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Partially randomized benchmarking of quantum operations

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МАГИСТЕРСКАЯ ДИССЕРТАЦИЯ

Неполное случайное тестирование квантовых операций

Магистерская образовательная программа: Математическая и Теоретическая Физика

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Partially randomized benchmarking of quantum operations

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Abstract

Randomized benchmarking is a widely used technique for estimating average fidelity of a set of quantum gates. To get a well-defined dependence of the fidelity on the number of gates in the set it is needed to average the set over the whole unitary, or Clifford group. Here we analyze a method of averaging fidelity over a subgroup of a two qubit unitary (Clifford) group specifically over tensor product of one qubit groups.

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Chapter 1

Introduction

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Quantum computing has been developing rapidly over the past few decades. Although, only few cases of an engineered quantum computer over-performing traditional ones (1) have been reported, in the future quantum computers may be able to solve problems that are beyond the reach of even the most powerful supercomputers. For example, Shor's (2) algorithm for integer factorization can be executed in polynomial time, in contrast to exponential time for a classical computer or Grover's (3) search algorithm takes $O(\sqrt{N})$ steps, instead of $O(N)$ steps for classical analog. Also nowadays, the size of transistors in a classical computer has reached its size limit at which quantum effects, such as electron tunneling, have to be taken into account.

A quantum computer is a system consisting of a large number of qubits, the elementary components of a quantum computer. A qubit is a quantum two-level system which maybe realized in various physical systems. Among the most promising are superconducting charge qubits, such as Xmon (4) or transmon (5). There are also qubits based on quantum dots (6), trapped ions (7) which have their advantages. The algorithm (8) on a quantum computer consists of preparing qubits in certain initial quantum states, then performing logical operations (gates) and finally measuring the final state of the qubits. In most algorithms only single and two-qubit gates are used, since they form the universal set (9).

In order to actually perform algorithms more efficiently than a classical computer is able now, one needs to create a system of thousands of logical qubits that keep coherence for sufficient period of time. There are considerable number of obstacles on the way to implement such a system. One of the problems is the errors occurring during algorithms realization. The errors come from the noise in the qubits, inaccuracies in the execution of operations. Therefore, one of the major tasks is to get rid of these errors or, at least, reduce them to such a level at which they will be insignificant. This level is dictated by fault tolerance theorem (10). That is why it is quite important to measure these errors to further understand their physical nature and diminish them.

Significant part of the total error comes from the quantum gates implementation. One of the methods of measuring quality of a gate is its complete tomography (11). In ideal case, a quantum gate is an unitary operator acting on the qubits' wave function. But due to the presence of the error, this operation maybe not quite unitary, thus its action should be described in terms of the density matrix. Formally, a real gate is a superoperator acting on the density matrix of the system. The tomography process studies the exact structure of such a superoperator. It is usually done in the following way: researchers prepare a qubit in some concrete initial state, then the studied gate is applied and they read out the result. These steps are done several times to get the sufficient statistics. Finally, the researches obtain a list of numbers which characterizes the superoperator of the studied gate. The list consists of 9 numbers for single-qubit gates and 225 for two-qubit gate. Single qubit DM has three independent parameters, thus a superoperator on it has $3^2 = 9$ parameters, for two-qubit DM one has 15 parameters, which implies $15^2 = 225$ independent parameters for a superoperator on it. There are a few disadvantages of such method. Firstly, it is difficult to measure this huge number of the parameters and further manage them. Secondly, according to the current state of gates implementation, the errors are around $10^{-1} - 10^{-2}$ for two-qubit gates and even smaller $10^{-3} - 10^{-4}$ for single-qubit gates, this fact complicates their measurement. Thirdly, there are also state preparation and measurement errors, so called SPAM errors, which somehow affect the results of the tomography. Finally, the tomography does not analyze the behaviour of the gate in the algorithm. Due to all these reasons, nowadays the experimenters often use different approach - randomized benchmarking, which tackles most of the issues.

Randomized benchmarking (RB) was initially introduced by E. Knill et al. (12) in 2007. It is a method based on the measurement of the error of randomly generated sequence of gates. It is possible to study a concrete gate with interleaved RB: the selected gate is mixed with random gates, so called reference gates and such sequences are studied. In this work we will discuss interleaved RB and its advantages. But this method rapidly becomes more and more complicated with the increase of the amount of qubits, even for two-qubit case it takes significant effort to correctly implement RB. Considering this complication, some experimentalists use a smaller set of more physically accessible gates for two qubit RB, so-called partial RB. But such method has its own drawbacks and may not give the same results as for the complete RB. Therefore, it was important for us to study partial RB for two qubit gates and obtain the results which may guide how to efficiently implement partial RB.

Chapter 2

Randomized benchmarking

2.1 General information

Among the advantages of this approach are the following. The overall error is enhanced, because multiple number of gates are performed, thus it becomes easier to measure it. Additionally, one should no longer account for the SPAM errors, since they become negligible with an increase of the length of the studied sequences. Also, it turns out that the fidelity of these sequences depends exponentially of its length, thus it allows to describe the quality by a single number - the value of the exponent rate.

Let us start by describing how exactly interleaved RB protocols are performed. Let V_i be gates from some chosen set of gates G and W is a studied gate. In standard RB the set is Clifford group [1] of m qubits - C_m. We begin by randomly picking n gates from the set G:

$$\{V_1, V_2, ...V_n\} \tag{2.1}$$

Each of V_i is randomly chosen from the set G. Then we prepare our system in an initial state described by the density matrix ρ_0 and apply a sequence of such form to it:

$$V_1 W V_2 W ... W V_n W F \tag{2.2}$$

In contrast to the common notation, here we mean that the first applied gate is V_1, then comes W and so on till the gate F. The latter is called the recovery gate and is chosen in such way that in the ideal case[2] the whole sequence becomes unity operator. Then we perform read out and check if the initial state described by the DM ρ_0 is preserved, it would be so in the ideal case, any deviation indicates presence of the errors. The probability to preserve initial state is called survival probability or fidelity. To estimate fidelity for each length of the sequence n (counting only number of V_i), we perform several sequences which differ from each other by the random choice of $\{V_1, V_2, ...V_n\}$ and recovery gate F. The set from which we pick V_i stays the same, so does the gate W. Eventually, for this particular n we will obtain the fidelity P. This averaged fidelity represents the statistics of

[1] More details about Clifford group maybe found in the appendix B

[2] In the absence of any errors.

the state:

$$\langle V_1 W V_2 W ... W V_n W F \rangle_{V_i \in G} [\rho_0] \tag{2.3}$$

Actually, we will show that such averaged sequence acts as dilatation of the initial DM and there is no need to additionally average over the initial state. Then, we increase n and perform all the steps for our new n. Finally, we find the fidelity $P(n)$ as a function of the length of the sequence n.

Which information does $P(n)$ contain? The error of the gate W summarized from all the entries of this gate into our sequence, this error is somehow averaged over various V_i, or in other words over the set from which we picked these V_i. Also, there are contributions from V_i, F and SPAM.

To estimate the error coming from gates V_i, the reference sequences are studied:

$$V_1 V_2 ... V_n F \tag{2.4}$$

From the fidelity of such sequences the average error of V_i is extracted.

Also, random sequences and their properties were studied in the work by the researches from Google (1), but they were not used to measure errors.

2.2 Experimental examples

For illustration, let us look at some examples of RB which is very common in experimental works: The figure 2.1 is taken from the work (13) done by Riggeti et. al. The experiment was conducted

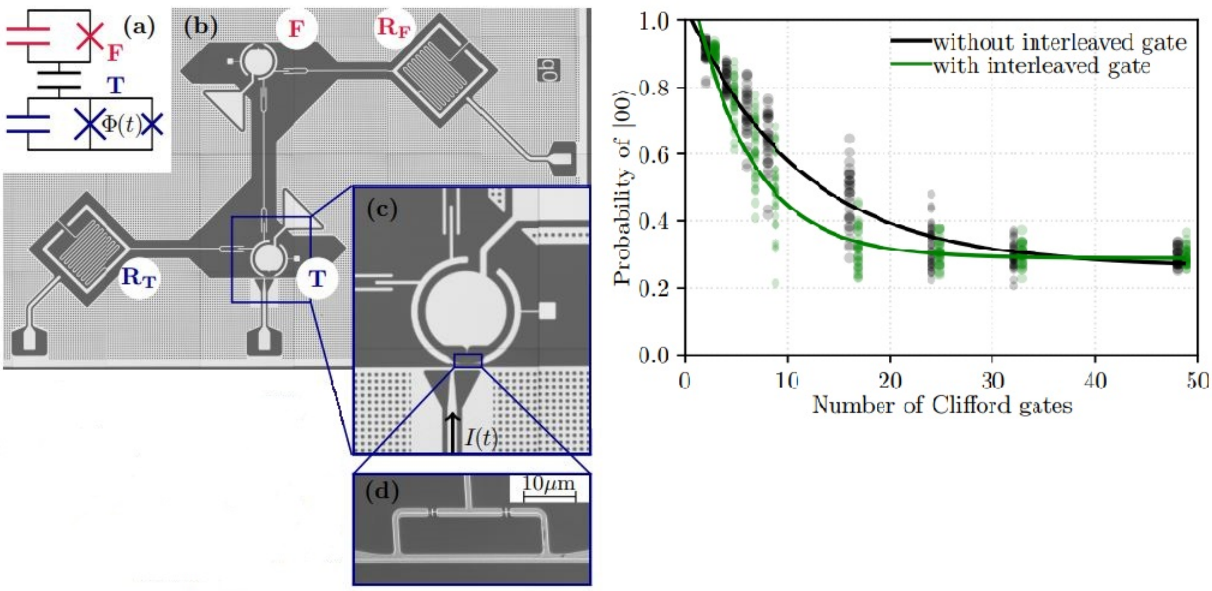

Figure 2.1: On the left two coupled transmon qubits, on the right RB of iSWAP gate performed here (13)

on two coupled Transmon qubits. The interleaved gate W was iSWAP[3], the set G was two-qubit Clifford group and sequence length n varied as $2, 4, 6, 8, 16, 24, 32, 48$. As one may see $P(n)$ starts from 1 and declines to the value 0.25, the latter corresponds to the case of DM with equal probability of each state.

