $j_{ac}\sin(\Omega\tau+\delta)$. If the average $\left\langle \sin(\Omega\tau+\delta)\,\dot{\varphi}_0^{-1}\right\rangle \neq 0$, then obtain the δ-dependent expression for j_1:

$$j_1 = -\frac{\left\langle \sin(\Omega\tau+\delta)\,\dot{\varphi}_0^{-1}\right\rangle}{\left\langle \dot{\varphi}_0^{-1}\right\rangle} \tag{5.57}$$

Because δ is immeasurable quantity it can take arbitrary value. Thus, δ parametrizes the Shapiro step, and width of this step determines by the difference between max and min value of (5.57) with respect to δ. After expansion $\sin(\Omega\tau+\delta)$ in (5.57), and rewriting we obtain the expression:

$$j_1 = \sqrt{\left(\frac{\left\langle \dot{\varphi}_0^{-1}\sin\Omega\tau\right\rangle}{\left\langle \dot{\varphi}_0^{-1}\right\rangle}\right)^2 + \left(\frac{\left\langle \dot{\varphi}_0^{-1}\cos\Omega\tau\right\rangle}{\left\langle \dot{\varphi}_0^{-1}\right\rangle}\right)^2}\,\sin\left(\delta + \arctan\frac{\left\langle \dot{\varphi}_0^{-1}\sin\Omega\tau\right\rangle}{\left\langle \dot{\varphi}_0^{-1}\cos\Omega\tau\right\rangle}\right) \tag{5.58}$$

from which one can clearly seen that first correction of the width (denoted j_1^W) defines by:

$$j_1^W = 2\sqrt{\left(\frac{\left\langle \dot{\varphi}_0^{-1}\sin\Omega\tau\right\rangle}{\left\langle \dot{\varphi}_0^{-1}\right\rangle}\right)^2 + \left(\frac{\left\langle \dot{\varphi}_0^{-1}\cos\Omega\tau\right\rangle}{\left\langle \dot{\varphi}_0^{-1}\right\rangle}\right)^2} \tag{5.59}$$

Formula (5.59) is a result of this section – width of the Shapiro step for the Josephson junction with arbitrary current-phase relation $j_S(\varphi)$ in terms of unperturbed solution $\varphi_0(\tau)$ (defined by eq (5.51), so the information about $j_S(\varphi)$ is hidden in $\varphi_0(\tau)$ in (5.59)) in the first order with respect to j_{ac}. Values of the $\omega(j_0)$ where expression (5.59) is not zero, defines the positions of the current steps. Direct calculation shows that in the case $j_S(\varphi) = \sin\varphi$, average $\left\langle \dot{\varphi}_0^{-1}\sin\Omega\tau\right\rangle \neq 0$ only when $\omega(j_0) = \Omega$. Average $\left\langle \dot{\varphi}_0^{-1}\cos\Omega\tau\right\rangle = 0$ for arbitrary $\omega(j_0)$. That correspond to arising of the first step [4]. Others Shapiro steps arises at the $\omega = n\Omega$ but corresponding non-zero terms arise only it the further orders of j_{ac}. At following section we will find the corrections for width of this step due to second harmonic using as a parameter small $\alpha \ll 1$ in $j_S(\varphi)$ defined at (2.8).

5.2 Calculation of φ_0 and Shapiro step correction in first order with respect to α

Now we will use the same approach as in 4.2 to find the corrections for φ_0. In order to find correction for the step width due to asymmetry, we should calculate corrections to the φ_0 using α as a small parameter:

$$\varphi_0(\tau) \approx \varphi_{\alpha 0} + \alpha\varphi_{\alpha 1} + O\left(\alpha^2\right) \tag{5.60}$$

where $\varphi_{\alpha 0}$ – solution (2.9a), of the (2.7) with respect to $\alpha = 0$.