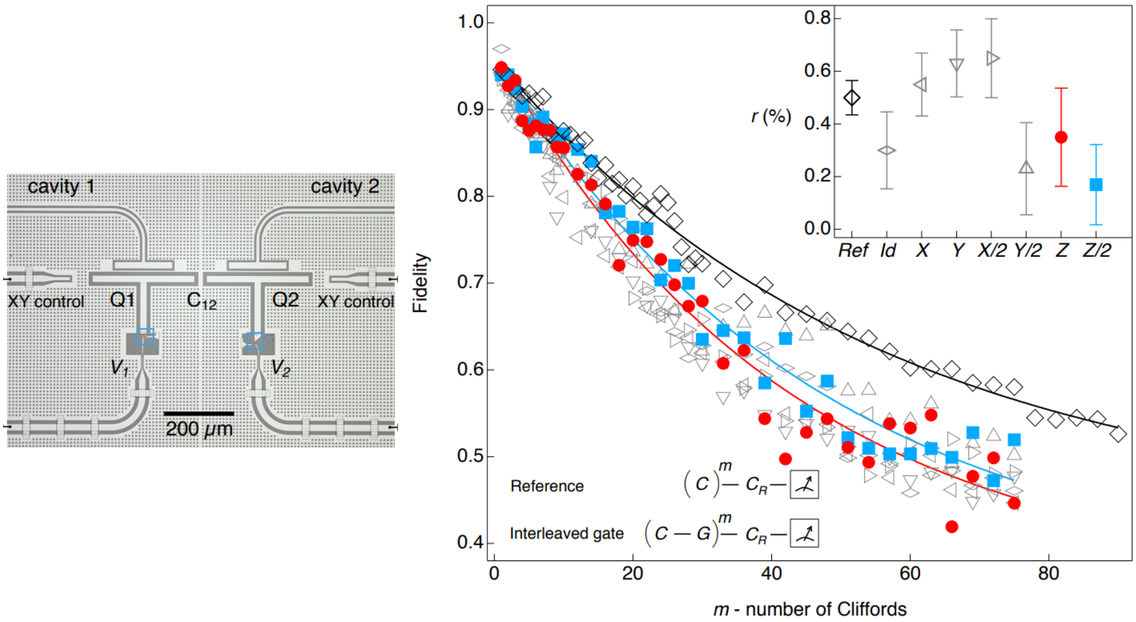

Figure 2.2: On the left two coupled gatemon qubits, on the right RB of single-qubit gates from the work by (14)

The figure 2.2 is from the work (14) by Casparis et al. In their experiment they had two coupled Gatemon qubits. On the graph they presented RB results for estimating the error of single-qubit gates performed on the second qubit. The upper curve is reference one, it decreases significantly slower than the others simply because it contains around two times less gates for the same length n (remember the definition of n - number of V_i in the sequence). The set was single Clifford group C_1 and the interleaved gates one may see at the inset.

The last picture in this section is from the work (15) about quantum teleportation, RB was performed on logical qubit implemented via superconducting microwave cavities. RB of single qubit gates from the inset was performed over C_1 group. The expression $2P_{correct} - 1$ is just the difference between the fidelity and its depolarized value 0.5 multiplied by factor of two. As one may see the graph has logarithmic scale.

[3] Basic information about common gates maybe be found the appendix.

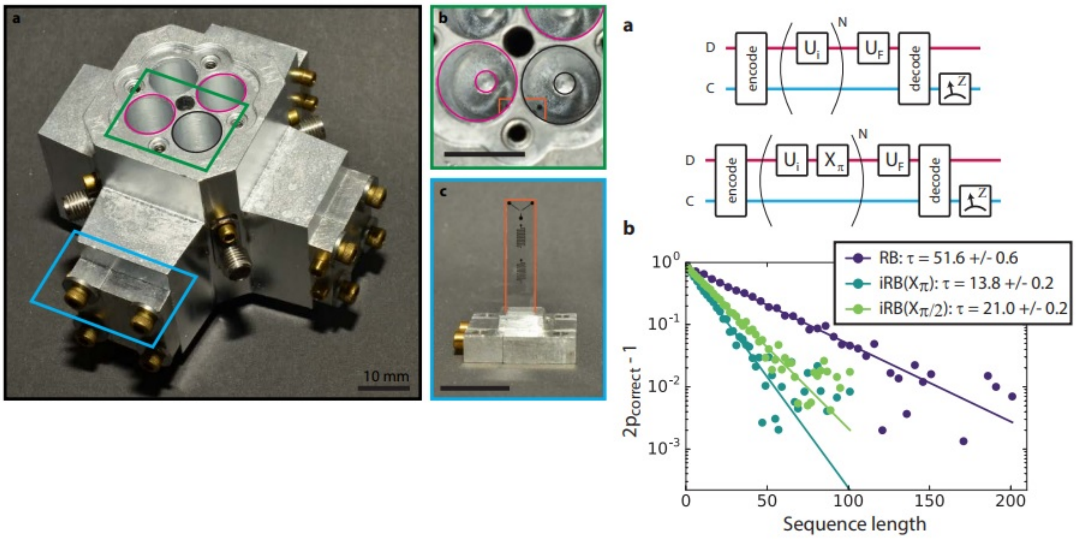

Figure 2.3: The photo of the full device and RB of single-qubit gates done here (15)

2.3 Exponential behaviour

In the next few sections we will show, that the fidelity dependence on the length of the sequence n has declining exponential form:

$$P(n) = A + Ba^n, \;\; a \ll 1 \tag{2.5}$$

For a detailed discussion of this formula one may find it in the work by Magesan et al. (16). The numbers A, B stand for SPAM errors and recovery gate F, which we do not study in the current work. The error of the gate W is described by a single parameter a. This is an advantage of RB, because we are able to extract the error of the selected gate from the other contributions.

Let us show that exponential dependence is indeed present. In our model we will assume that the error of V_i is negligible compared to the error of gate W, this assumption would be justified when we come to partial RB section. The errors of the gates in the sequence are not correlated. Also, the set G is some group. Let us begin with the case when gate W is unity in ideal case (absence of any errors), at the end of the section we will consider any W and show that our results will still be valid. Also, the set from which we pick gates V_i is group G. Before we proceed with the proof, let us introduce common notations in the work.

1) According to our assumption, V_i is unitary gate, thus acts on the density matrix via conjugation:

$$V_i[\rho] = V_i \rho V_i^\dagger \tag{2.6}$$

2) Gate W is a superoperator with some non-trivial action on the density matrix.

$$W[\rho] \tag{2.7}$$

3) Action of such sequences, for instance, V_1WV_2 expands like:

$$(V_1WV_2)[\rho] = V_2W[V_1\rho V_1^\dagger]V_2^\dagger \tag{2.8}$$

For example, the sequence $V_1V_1^\dagger W = W$, while the sequence $V_1WV_1^\dagger$ can not be simplified, because of non-trivial action of W. Let us begin with the sequence:

$$V_1WV_2W...V_nWF \tag{2.9}$$

The recovery gate has the form:

$$F = (V_1V_2...V_n)^\dagger \tag{2.10}$$

One may insert several unity gates and rewrite the sequence:

$$V_1WV_2W...V_nWF = [V_1WV_1^\dagger][(V_1V_2)W(V_1V_2)^\dagger]...[(V_1V_2...V_n)W(V_1V_2...V_n)^\dagger] \tag{2.11}$$

Introduce $U_k = V_1V_2...V_k$ for each $k = \overline{1,\ n}$.

$$V_1WV_2W...V_nWF = (U_1WU_1^\dagger)(U_2WU_2^\dagger)...(U_nWU_n^\dagger) \tag{2.12}$$

Let's average this line over V_n, while all other V_k are fixed. In terms of U_k it means that U_n covers the whole group, while the others U_k being fixed. Thus we can simplify:

$$(U_1WU_1^\dagger)(U_2WU_2^\dagger)...(U_nWU_n^\dagger) = (U_1WU_1^\dagger)(U_2WU_2^\dagger)...(U_{n-1}WU_{n-1}^\dagger)\overline{W} \tag{2.13}$$

Here we defined new operation $\overline{W} = \langle VWV^\dagger\rangle_{V\in G}$, further in the work we will closely study this basic element. Continue by averaging over V_{n-1}, by the same logic we can further simplify:

$$(U_1WU_1^\dagger)...(U_{n-1}WU_{n-1}^\dagger)\overline{W} = (U_1WU_1^\dagger)...(U_{n-2}WU_{n-2}^\dagger)\overline{W}^2 \tag{2.14}$$

Following this logic for the smaller indexes, we get:

$$\langle V_1WV_2W...V_nWF\rangle_{V_1,\ V_2...V_n} = \overline{W}^n \tag{2.15}$$

What did we achieve? We were able to reduce long sequence of the gates to only one element to the power on n. From now on we shall continue by investigating this new operation $\overline{W}$. Its action on the density matrix ρ:

$$\overline{W}[\rho] = \langle V^\dagger W[V\rho\, V^\dagger]V\rangle_V \tag{2.16}$$

Some obvious properties of $\overline{W}$ are linearity, which follows from linearity of superoperator W, and $\overline{W}[I] = I$, which represents DM trace conservation. Also, operation $\overline{W}$ has a very important property, we call it "isotropy over group G" or simply "isotropy". The property looks like:

$$\overline{W}[U\rho\, U^\dagger] = U\overline{W}[\rho]U^\dagger \;\; \forall U \in G \tag{2.17}$$

Or in terms of our sequence notation it is commutation of $\overline{W}$ with reference gates:

$$V\overline{W} = \overline{W}V \tag{2.18}$$

Let us show why it is true, using the definition of $\overline{W}$:

$$\overline{W}[U\rho\, U^\dagger] = \langle V^\dagger W[VU\rho\, U^\dagger V^\dagger]V\rangle_V \tag{2.19}$$

We may change the averaging variable $V \to VU^\dagger$, such replacement is possible, because it is automorphism of group G. It yields:

$$\langle UV^\dagger W[V\rho\, V^\dagger]VU^\dagger\rangle_V = U\langle V^\dagger W[V\rho\, V^\dagger]V\rangle_V U^\dagger = U\overline{W}[\rho]U^\dagger \tag{2.20}$$

Isotropy condition severely constrains the action of $\overline{W}$ - the larger group G, the more equations are hidden in it. Now we are ready to show that if G is Clifford group of m qubits C_m and W is a m-qubit gate, then the RB is a dilatation of the DM:

$$\overline{W}[\rho] = a\cdot\rho + \frac{1-a}{2^m}\cdot\hat{I} \tag{2.21}$$

We will start with single qubit case, since it is most straightforward to check. Our isotropy condition states:

$$\overline{W}[U\rho\, U^\dagger] = U\overline{W}[\rho]U^\dagger \;\; \forall U \in G \tag{2.22}$$

Firstly, quick observation: for any U works:

$$\overline{W}[\rho] = a\cdot\rho + \frac{1-a}{2}\cdot\hat{I}, \text{ where } a \in [-1,\ 1] \tag{2.23}$$

Fact that a lies in $[-1,\ 1]$ comes from the density matrix spectrum properties. Using common representation of two-level system DM we may write:

$$\hat{\rho} = \frac{1}{2}(\hat{I} + (\mathbf{s}, \hat{\boldsymbol{\sigma}})), \tag{2.24}$$

Let us check the action on the components of the DM separately. Taking for U operators $\hat{\sigma}_\alpha$, where $\alpha \in \{x, y, z\}$, one may obtain following relations:

$$\hat{\sigma}_\alpha \overline{W}[\hat{\sigma}_\beta] = -\overline{W}[\hat{\sigma}_\beta]\hat{\sigma}_\alpha,\ \alpha \neq \beta \tag{2.25}$$

$$\hat{\sigma}_\alpha \overline{W}[\hat{\sigma}_\alpha] = \overline{W}[\hat{\sigma}_\alpha]\hat{\sigma}_\alpha \text{ (no sum here)} \tag{2.26}$$

These equations yield: $\overline{W}[\hat{\sigma}_\alpha] = a \cdot \hat{\sigma}_\alpha$. Now let us take a look at: $\overline{W}[\hat{\sigma}_x + \hat{\sigma}_y]$ and $U = \exp{(-i\sigma_z \frac{\pi}{4})}$:

$$\exp{(-i\sigma_z \frac{\pi}{4})} \cdot \sigma_x \cdot \exp{(i\sigma_z \frac{\pi}{4})} = \sigma_y \tag{2.27}$$

$$\exp{(-i\sigma_z \frac{\pi}{4})} \cdot \sigma_y \cdot \exp{(i\sigma_z \frac{\pi}{4})} = -\sigma_x \tag{2.28}$$

$$U\overline{W}[\sigma_x + \sigma_y]U^\dagger = a_1 \cdot \sigma_y - a_2 \cdot \sigma_x \tag{2.29}$$

$$\overline{W}[U(\sigma_x + \sigma_y)U^\dagger] = a_2 \cdot \sigma_y - a_1 \cdot \sigma_x \tag{2.30}$$

Therefore, $a_1 = a_2$ and $\overline{W}[\sigma_x + \sigma_y] = a \cdot (\sigma_x + \sigma_y)$. Following this procedure, one can get the same relation for the entry $\sigma_y + \sigma_z$ and thus:

$$\overline{W}[(\mathbf{s}, \boldsymbol{\sigma})] = a \cdot (\mathbf{s}, \boldsymbol{\sigma}). \tag{2.31}$$

The proof for two-qubit case will be shown in the next chapter.

2.3.1 Intermediate results

Let us sum up the achieved results, we started from the sequence:

$$\langle V_1 W V_2 W ... W V_n W F \rangle_{V_i \in G} \tag{2.32}$$

Showed that it equals to:

$$\langle V_1 W V_2 W ... W V_n W F \rangle_{V_i \in G} = \overline{W}^n \tag{2.33}$$

If the group G is at least the m-qubit Clifford group C_m, then $\overline{W}$ is dilatation:

$$\overline{W}[\rho] = a \cdot \rho + \frac{1-a}{2^m} \cdot \hat{I} \tag{2.34}$$

And our initial sequence acting on the density matrix ρ:

$$\langle V_1 W V_2 W ... W V_n W F\rangle_{V_i \in C_m}[\rho] = \overline{W}^n[\rho] = a^n \cdot \rho + \frac{1-a^n}{2^m} \cdot \hat{I} \tag{2.35}$$

So, the density matrix exponentially shrinks to the completely depolarized one and thus the probability of each state is equal. Therefore, the fidelity exponentially decreases to the value of $1/2^m$. Let's look at some physical examples.

2.3.2 Applications of unity interleaved gate

Here we will provide a few examples of W close to unity. 1) Let the qubit relax for time period τ, so Hamiltonian reads: $H = -B_0 \cdot \sigma_z + H_{\text{bath}}$. In such case the qubit evolution can be qualitatively described by two characteristic times: $T_1,\ T_2$. So called, thermalization and dephasing times. In this case parameter a is equal to:

$$a = \frac{1}{3}\exp\left(-\frac{\tau}{T_1}\right) + \frac{2}{3}\exp\left(-\frac{\tau}{T_2}\right) \tag{2.36}$$

2) Imagine we have noise along the Z axis, so $W = \exp\left(i\frac{\phi}{2}\sigma_z\right)$, where ϕ is a small angle, randomly distributed with the density function $P(\phi)$. Let it be Gaussian distribution function with the disperse A:

$$P(\phi) = \frac{1}{\sqrt{2\pi A}}\exp\left(-\frac{\phi^2}{2A}\right) \tag{2.37}$$

Then a might be calculated to be

$$a = \int d\phi P(\phi)\frac{2\cos\phi + 1}{3} = \frac{2e^{-\frac{A}{2}} + 1}{3} \tag{2.38}$$

The calculation of both examples may be found in the appendix E.

2.3.3 Arbitrary interleaved gate

Till this point, we only studied gates W which are close to unity, which may seem strange. But now we will quickly show using the derived formulas, that this condition is not necessary to obtain exponential behaviour. Let now W be any gate. In our model, we will assume that we can decompose $W = \Lambda W_0$, where Λ is a superoperator representing the error effect[4] and W_0 is unitary operator from SU(2^m). Performing standard RB procedure we apply the sequence to the system:

$$\langle V_1 W V_2 W ... V_n W F\rangle_{V_i \in C_m} \tag{2.39}$$

[4] We assume that averaging over the noise happens faster than averaging over the sequences

One may notice that now the recovery gate is also some special gate, which depends on W_0:

$$F = (V_1 W_0 ... V_n W_0)^\dagger \tag{2.40}$$

It does not imply that one should perform additional $2n$ gates. Usually, it is calculated manually and then applied as a single physical pulse. Therefore, for long enough sequences, its error would become negligible compared to the overall error. Using the mentioned form of F, let us expand the sequence:

$$\langle V_1 \Lambda W_0 ... V_n \Lambda W_0 W_0^\dagger V_n^\dagger ... W_0^\dagger V_1^\dagger \rangle_{V_i \in C_m} = \langle V_1 \Lambda W_0 ... V_{n-1} \Lambda W_0 \overline{\Lambda} W_0^\dagger V_{n-1}^\dagger ... W_0^\dagger V_1^\dagger \rangle_{V_i \in C_m} \tag{2.41}$$

Here we defined $\overline{\Lambda} = \langle V \Lambda V^\dagger \rangle_{V \in C_m}$. We already know that its action on the DM has the form:

$$\overline{\Lambda}[\rho] = a \cdot \rho + \frac{1-a}{2^m} \cdot \hat{I} \tag{2.42}$$

Such form is, evidently, isotropic over SU(2^m) group. So, in the sequence we can commute $\overline{\Lambda} W_0$ $= W_0 \overline{\Lambda}$. This allows us to simplify further:

$$\langle V_1 W V_2 W ... V_n W F \rangle_{V_i \in C_m} = \langle V_1 \Lambda W_0 ... V_{n-2} \Lambda W_0 \overline{\Lambda}^2 W_0^\dagger V_{n-2}^\dagger ... W_0^\dagger V_1^\dagger \rangle_{V_i \in C_m} = \overline{\Lambda}^n \tag{2.43}$$

Thus, the action of the sequence on the DM is:

$$\langle V_1 W V_2 W ... V_n W F \rangle_{V_i \in C_m}[\rho] = a^n \cdot \rho + \frac{1-a^n}{2^m} \cdot \hat{I} \tag{2.44}$$

The single exponent should be expected on the graph for any interleaved gate if one average the interleaved gate over Clifford group.