Zero order result If we limit ourselves by zero order, as it was mentioned at the end of the previous section, after direct calculation of the averages we will obtain following results:

$$\left\langle \dot{\varphi}_{\alpha 0}^{-1}\right\rangle = \frac{j_0}{j_0^2 - 1} \tag{5.61a}$$

$$\left\langle \dot{\varphi}_{\alpha 0}^{-1}\cdot\sin\omega\tau\right\rangle = \frac{1}{2\left(j_0^2-1\right)} \tag{5.61b}$$

$$\left\langle \dot{\varphi}_{\alpha 0}^{-1}\cdot\cos\omega\tau\right\rangle = 0 \tag{5.61c}$$

which lead to the Thompson's result [4] for the symmetric Josephson junction:

$$j_1^W = 2\left|\frac{\left\langle \dot{\varphi}_{\alpha 0}^{-1}\cdot\sin\omega\tau\right\rangle}{\left\langle \dot{\varphi}_{\alpha 0}^{-1}\right\rangle}\right| = \frac{1}{|j_0|} \tag{5.62}$$

First order result At zero order only first term under root in (5.59) give a contribution. Term $\left\langle\dot{\varphi}_{\alpha 0}^{-1}\cos\Omega\tau\right\rangle$ in (5.59) has $\sim O(\alpha)$ smallness with respect to α, which one can conclude form the result (5.61c). That is why only $\left\langle\dot{\varphi}_{\alpha 0}^{-1}\sin\Omega\tau\right\rangle$ term can produce the contributiont to the first order. The substitution (5.60) into (5.59) give the following result:

$$\begin{aligned} j_1^W &= 2\left|\frac{\left\langle\left(\varphi_{\alpha 0}+\alpha\varphi_{\alpha 1}\right)^{-1}\sin\omega\tau\right\rangle}{\left\langle\left(\varphi_{\alpha 0}+\alpha\varphi_{\alpha 1}\right)^{-1}\right\rangle}\right| = \\ &= 2\left|\frac{\left\langle\dot{\varphi}_{\alpha 0}^{-1}\cdot\sin\omega\tau\right\rangle}{\left\langle\dot{\varphi}_{\alpha 0}^{-1}\right\rangle}+\alpha\left(\left\langle\dot{\varphi}_{\alpha 0}^{-1}\cdot\sin\omega\tau\right\rangle\frac{\left\langle\dot{\varphi}_{\alpha 1}\dot{\varphi}_{\alpha 0}^{-2}\right\rangle}{\left\langle\dot{\varphi}_{\alpha 0}^{-1}\right\rangle^2}-\frac{\left\langle\dot{\varphi}_{\alpha 1}\dot{\varphi}_{\alpha 0}^{-2}\cdot\sin\omega\tau\right\rangle}{\left\langle\dot{\varphi}_{\alpha 0}^{-1}\right\rangle}\right)\right| \end{aligned} \tag{5.63}$$

Now, we turn to the calculation of the first correction $\varphi_{\alpha 1}$. Using formula (4.39) and result (4.44) we can rewrite $\varphi_{\alpha 1}$:

$$\varphi_{\alpha 1}=\dot{\varphi}_{\alpha 0}\sin\left(2\pi\frac{\Phi}{\Phi_0}\right)\left\{\int^{\tau}\frac{\cos 2\varphi_{\alpha 0}}{\dot{\varphi}_{\alpha 0}}d\tau'-\left\langle\frac{\cos 2\varphi_{\alpha 0}}{\dot{\varphi}_{\alpha 0}}\right\rangle\frac{\int^{\tau}\dot{\varphi}_{\alpha 0}^{-1}d\tau'}{\left\langle\dot{\varphi}_{\alpha 0}^{-1}\right\rangle}\right\}-\dot{\varphi}_{\alpha 0}\cos\left(2\pi\frac{\Phi}{\Phi_0}\right)\int^{\tau}\frac{\sin 2\varphi_{\alpha 0}}{\dot{\varphi}_{\alpha 0}}d\tau' \tag{5.64}$$

Note that this formula give a clear illustration how the current correction compensate the growing terms in $\varphi_{\alpha 1}$: from this form its clearly seen that averaging the expression in curly bracket will give zero. Explicit form of the $\varphi_{\alpha 1}$ is given by following expressions:

$$\int^{\tau}\frac{\cos 2\varphi_{\alpha 0}}{\dot{\varphi}_{\alpha 0}}d\tau'-\left\langle\frac{\cos 2\varphi_{\alpha 0}}{\dot{\varphi}_{\alpha 0}}\right\rangle\frac{\int^{\tau}\dot{\varphi}_{\alpha 0}^{-1}d\tau'}{\left\langle\dot{\varphi}_{\alpha 0}^{-1}\right\rangle}=2\left(\omega\tau-2\pi n\right)-4\arctan\frac{j_0\tan\left(\omega\tau/2\right)+1}{\omega}+2\left(\frac{2}{\omega}-\frac{1}{j_0}\right)\cos\omega\tau \tag{5.65a}$$

$$\omega\tau\in\left(\pi\left(2n-1\right);\pi\left(2n+1\right)\right]$$

$$\int^{\tau}\frac{\sin 2\varphi_{\alpha 0}}{\dot{\varphi}_{\alpha 0}}d\tau'=2\left\{\log\left(j_0+\sin\omega\tau\right)+\frac{j_0}{\omega^2}\sin\omega\tau\right\} \tag{5.65b}$$

$$\dot{\varphi}_{\alpha 0}=\frac{\omega^2}{j_0+\sin\omega\tau} \tag{5.65c}$$

where after derivation of (2.9a) we take for simplicity $\tau_0=-\left(j^2-1\right)^{-1/2}\arcsin\left(j^{-1}\right)$ and the addition term $2\pi n$ in (5.65a) has the same origin as the addition term in (4.23) and represents growing part of the (5.64), whereas $\omega\tau$ term provide the needed compensation of this growing part. Next, straightforward calculation shows that the averages in (5.63) which include terms in (5.64) which contain $\cos 2\pi\Phi/\Phi_0$ vanishes:

$$\left\langle\dot{\varphi}_{\alpha 0}^{-2}\frac{\partial}{\partial\tau}\left(\dot{\varphi}_{\alpha 0}\int^{\tau}\frac{\sin 2\varphi_{\alpha 0}}{\dot{\varphi}_{\alpha 0}}d\tau'\right)\right\rangle=0 \tag{5.66a}$$

$$\left\langle\sin\omega\tau\cdot\dot{\varphi}_{\alpha 0}^{-2}\frac{\partial}{\partial\tau}\left(\dot{\varphi}_{\alpha 0}\int^{\tau}\frac{\sin 2\varphi_{\alpha 0}}{\dot{\varphi}_{\alpha 0}}d\tau'\right)\right\rangle=0 \tag{5.66b}$$

what means that in $\varphi_{\alpha 1}$ only first term with $\sin 2\pi\Phi/\Phi_0$ will give a contribution into the width correction. That fact agrees with the general statement of absence any asymmetry in case $\Phi=n\Phi_0$. That's mean than the averages which we need to calculate has a form:

$$\begin{aligned} \left\langle\dot{\varphi}_{\alpha 1}\dot{\varphi}_{\alpha 0}^{-2}\right\rangle=\sin\left(2\pi\frac{\Phi}{\Phi_0}\right)\Bigg(\left\langle\dot{\varphi}_{\alpha 0}^{-2}\frac{\partial}{\partial\tau}\left\{\dot{\varphi}_{\alpha 0}\int^{\tau}\frac{\cos 2\varphi_{\alpha 0}}{\dot{\varphi}_{\alpha 0}}d\tau'\right\}\right\rangle- \\ -\frac{\left\langle\dot{\varphi}_{\alpha 0}^{-1}\cdot\cos 2\dot{\varphi}_{\alpha 0}\right\rangle}{\left\langle\dot{\varphi}_{\alpha 0}^{-1}\right\rangle}\left\langle\dot{\varphi}_{\alpha 0}^{-2}\frac{\partial}{\partial\tau}\left\{\dot{\varphi}_{\alpha 0}\int^{\tau}\dot{\varphi}_{\alpha 0}^{-1}d\tau'\right\}\right\rangle\Bigg) \end{aligned} \tag{5.67}$$