Chapter 3

Partial randomized benchmarking

In the previous section we showed that to describe the error of the benchmarked gate with single parameter, one need to average over Clifford group. For one qubit case, when the benchmarked gate is from SU(2) group, the researchers should be able to perform group C_1. There are 24 elements in it, these elements are well-known and might be implemented with high precision. But for the two-qubit case, the picture is not so clear. The group C_2 order is 11520[1]. Its elements are some non-trivial unitary operations from SU(4), it takes significant effort to physically implement them. For example, assume the qubits coupling allows to implement CNOT gate. To benchmark by interleaving it with the elements from C_2, one needs to, firstly, construct each of 11520 elements from CNOT and single-qubit gates. Even though, in our sequence, we simply write V_i, in reality behind this V_i stands some combination of CNOT gate and single-qubit gates. It severely complicates RB procedure for two-qubit gates.

Considering these obstacles one may use smaller group to average over: not the whole C_2 group, but only single-qubit gates, in group notation $C_1 \times C_1$. The advantages are the following. Firstly, single-qubit gates are more accessible to implement. Secondly, nowadays single-qubit gates have the lower error by a few orders of magnitude than a two-qubit gates [2], thus the major part of the total error would be coming from benchmarked two-qubit gate. Thirdly, the group $C_1 \times C_1$ contains only $24^2 = 576$ elements and its structure is clear.

For example, in the work (14), they performed an unusual iRb of CZ gate: random single-qubit Clifford gates were applied to the target qubit and the control qubit was randomly prepared in one of the states $|0\rangle$ or $|1\rangle$ and returned to the state $|0\rangle$ after each CZ gate. This averaging is even smaller than single qubit rotations. Another example, in this work (17) by Chen et al. they studied coupling between superconducting qubits. To benchmark gate CZ, they interleaved it with the reference gates from $C_1 \times C_1$, the fidelity graph is shown on the figure 3.1. In the work, they approximated the dependence with the exponent as for the case of full Clifford group averaging. But how accurate is such approximation?

The main question studied in the present work if the averaged over single qubit rotations $C_1 \times C_1$ sequence acts as a dilatation on the DM, which results in the exponential behaviour of the

[1] The additional info about Clifford group is in the appendix

[2] This fact justifies our assumption about perfectness of V_i

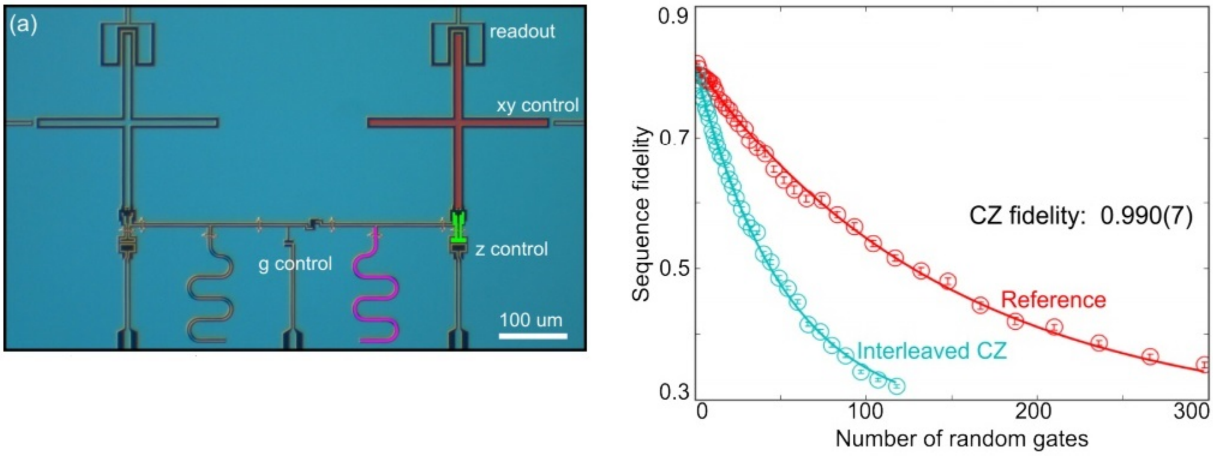

Figure 3.1: Image of two inductively coupled gmon qubits and iRB of CZ gate via single-qubit gates

fidelity. And if so which is the rate of the exponent and how it corresponds to the rate obtained by the two qubit Clifford group C_2 averaging. We will show that for the major part of the gates exponential behaviour is present with the rate close to the full averaging case. Except for the gates close to unity and SWAP. Note that related questions for generic gates were discussed in (21),(22).

3.1 Simple case of partial RB

Before we proceed with averaging over $C_1 \times C_1$, let us discuss an instructive single-qubit case, but with averaging over the z-axis rotations. Suppose we average over the group consisting of the elements:

$$R[\phi] = \exp[\frac{i}{2}\sigma_z\phi] \tag{3.1}$$

We will show that the action of $\overline{W}$ on $\rho = \frac{1}{2}(\sigma_0 + \boldsymbol{n}\boldsymbol{\sigma})$ has the form:

$$\overline{W}[\rho] = \frac{1}{2}[\sigma_0 + a(n_x\sigma_x + n_y\sigma_y) + b(n_x\sigma_y - n_y\sigma_x) + cn_z\sigma_z] \tag{3.2}$$

with three parameters a, b, c which characterize errors. In the ideal case $a = c = 1$ and $b = 0$, but in the presence of the error there are deviations. Using the conditions of isotropy:

$$U^\dagger\overline{W}[\rho]U = \overline{W}[U^\dagger\rho U], \tag{3.3}$$

and some useful formulas:

$$R[-\phi]\sigma_x R[\phi] = \cos\phi \cdot \sigma_x + \sin\phi \cdot \sigma_y \tag{3.4}$$

$$R[-\phi]\sigma_y R[\phi] = \cos\phi \cdot \sigma_y - \sin\phi \cdot \sigma_x \tag{3.5}$$

$$R[-\phi]\sigma_z R[\phi] = \sigma_z \tag{3.6}$$

Most general form of the action:

$$\overline{W}[\sigma_\alpha] = \sum_{\beta=1}^{3} a_{\alpha\beta}\sigma_\beta \tag{3.7}$$

1) Taking $U = i\sigma_z$ and the argument of $\overline{W}$ as σ_x or σ_y, one gets $a_{xz} = a_{yz} = 0$

2) Taking $U = i\sigma_z$ and the argument to be σ_z, one gets $a_{zx} = a_{zy} = 0$

3) Taking $U = R[\pi/2]$ and σ_x or σ_y, one gets $a_{xx} = a_{yy},\ a_{xy} = -a_{yx}$

Therefore we have:

$$a_{\alpha\beta} = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix} \tag{3.8}$$

So, the fidelity function $P(n)$ contains three exponents $(a \pm ib)^n$ and c^n given by the eigenvalues of this matrix.

3.2 Unity interleaved gate

Here we will show that two qubit gate averaged over single qubit rotations $\overline{W}[\rho]$ is described by three parameters close to one if the gate W is close to unity[3]:

$$\overline{W}[\rho] = \langle V^\dagger W[V\rho\, V^\dagger]V\rangle_{V\in C_1\times C_1} \tag{3.9}$$

The constraints of this action should be obtained from the isotropy over $C_1 \times C_1$:

$$\overline{W}[U\rho\, U^\dagger] = U\overline{W}[\rho]U^\dagger \ \ \forall U \in C_1 \times C_1 \tag{3.10}$$

In the appendix E we will show by the direct calculations that $\overline{W}$ acts differently on the different Pauli matrices:

$$\overline{W}[\sigma_\alpha \otimes \sigma_0] = a \cdot \sigma_\alpha \otimes \sigma_0,\ \ \alpha \in x, y, z, \tag{3.11}$$

$$\overline{W}[\sigma_0 \otimes \sigma_\beta] = b \cdot \sigma_0 \otimes \sigma_\beta,\ \ \beta \in x, y, z, \tag{3.12}$$

$$\overline{W}[\sigma_\alpha \otimes \sigma_\beta] = c \cdot \sigma_\alpha \otimes \sigma_\beta,\ \ \alpha,\ \beta \in x, y, z, \tag{3.13}$$

In total, there are three independent parameters. Such form can be understood by the fact that single qubit rotations do not mix parts $\sigma_\alpha \otimes \sigma_0$, $\sigma_0 \otimes \sigma_\beta$ and $\sigma_\alpha \otimes \sigma_\beta$ with each other.

[3]For instance, one can study decoherence of two qubit state with such approach

General form of the two-qubit density matrix is:

$$\rho = \frac{1}{4} \cdot \sigma_0 \otimes \sigma_0 + n_\alpha \cdot \sigma_\alpha \otimes \sigma_0 + m_\beta \cdot \sigma_0 \otimes \sigma_\beta + k_{\alpha\beta} \cdot \sigma_\alpha \otimes \sigma_\beta, \ \{\alpha, \ \beta\} \in x, y, z \tag{3.14}$$

Then the action of $\overline{W}$ on the DM has the form:

$$\overline{W}[\rho] = \frac{1}{4} \cdot \sigma_0 \otimes \sigma_0 + a \cdot (n_\alpha \cdot \sigma_\alpha \otimes \sigma_0) + b \cdot (m_\beta \cdot \sigma_0 \otimes \sigma_\beta) + c \cdot (k_{\alpha\beta} \cdot \sigma_\alpha \otimes \sigma_\beta), \ \{\alpha, \ \beta\} \in x, y, z, \tag{3.15}$$

So, unlike, for the whole Clifford group C_2, where we would have only one parameter, in our case of a smaller group there are three independent parameters instead of one. If we naively plotted the survival probability, the drawn graph would be sum of three exponents and approximating it with only one exponent would be a mistake. Let us show how we can manage with these three exponents in a case of W close to unity.