and similarly for $\left\langle\dot{\varphi}_{\alpha 1}\dot{\varphi}_{\alpha 0}^{-2}\cdot\sin\omega\tau\right\rangle$. After some algebra we will obtain the following result:

$$\left\langle\dot{\varphi}_{\alpha 1}\dot{\varphi}_{\alpha 0}^{-2}\right\rangle=\sin\left(2\pi\frac{\Phi}{\Phi_0}\right)\left(\frac{\operatorname{sgn}\left(j_0\right)}{j_0}\frac{4j_0^2+2}{\sqrt{j_0^2-1}}-\frac{4j_0^2}{j_0^2-1}\right) \tag{5.68a}$$

$$\left\langle\dot{\varphi}_{\alpha 1}\dot{\varphi}_{\alpha 0}^{-2}\cdot\sin\omega\tau\right\rangle=\sin\left(2\pi\frac{\Phi}{\Phi_0}\right)\left(\operatorname{sgn}\left(j_0\right)\frac{5-2j_0^2}{2\sqrt{j_0^2-1}}+j_0\frac{j_0^2-3}{j_0^2-1}\right) \tag{5.68b}$$

and after substitution (5.68a) and (5.68b) into (5.63) we obtain the following result:

$$j_1^W = \left| \frac{1}{j_0} + \alpha \sin\left(2\pi\frac{\Phi}{\Phi_0}\right)\left(\frac{\operatorname{sgn}(j_0)}{j_0^3}\frac{2j_0^6 - 3j_0^4 + 3j_0^2 - 2}{\sqrt{j_0^2-1}} - 2\left(j_0^2-1\right)\right)\right| \tag{5.69}$$

Formula (5.69) (after returning to the initial correction $j_1(j_0, \delta)$) by the combining with the result of 4.2 (formula (4.47)) give us the whole picture of the first Shapiro step (CVC in case when $\omega = \pm\Omega$):

$$\begin{cases} \pm\Omega = \operatorname{sgn}(j_0)\left(\sqrt{j_0^2-1} + \alpha\sin\left(2\pi\frac{\Phi}{\Phi_0}\right)\frac{2j_0^3 - 3j_0 - 2\cdot\operatorname{sgn}(j_0)\left(j_0^2-1\right)^{3/2}}{\sqrt{j_0^2-1}}\right) \\ j\left(j_0\right) = j_0 + \frac{j_{ac}}{2}\left(\frac{1}{j_0} + \alpha\sin\left(2\pi\frac{\Phi}{\Phi_0}\right)\left(\frac{\operatorname{sgn}(j_0)}{j_0^3}\frac{2j_0^6-3j_0^4+3j_0^2-2}{\sqrt{j_0^2-1}} - 2\left(j_0^2-1\right)\right)\right)\cos\delta \end{cases} \tag{5.70}$$

from where, after exception j_0 in the first order with respect to α, we obtain the result:

$$\boxed{j\left(\pm\Omega\right) = j_{CVC}\left(\pm\Omega\right) + \frac{j_{ac}\cos\delta}{2} j_{step}^W\left(\pm\Omega\right) + O\left(j_{ac}^2\right)} \tag{5.71}$$