3.2.1 Complete averaging

In the first chapter we already mentioned that averaging over the full Clifford group leads to the dilatation of the DM. Let us quickly show how the presence of a gate from the two qubit Clifford groupC_2 leads to the equation $a = b = c$, therefore only one independent parameter. For instance, we will take CNOT $\in C_2$, on the one hand:

$$\overline{W}[\text{CNOT}^\dagger \cdot \sigma_x \otimes \sigma_0 \cdot \text{CNOT}] = \overline{W}[\sigma_x \otimes \sigma_x] = c \cdot \sigma_x \otimes \sigma_x \tag{3.16}$$

On the other hand, if we assume averaging over C_2, thus commutation of $\overline{W}$ with CNOT:

$$\overline{W}[\text{CNOT}^\dagger \cdot \sigma_x \otimes \sigma_0 \cdot \text{CNOT}] = \text{CNOT}^\dagger \cdot \overline{W} \cdot \text{CNOT} = a \cdot \sigma_x \otimes \sigma_x \tag{3.17}$$

Which yields $a = c$. By the same procedure with $\sigma_z \otimes \sigma_z = \text{CNOT}^\dagger \cdot \sigma_0 \otimes \sigma_z \cdot \text{CNOT}$, we get $b = c$. Thus, the action of completely averaged $\overline{W}$ on the DM has the form of a dilatation:

$$\overline{W}[\rho] = \frac{1}{4} \cdot \sigma_0 \otimes \sigma_0 + a_{full} \cdot (n_\alpha \cdot \sigma_\alpha \otimes \sigma_0 + \cdot m_\beta \cdot \sigma_0 \otimes \sigma_\beta + \cdot k_{\alpha\beta} \cdot \sigma_\alpha \otimes \sigma_\beta), \ \{\alpha, \ \beta\} \in x, y, z, \tag{3.18}$$

3.2.2 Extraction of the three exponents

As the first step, we choose a basis, for instance:

$$|\uparrow\uparrow\rangle, \ |\uparrow\downarrow\rangle, \ |\downarrow\uparrow\rangle, \ |\downarrow\downarrow\rangle \tag{3.19}$$

Then we prepare our system in the state $|\uparrow\uparrow\rangle$, the density matrix corresponding to such state:

$$\rho_0 = \frac{1}{4}(\sigma_0 + \sigma_z) \otimes (\sigma_0 + \sigma_z) = \frac{1}{4}(\sigma_0 \otimes \sigma_0 + \sigma_z \otimes \sigma_0 + \sigma_0 \otimes \sigma_z + \sigma_z \otimes \sigma_z) \tag{3.20}$$

And perform our partial RB procedure, as we have shown it is the same as applying operation $\overline{W}$ the length of the sequence times. For the sequence of length n we get:

$$\langle V_1 W V_2 W ... W V_n W F \rangle_{V_i \in C_1 \times C_1}[\rho_0] = \frac{1}{4}(\sigma_0 \otimes \sigma_0 + a^n \cdot \sigma_z \otimes \sigma_0 + b^n \cdot \sigma_0 \otimes \sigma_z + c^n \cdot \sigma_z \otimes \sigma_z) \tag{3.21}$$

The probabilities to end up in one of the four basis states $P_{\uparrow\uparrow},\ P_{\uparrow\downarrow},\ P_{\downarrow\uparrow},\ P_{\downarrow\downarrow}$ are measured in the experiment. Also, we can extract them from the DM of the final state. Let us express our parameters through these probabilities:

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} P_{\uparrow\uparrow} \\ P_{\uparrow\downarrow} \\ P_{\downarrow\uparrow} \\ P_{\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} c^n \\ b^n \\ a^n \\ 1 \end{pmatrix} \tag{3.22}$$

So, instead of naively plotting the survival probability $P_{\uparrow\uparrow}$, one should plot derived linear combinations and approximate them with the standard single exponential form. From such graphs each of three parameters could be extracted.

3.3 Arbitrary interleaved gate

Now let us consider general case when W is any two-qubit gate. This case is the main part of our work. According to our model, we can decompose $W = \Lambda W_0$, where Λ is a superoperator representing the error effect and W_0 is unitary operator from SU(4). Performing standard RB procedure we apply the sequence to the system, denote it as $\overline{\Lambda}_n$:

$$\overline{\Lambda}_n \equiv \langle V_1 W V_2 W ... V_n W F \rangle_{V_i \in C_1 \times C_1} \tag{3.23}$$

Recovery gate F is calculated manually and has the form:

$$F = (V_1 W_0 ... V_n W_0)^\dagger \tag{3.24}$$

The sequence represents the averaged error from the sequence of the length n.

3.3.1 Recurrent relation

Let us begin working with the sequence. For simplification purposes, we will start with the length of two and will try to notice some patterns:

$$\overline{\Lambda}_2 = \langle V_1 W V_2 W F \rangle_{V_i} = \langle V_1 \Lambda W_0 V_2 \Lambda W_0 F \rangle_{V_i} = \langle V_1 \Lambda W_0 (V_2 \Lambda V_2^\dagger) W_0^\dagger V_1^\dagger \rangle_{V_i} \tag{3.25}$$

Denoting $\overline{\Lambda} \equiv \overline{\Lambda}_1 = \langle V \Lambda V^\dagger \rangle_V$, we have the relation:

$$\overline{\Lambda}_2 = \langle V \Lambda W_0 \overline{\Lambda} W_0^\dagger V^\dagger \rangle_V \tag{3.26}$$

To proceed let us transform the sequence of the length three:

$$\overline{\Lambda}_3 = \langle V_1 \Lambda W_0 V_2 \Lambda W_0 V_3 \Lambda W_0 F \rangle_{V_i} = \langle V_1 \Lambda W_0 V_2 \Lambda W_0 (V_3 \Lambda V_3^\dagger) W_0^\dagger V_2^\dagger W_0^\dagger V_1^\dagger \rangle_{V_i} \tag{3.27}$$

Recognising known patterns, we may simplify:

$$\overline{\Lambda}_3 = \langle V_1 \Lambda W_0 V_2 \Lambda W_0 \overline{\Lambda} W_0^\dagger V_2^\dagger W_0^\dagger V_1^\dagger \rangle_{V_i} = \langle V \Lambda W_0 \overline{\Lambda}_2 W_0^\dagger V^\dagger \rangle_V \tag{3.28}$$

The relation between $\overline{\Lambda}_n$ and $\overline{\Lambda}_{n+1}$ is evident:

$$\overline{\Lambda}_{n+1} = \langle V \Lambda W_0 \overline{\Lambda}_n W_0^\dagger V^\dagger \rangle_V \tag{3.29}$$

This recurrent relation dictates linear dependence between the error $\overline{\Lambda}_n$ on the step n and the error $\overline{\Lambda}_{n+1}$ on the step $n+1$. To find it, let us define the action of $\overline{\Lambda}_n$ on the density matrix as:

$$\overline{\Lambda}_n[\rho] = \frac{1}{4} \cdot \sigma_0 \otimes \sigma_0 + f_n^{(1)} \cdot n_\alpha \sigma_\alpha \otimes \sigma_0 + f_n^{(2)} \cdot m_\beta \sigma_0 \otimes \sigma_\beta + f_n^{(3)} \cdot k_{\alpha\beta} \sigma_\alpha \otimes \sigma_\beta \tag{3.30}$$

By such way we defined vector $\boldsymbol{f_n}$. Then the recurrent relation between the errors after the step n and $n+1$ can be described by the 3 by 3 matrix:

$$\boldsymbol{f_{n+1}} = \hat{M} \boldsymbol{f_n} \tag{3.31}$$

Since, the vector $\boldsymbol{f_0}$ is just $\boldsymbol{f_0} = (1,\ 1,\ 1)^T$. The vector describing the action of $\overline{\Lambda}_n$ is:

$$\boldsymbol{f_n} = \hat{M}^n \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.32}$$

So, in this case as the result of the partial RB one should always expect three exponents corresponding to the three eigenvalues of the matrix $\hat{M}$. But if, for instance, the eigenvalues are $0.99,\ 0.5,\ 0.5$, the latter two would not be seen on the fidelity from the length plots for the long enough sequences. Thus, in reality only one exponent is present. The general property of the matrix M is that there always is at least a single eigenvalue close to 1, which relates to our sequence being unity in the ideal case. The other two maybe any numbers with absolute value smaller than one. To find all the gates for which only one exponent is seen on the plot, one needs to further research matrix $\hat{M}$.

3.3.2 Further notation

In this section we will develop method of calculating matrix M in general case for various two-qubits gates W. If we would observe that the eigenvalues are such that there is only one close to 1 by the order of the error thus we may state that one-exponent approximation is valid. Main approach here would be representation of the action of unitary two-qubit gates in terms of the action on the density matrix written as a linear combination of $\sigma_i \otimes \sigma_j$ with real coefficients. It is helpful to describe the action of unitary operator W_0 on it via:

$$W_0 \sigma_i \otimes \sigma_j W_0^\dagger = \sum_{k,l=0}^{3} W_{ij\to kl} \sigma_k \otimes \sigma_l \tag{3.33}$$

Here we omitted index 0 on the r.h.s. of the equation. Further in the work if we write $W_{ij\to kl}$, these elements correspond to W_0. They are real numbers and equal to:

$$W_{ij\to kl} = \frac{1}{4}\mathrm{Tr}(W \sigma_i \otimes \sigma_j W^\dagger \sigma_k \otimes \sigma_l) \tag{3.34}$$

Let us introduce basis of length fifteen with the elements:

$$\{\mathbf{X0}, \mathbf{Y0}, \mathbf{Z0}, \mathbf{0X}, \mathbf{0Y}, \mathbf{0Z}, \mathbf{XX}, \mathbf{XY}, \mathbf{XZ}, \mathbf{YZ}, \mathbf{YY}, \mathbf{YZ}, \mathbf{ZX}, \mathbf{ZY}, \mathbf{ZZ}\} \tag{3.35}$$

Where we used short-hand notation, for example, the variables **X0**, **0X**, **XX** stand for $\sigma_x \otimes \sigma_0$, $\sigma_0 \otimes \sigma_x$, $\sigma_x \otimes \sigma_x$ and so on. In such representation

$$W_{ij\to kl} \equiv w_{\alpha\beta} \tag{3.36}$$

becomes a 15 by 15 matrix. It can be checked that the matrix $w_{\alpha\beta}$ is orthogonal. The 15 by 15 matrix for Λ is denoted as

$$\Lambda_{ij\to kl} \equiv \lambda_{\alpha\beta} \tag{3.37}$$

Also, let us visually divide our vector into three sets. First set (1) is **X0**, **Y0**, **Z0**, second set (2) is **0X**, **0Y**, **0Z** and the third set (3) **XX**, **XY**, **XZ**, **XY**, **YY**, **YZ**, **ZX**, **ZY**, **ZZ**.

3.3.3 Matrix M for unity gate

Here we will obtain the expression for the iteration matrix M for the already studied case of W close to unity. In this case we have:

$$\overline{\Lambda}_2 = \langle V \Lambda W_0 \overline{\Lambda} W_0^{\dagger} V^{\dagger} \rangle_V = \langle V \Lambda \overline{\Lambda} V^{\dagger} \rangle_V = \overline{\Lambda}^2 \tag{3.38}$$

And in general:

$$\overline{\Lambda}_n = \overline{\Lambda}^n \tag{3.39}$$

So, if we describe the action of $\overline{\Lambda}$ with the three parameters a, b, c, multiplying each of the three sets correspondingly. Then, iteration matrix M has the form:

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \tag{3.40}$$

But, let us deliberately calculate it more formally. Firstly, we have unitary gate $V \equiv V_a \otimes V_b$ from SU(2)$\times$SU(2), define $\mathcal{V}_{\beta\alpha}$ and $\mathcal{V}^{\beta\alpha}$ as:

$$\sum_{\beta} \mathcal{V}_{\beta\alpha} \sigma_{\beta} \equiv V_a \sigma_{\alpha} V_a^{\dagger} \tag{3.41}$$

$$\sum_{\beta} \mathcal{V}^{\beta\alpha} \sigma_{\beta} \equiv V_b \sigma_{\alpha} V_b^{\dagger} \tag{3.42}$$

Here the 3 by 3 orthogonal matrices $\mathcal{V}_{\beta\alpha}$ and $\mathcal{V}^{\beta\alpha}$ are completely independent and correspond to the different spaces of the first qubit and of the second qubit. Applying this notation to the action of V on the $\sigma_i \otimes \sigma_j$ we have:

$$V[\sigma_{\alpha} \otimes \sigma_0] = V \sigma_{\alpha} \otimes \sigma_0 V^{\dagger} = (V_a \sigma_{\alpha} V_a^{\dagger}) \otimes \sigma_0 = \sum_{\beta} \mathcal{V}_{\beta\alpha} \sigma_{\beta} \otimes \sigma_0 \tag{3.43}$$

$$V[\sigma_0 \otimes \sigma_{\alpha}] = V \sigma_0 \otimes \sigma_{\alpha} V^{\dagger} = \sigma_0 \otimes (V_b \sigma_{\alpha} V_b^{\dagger}) = \sum_{\beta} \mathcal{V}^{\beta\alpha} \sigma_0 \otimes \sigma_{\beta} \tag{3.44}$$

$$V[\sigma_{\alpha} \otimes \sigma_{\beta}] = V \sigma_{\alpha} \otimes \sigma_{\beta} V^{\dagger} = (V_a \sigma_{\alpha} V_a^{\dagger}) \otimes (V_b \sigma_{\beta} V_b^{\dagger}) = \sum_{\beta,\gamma} \mathcal{V}_{\gamma\alpha} \mathcal{V}^{\delta\beta} \sigma_{\gamma} \otimes \sigma_{\delta} \tag{3.45}$$

When we average over $V \in C_1 \times C_1$, each of V_a and V_b is averaged over C_1, which is the same as averaging over SU(2). Therefore the matrices $\mathcal{V}_{\beta\alpha}$ and $\mathcal{V}^{\beta\alpha}$ are averaged over SO(3) independently, we have:

$$\langle \mathcal{V}_{\alpha\beta}(\mathcal{V}^{-1})_{\gamma\delta}\rangle = \frac{1}{3}\delta_{\alpha\delta}\delta_{\beta\gamma} \tag{3.46}$$

$$\langle \mathcal{V}^{\alpha\beta}(\mathcal{V}^{-1})^{\gamma\delta}\rangle = \frac{1}{3}\delta_{\alpha\delta}\delta_{\beta\gamma} \tag{3.47}$$

For the average not to be zero, we need to have two matrices from the same space:

$$\langle \mathcal{V}_{\alpha\beta}\rangle = \langle \mathcal{V}^{\alpha\beta}\rangle = \langle (\mathcal{V}^{-1})_{\alpha\beta}\rangle = \langle (\mathcal{V}^{-1})^{\alpha\beta}\rangle = \langle \mathcal{V}_{\alpha\beta}(\mathcal{V}^{-1})^{\gamma\delta}\rangle = \langle \mathcal{V}^{\alpha\beta}(\mathcal{V}^{-1})_{\gamma\delta}\rangle = 0 \tag{3.48}$$

Using this notation we may now write the action of $\overline{\Lambda}$ to express the parameters a, b, c. Let us begin with a:

$$\overline{\Lambda}[\sigma_x \otimes \sigma_0] = \langle V\Lambda V^\dagger\rangle_V[\sigma_x \otimes \sigma_0] = \sum_\alpha \langle \mathcal{V}_{\alpha x}\Lambda_{\alpha 0\to ij}(V^\dagger[\sigma_i \otimes \sigma_j])\rangle \tag{3.49}$$

Now we should consider three cases: $\sigma_i \otimes \sigma_j$ is from the first set, from the second set or from the third set.

$$\sum_{(ij)\in(1)} \langle \mathcal{V}_{\alpha x}\Lambda_{\alpha 0\to i0}(\mathcal{V}^{-1})_{\beta i}\rangle \sigma_\beta \otimes \sigma_0 = \sigma_x \otimes \sigma_0 \sum_{\alpha=1}^{3}\frac{1}{3}\Lambda_{\alpha 0\to\alpha 0} \tag{3.50}$$

$$\sum_{(ij)\in(2)} \langle \mathcal{V}_{\alpha x}\Lambda_{\alpha 0\to 0j}(\mathcal{V}^{-1})^{\beta j}\rangle \sigma_0 \otimes \sigma_\beta = 0 \tag{3.51}$$

$$\sum_{(ij)\in(3)} \langle \mathcal{V}_{\alpha x}\Lambda_{\alpha 0\to\beta\gamma}(\mathcal{V}^{-1})_{\beta i}(\mathcal{V}^{-1})^{\gamma j}\rangle \sigma_\beta \otimes \sigma_\gamma = 0 \tag{3.52}$$

Thus, we conclude in our 15 by 15 matrices notation:

$$a = \sum_{\alpha=1}^{3}\frac{1}{3}\Lambda_{\alpha 0\to\alpha 0} = \sum_{\gamma\in(1)}\frac{1}{3}\lambda_{\gamma\gamma} \tag{3.53}$$

Same can be done to determine the other two parameters:

$$b = \sum_{\gamma\in(2)}\frac{1}{3}\lambda_{\gamma\gamma}, \quad c = \sum_{\gamma\in(3)}\frac{1}{9}\lambda_{\gamma\gamma} \tag{3.54}$$

The complete averaging over two qubit Clifford group yields a single parameter, which is the trace of the superoperator Λ:

$$a_{full} = \frac{1}{15}\sum_\gamma \lambda_{\gamma\gamma} \tag{3.55}$$

Thus, we have the relation:

$$a_{full} = \frac{a+b+3c}{5} \tag{3.56}$$

3.3.4 Expression of the matrix M

Now let us return to the case of any $W = \Lambda W_0$:

$$\overline{\Lambda}_2 = \langle V \Lambda W_0 \overline{\Lambda} W_0^\dagger V^\dagger \rangle \tag{3.57}$$

To calculate the first row of matrix M we should look at the action on the first set, defined as $f_2^{(1)}$. On the one hand, it is equal to:

$$f_2^{(1)} = M_{11} \cdot a + M_{12} \cdot b + M_{13} \cdot c \tag{3.58}$$

On the other hand, it is the coefficient before $\sigma_\alpha \otimes \sigma_0$. For instance, let us take $\sigma_x \otimes \sigma_0$:

$$\overline{\Lambda}_2[\sigma_x \otimes \sigma_0] = \sum_{\alpha=1}^{3} \sum_{(ij)} \langle \mathcal{V}_{\alpha x} \Lambda_{\alpha 0 \to ij} (W_0 \overline{\Lambda} W_0^\dagger V^\dagger [\sigma_i \otimes \sigma_j]) \rangle =$$

$$= \sum_{\alpha=1}^{3} \sum_{(ij)} \sum_{(kl)} \langle \mathcal{V}_{\alpha x} \Lambda_{\alpha 0 \to ij} W_{ij \to kl} (\overline{\Lambda} W_0^\dagger V^\dagger [\sigma_k \otimes \sigma_l]) \rangle \tag{3.59}$$

At this stage we can already distinguish matrix elements of M, we know that $\overline{\Lambda}$ acts by multiplying the argument by one of the three parameters which correspond to the vector $\boldsymbol{f_1}$. So, when (kl) are from the first set, the operator $\overline{\Lambda}$ multiplies by a and we may extract M_{11}:

$$\sum_{\alpha=1}^{3} \sum_{(ij)} \sum_{(kl) \in (1)} \langle \mathcal{V}_{\alpha x} \Lambda_{\alpha 0 \to ij} W_{ij \to kl} (W^{-1})_{kl \to mn} (V^\dagger [\sigma_m \otimes \sigma_n]) \rangle \tag{3.60}$$