This formula defines the whole position (the center and the edges) of the first Shapiro step in the region of positive and negative current (which correspond $\omega = +\Omega$ and $\omega = -\Omega$ respectively) where $j_{CVC}(\Omega)$ – term which define the center of first the Shapiro step, taking into account asymmetric correction:

$$j_{CVC}\left(\pm\Omega\right) = \pm\sqrt{\Omega^2+1} + \alpha\sin\left(2\pi\frac{\Phi}{\Phi_0}\right)\left(1 - 2\Omega^2 + \frac{2\Omega^3}{\sqrt{\Omega^2+1}}\right) + O\left(\alpha^2\right) \tag{5.72}$$

and $j_{step}^W(\Omega)$ is the actual width of the first Shapiro step in the units j_{ac}:

$$j_{step}^W\left(\pm\Omega\right) = \left|\pm\frac{1}{\sqrt{\Omega^2+1}} + \frac{\alpha\sin 2\pi\Phi/\Phi_0}{\left(\Omega^2+1\right)^{3/2}}\left(2\Omega^5 + \Omega^3 + 3\Omega - \sqrt{\Omega^2+1}\left(2\Omega^4+1\right)\right)\right| + O\left(\alpha^2\right) \tag{5.73}$$

Part IV
Overview of results

1. We have developed several perturbation methods:
   (a) Perturbation theory for arbitrary current-phase relation $j_S(\varphi)$ in case when current much higher than critical $|j| \gg |j_c^{\pm}|$ (Harmonic perturbation theory)
   (b) Perturbation theory for sine-like current-phase relation in presence of small ($\alpha \ll 1$) second harmonic ($j_S(\varphi)$ defined in (2.8)) was developed by using direct integration
   (c) Perturbation method stated by Thompson in [4] was generalized for the case of small ($\alpha \ll 1$) second harmonic.
2. Using these methods we have calculated correction to the CVC of asymmetric Josephson junction if there is a small parameter (α or j^{-1}). These calculations have shown the evidence of asymmetric influence on the CVC by the second harmonic shifted with magnetic flux. The results are collected in Table. 1
3. Using method of direct integration 1b we have found the inverse phase-time dependence $t(\varphi)$ in presence of second harmonic (formula (4.23))
4. Thompson's method for calculating Shapiro steps has been generalized for the case of arbitrary $j_S(\varphi)$.
5. First asymmetric correction for the first Shapiro step width, was calculated in first order with respect to incident irradiation amplitude j_{ac} using Thompson's method (formula (5.71)) for the case of small ($\alpha \ll 1$) second harmonic ($j_S(\varphi)$ defined in (2.8)).

Table 1: Result of the part II

Parametric region	Expression for the average voltage
$\lvert j\rvert \gg j_c^{\pm}$ α, $\Phi -$ arbitrary	$j - \frac{\alpha^2+1}{2j} - \frac{3\alpha}{4j^2}\sin\left(2\pi\frac{\Phi}{\Phi_0}\right) - \frac{1+8\alpha^2+\alpha^4}{8j^3} + O\left(j^{-4}\right)$
$\alpha \ll 1$ $\lvert j\rvert > 1$ $\Phi -$ arbitrary	$\operatorname{sgn}(j)\left(\sqrt{j^2-1} + \alpha\sin\left(2\pi\frac{\Phi}{\Phi_0}\right)\frac{2j^3-3j-2\cdot\operatorname{sgn}(j)\left(j^2-1\right)^{3/2}}{\sqrt{j^2-1}}\right) + O\left(\alpha^2\right)$
$\alpha \ll 1$ $\lvert j\rvert > 1$ $\Phi = n\Phi_0$ where $n \in \mathbb{Z}$	$\operatorname{sgn}(j)\sqrt{j^2-1} + 2\alpha^2\left\{\frac{6j^4-9j^2+2}{\operatorname{sgn}(j)\sqrt{j^2-1}} - 6j\left(j^2-1\right)\right\} + + \frac{2\alpha^4 j}{j^2-1}\left\{\frac{-136j^7+328j^5-249j^3+56j}{\operatorname{sgn}(j)\sqrt{j^2-1}} + +136j^6-260j^4+136j^2-12\right\} + O\left(\alpha^5\right)$

6 Bibliography

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