For the average not to be zero (mn) should belong to the first set:

$$\sum_{\alpha,\beta=1}^{3} \sum_{(ij)} \sum_{(kl) \in (1)} \sum_{(mn) \in (1)} \frac{1}{3} \delta_{x\beta} \delta_{m\alpha} \Lambda_{\alpha 0 \to ij} W_{ij \to kl} (W^{-1})_{kl \to mn} \sigma_\beta \otimes \sigma_0 \tag{3.61}$$

Thus, the coefficient before $\sigma_x \otimes \sigma_0$ corresponding to the part with the parameter a is M_{11}:

$$M_{11} = \sum_{(ij)} \sum_{(kl) \in (1)} \sum_{(mn) \in (1)} \frac{1}{3} \Lambda_{mn \to ij} W_{ij \to kl} (W^{-1})_{kl \to mn} \tag{3.62}$$

Or in terms of 15 by 15 matrices λ, w and $w^{-1} = w^T$ we can shortly write:

$$M_{11} = \sum_{\gamma} \sum_{\alpha \in (1)} \sum_{\beta \in (1)} \frac{1}{3} \lambda_{\gamma\alpha} w_{\beta\gamma} (w^{-1})_{\alpha\beta} = \sum_{\gamma} \sum_{\alpha \in (1)} \sum_{\beta \in (1)} \frac{1}{3} \lambda_{\gamma\alpha} w_{\beta\gamma} w_{\beta\alpha} \tag{3.63}$$

The index α corresponds to (mn), the index β to (kl), while γ runs through all the values corresponds to (ij). The other eight elements can be found by the same procedure:

$$M_{12} = \sum_{\gamma,\ \alpha \in (1),\ \beta \in (2)} \frac{1}{3} \lambda_{\gamma\alpha} w_{\beta\gamma} w_{\beta\alpha},\ M_{13} = \sum_{\gamma,\ \alpha \in (1),\ \beta \in (3)} \frac{1}{3} \lambda_{\gamma\alpha} w_{\beta\gamma} w_{\beta\alpha} \tag{3.64}$$

$$M_{21} = \sum_{\gamma,\ \alpha \in (2),\ \beta \in (1)} \frac{1}{3} \lambda_{\gamma\alpha} w_{\beta\gamma} w_{\beta\alpha},\ M_{22} = \sum_{\gamma,\ \alpha \in (2),\ \beta \in (2)} \frac{1}{3} \lambda_{\gamma\alpha} w_{\beta\gamma} w_{\beta\alpha} \tag{3.65}$$

$$M_{23} = \sum_{\gamma,\ \alpha \in (2),\ \beta \in (3)} \frac{1}{3} \lambda_{\gamma\alpha} w_{\beta\gamma} w_{\beta\alpha},\ M_{31} = \sum_{\gamma,\ \alpha \in (3),\ \beta \in (1)} \frac{1}{9} \lambda_{\gamma\alpha} w_{\beta\gamma} w_{\beta\alpha}, \tag{3.66}$$

$$M_{32} = \sum_{\gamma,\ \alpha \in (3),\ \beta \in (2)} \frac{1}{9} \lambda_{\gamma\alpha} w_{\beta\gamma} w_{\beta\alpha},\ M_{33} = \frac{1}{9} \sum_{\gamma,\ \alpha \in (3),\ \beta \in (3)} \lambda_{\gamma\alpha} w_{\beta\gamma} w_{\beta\alpha} \tag{3.67}$$

3.4 Zero order approximation

Our main goal is to find the eigenvales of the iteration matrix M, in order to understand the form of the fidelity dependence. To calculate the eigenvalues with the precision of the order of the error of gate W, one may study this matrix in the zero order approximation $\lambda_{\alpha\beta} = \delta_{\alpha\beta}$. So, if we get the two eigenvalues sufficiently less than one, we should conclude that only one exponent in the fidelity expression would be important. In this section we would express zero order $M^{(0)}$ through the local invariants [4] of the gate W_0, which are used in algorithms optimization.

3.4.1 Linear equations for the elements of $M_{ij}^{(0)}$

In the case of $\lambda_{\alpha\beta} = \delta_{\alpha\beta}$, the equations on the matrix elements M are the sum of the squares of the elements w over the appropriate sets. The orthogonality of 15 by 15 matrix w implies that the sum of the squares of the elements in each row and column is equal to 1. Therefore, we obtain six trivial equations on the matrix elements of zero order $M^{(0)}$:

$$M_{11}^{(0)} + M_{21}^{(0)} + 3M_{31}^{(0)} = 1;\ \ M_{12}^{(0)} + M_{22}^{(0)} + 3M_{32}^{(0)} = 1;\ \ M_{13}^{(0)} + M_{23}^{(0)} + 3M_{33}^{(0)} = 3 \tag{3.68}$$

[4] These invariants were introduced here (18), detailed information about them maybe found in the appendix C

Also, we know that there always is the eigenvalue 1 with the eigenvector $(1,1,1)^T$. Which follows from our sequence being unity in the absence of the error.

$$M_{11}^{(0)} + M_{12}^{(0)} + M_{13}^{(0)} = 1; \;\; M_{21}^{(0)} + M_{22}^{(0)} + M_{23}^{(0)} = 1; \;\; M_{31}^{(0)} + M_{32}^{(0)} + M_{33}^{(0)} = 1 \tag{3.69}$$

Actually, only 5 of them are independent.

3.4.2 Invariance under rotations from SU(2)xSU(2)

Here we will show that zero order iteration matrix $M^{(0)}$ is the same for different two qubit gates that can be transformed into each other only with single qubit rotations. Let U_1, U_2 be any rotations from SU(2)xSU(2). If we substitute $W_0 \longrightarrow U_1 W_0 U_2$ instead of W_0 in the initial expression of $\overline{\Lambda}_2$, but with the zero order error approximation, we have:

$$\langle V W_0 \overline{\Lambda} W_0^\dagger V^\dagger \rangle \longrightarrow \langle V (U_1 W_0 U_2) \overline{\Lambda} (U_2^\dagger W_0^\dagger U_1^\dagger) V^\dagger \rangle \tag{3.70}$$

Firstly, we can commute $\overline{\Lambda}$ with U_2 according to the isotropy condition. Secondly, we may change the averaging variable from V to VU_1. It is obvious, if we would average over SU(2) group. But as we will discuss in the appendix B, averaging over C_1 gives the same results as averaging over SU(2). Thus, two-qubit gates which can be transformed from one to another by such multiplication by single-qubit rotations have the same zero order matrices $M^{(0)}$.

On the other hand, it has been shown (18) that such two qubit gates have the same local invariants G_1 and G_2, which are used in the algorithms optimization. These are two numbers G_1 is complex and G_2 is always real, they may be directly calculated for any gate from SU(4). The statement is that different two-qubit gates can be transformed from one to the other via the multiplication by single-qubit gates from both sides if and only if the local invariants G_1, G_2 for each of them coincide. The general information about these invariants are in the appendix C. Thus, zero order matrix $M^{(0)}$ for gate W_0 depends only on the local invariants G_1, G_2 of this gate.

3.4.3 Symmetry under transposition of the qubits

Another property of zero order matrix $M^{(0)}$ is that it does not change if the qubits are transposed. To show that one should notice that the local invariants G_1, G_2 of the gate W_0 and $\mathrm{SWAP}^\dagger \cdot W \cdot \mathrm{SWAP}$ are the same. We checked it straightforwardly in the appendix C. Thus for the matrix elements of the zero order iteration matrix $M^{(0)}$ we have the equations:

$$M_{12}^{(0)} = M_{21}^{(0)}; \;\; M_{11}^{(0)} = M_{22}^{(0)}; \;\; M_{13}^{(0)} = M_{23}^{(0)}; \;\; M_{31}^{(0)} = M_{32}^{(0)}; \tag{3.71}$$

3.4.4 General form of M_0

Collecting all the obtained relations between the matrix elements of zero order $M^{(0)}$, we have the general form:

$$M^{(0)} = \begin{pmatrix} m_1 & m_2 & 1-m_1-m_2 \\ m_2 & m_1 & 1-m_1-m_2 \\ \frac{1-m_1-m_2}{3} & \frac{1-m_1-m_2}{3} & \frac{1+2m_1+2m_2}{3} \end{pmatrix} \tag{3.72}$$

Now, we may express m_1 and m_2 through the parameters G_1, G_2. Firstly, we found the following expressions:

$$m_1 = \frac{2|G_1| + G_2 + 1}{6} \tag{3.73}$$

$$m_2 = \frac{2|G_1| - G_2 + 1}{6} \tag{3.74}$$

In order to check if these expressions work for any gate W_0 from SU(4), it is sufficient to check them for the gates with all possible G_1, G_2. It is known that the gates of the form [5]:

$$W_0 = \exp[i(c_1 \cdot \mathbf{XX} + c_2 \cdot \mathbf{YY} + c_3 \cdot \mathbf{ZZ})], \tag{3.75}$$

by varying $c_1,\ c_2,\ c_3$ run through any possible pair of G_1, G_2. And therefore such gates represent the whole set of gates with different local invariants $G_1,\ G_2$. For the gates of such a form we directly calculated matrix $M^{(0)}$ in terms of $c_1,\ c_2,\ c_3$ by finding 15 by 15 matrix w. We observed that the mentioned expressions of m_1 and m_2 is valid for any $c_1,\ c_2,\ c_3$, thus in general for any gate from SU(4).

3.4.5 Spectrum analysis

Using the explicit form of the zero order iteration matrix $M^{(0)}$, one may find its eigenvalues:

$$1; \ \ m_1 - m_2; \ \ \frac{5m_1 + 5m_2 - 2}{3} \tag{3.76}$$

To further enhance our understanding of the matrix $M^{(0)}$, we plotted numbers ($m1$, $m2$) on the plane for randomly generated[6] gates W_0 from U(4). In the figure 3.2 each point corresponds to some unitary gate. The ordinate is m_2 and the abscissa is m_1. There are four boundaries restricting the area of all possible m_1, m_2. The blue color boundaries $m_1, m_2 \geq 0$ immediately follow from the expressions through matrix elements of w. The red colour boundaries are the conditions $m_1 + m_2 \geq \frac{1}{3}$ for the bottom one and $\sqrt{m_1} + \sqrt{m_2} \leq 1$ for the upper one. In terms of G_1 and G_2 these conditions

[5] Explanation of this fact is in the appendix C

[6] We took matrices of the form exp $(2i\pi H)$, where H is distributed by GUE

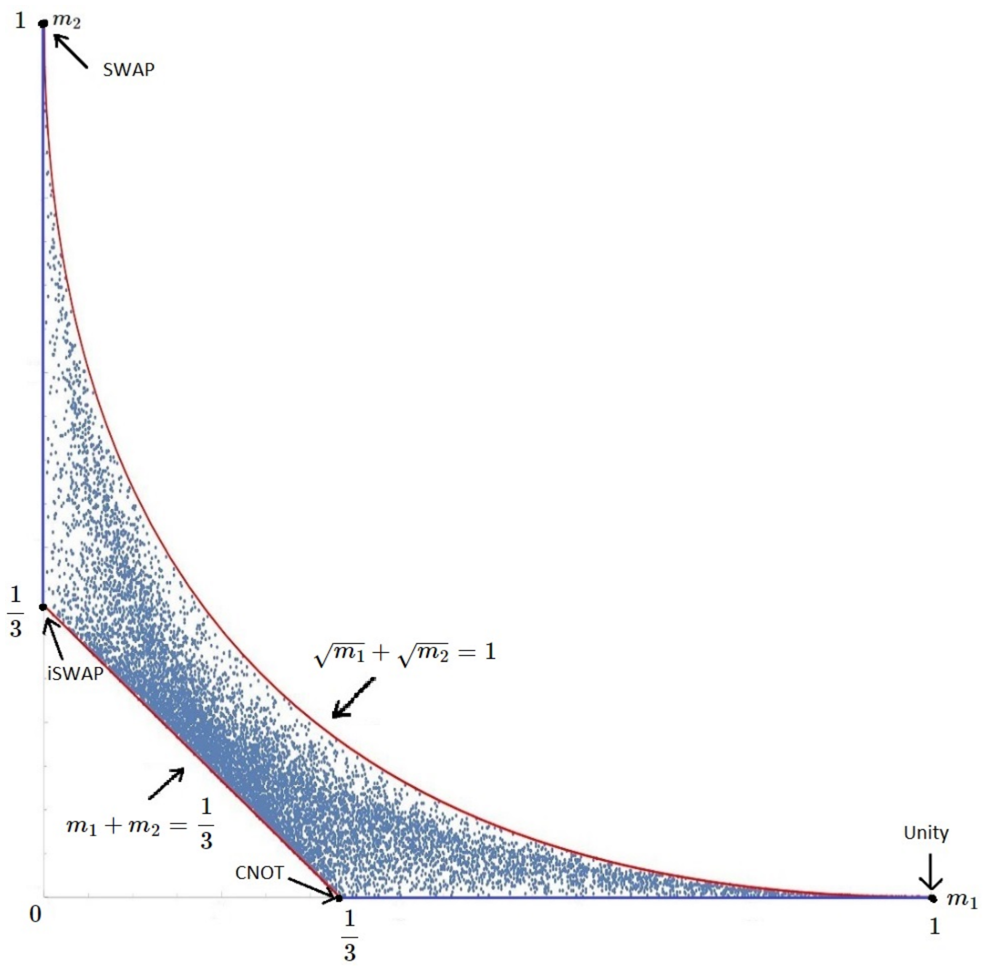

Figure 3.2: Points (m_1, m_2) span this area in the plane

look like $|G_1| \geq 0$ for the lower boundary and $(G_2)^2 + 3 \geq 12|G_1|$ for the upper boundary, while the first one is trivial, the second one is not so obvious and we will prove it in the appendix C. Now having the visual representation of how values m_1 and m_2 are distributed over the plane, we should seek for the cases when there are at least two of them being equal to ± 1. So, we solve $m_1 - m_2 = \pm 1$ or $\frac{5m_1 + 5m_2 - 2}{3} = \pm 1$. The intersections with these lines happen at the points (1,0) which is unity gate and (0,1) which is SWAP gate. Matrix $M^{(0)}$ for unity gate has the spectrum $\{1, 1, 1\}$ and for SWAP gate it has the spectrum $\{1, 1, -1\}$. Any other gate has only one eigenvalue being equal to 1, thus after accounting for the small error correction, one should expect single valuable (close to 1) exponent. The other two would be some numbers less than one to the power of n, thus they will disappear from the graph very quickly.

3.5 Perturbation by the error

Here we will study the corrections to the zero order iteration matrix M^0, which occur due to the error superoperator Λ.

3.5.1 Degenerate case

To begin with we will consider a degenerate case, where Λ is already isotropic over single qubit rotations $C_1 \times C_1$, described by three parameters $a,\ b,\ c$. So, the expression for $\overline{\Lambda_2}$ in this case:

$$\overline{\Lambda}_2 = \langle V\overline{\Lambda}W_0\overline{\Lambda}W_0^\dagger V^\dagger\rangle_V = \overline{\Lambda}\langle VW_0\overline{\Lambda}W_0^\dagger V^\dagger\rangle_V \tag{3.77}$$

Our assumption allowed us to commute V with $\overline{\Lambda}$ and thus decompose matrix M into the two matrices:

$$\begin{pmatrix} 1-\alpha & 0 & 0 \\ 0 & 1-\beta & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix} \cdot \begin{pmatrix} m_1 & m_2 & 1-m_1-m_2 \\ m_2 & m_1 & 1-m_1-m_2 \\ \frac{1-m_1-m_2}{3} & \frac{1-m_1-m_2}{3} & \frac{1+2m_1+2m_2}{3} \end{pmatrix} \tag{3.78}$$

Here we denoted $a = 1-\alpha$, $b = 1-\beta$, $c = 1-\gamma$, where the Greek letters represent small numbers. To use standard perturbation theory, one firstly needs to change the current basis to the eigenbasis of M_0, in the new basis we have the perturbation problem of the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & m_1-m_2 & 0 \\ 0 & 0 & \frac{5m_1+5m_2-2}{3} \end{pmatrix} \tag{3.79}$$

With the perturbation:

$$S^{-1}\mathcal{E}S = \begin{pmatrix} -\frac{\alpha+\beta+3\gamma}{5} & \frac{(m1-m2)(\alpha-\beta)}{5} & \frac{(-2+5m1+5m2)(\alpha+\beta-2\gamma)}{10} \\ \frac{\alpha-\beta}{2} & -\frac{(m_1-m_2)(\alpha+\beta)}{2} & -\frac{(-2+5m_1+5m_2)(\alpha-\beta)}{4} \\ \frac{\alpha+\beta-2\gamma}{5} & \frac{(m_1-m_2)(\alpha-\beta)}{5} & -\frac{(-2+5m_1+5m_2)(3\alpha+3\beta+4\gamma)}{30} \end{pmatrix} \tag{3.80}$$

Where S stands for the transformation matrix. Using known formulas one may obtain the correction to the largest eigenvalue:

$$\mu \approx 1 - \frac{\alpha+\beta+3\gamma}{5} + \frac{(m_1-m_2)(\alpha-\beta)^2}{10(1-m1+m2)} - \frac{3(-2+5m_1+5m_2)(\alpha+\beta-2\gamma)^2}{250(-1+m_1+m_2)} \tag{3.81}$$

So, to the first order it coincides with the single parameter for complete C_2 averaging, however there is a second order correction.

3.5.2 Generic form of Λ

Now let us move on to the general case of arbitrary Λ. We will denote $\epsilon_{\alpha\beta} = \lambda_{\alpha\beta} - \delta_{\alpha\beta}$, where $\epsilon_{\alpha\beta}$ is a 15 by 15 matrix with small elements. From the equations for the matrix elements M, one may obtain the perturbation 3 by 3 matrix $\mathcal{E}$, which expresses through $\epsilon_{\alpha\beta}$. Then we should transform it to the eigenbasis of M_0 and the element (1,1) would be the correction to the main eigenvalue. The transformation matrix is (it does not depend on the values of m_1, m_2):

$$S = \begin{pmatrix} 1 & -1 & -3/2 \\ 1 & 1 & -3/2 \\ 1 & 0 & 1 \end{pmatrix} \quad S^{-1} = \begin{pmatrix} 1/5 & 1/5 & 3/5 \\ -1/2 & 1/2 & 0 \\ -1/5 & -1/5 & 2/5 \end{pmatrix} \tag{3.82}$$

Thus, one needs to find:

$$\frac{1}{5}\begin{pmatrix} 1 & 1 & 3 \end{pmatrix} \mathcal{E} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.83}$$

The factor of 3 in the first vector is well combined with the factor of $\frac{1}{9}$ which stands before the third row elements for the matrix M expressions. All the partial sums over the sets combine into one sum over full 15 by 15 space. We get:

$$\mu \approx 1 + \frac{1}{15}\epsilon_{\gamma\alpha} w_{\beta\gamma} (w^{-1})_{\alpha\beta} = 1 + \frac{1}{15}\mathrm{Tr}(w\epsilon w^{-1}) = 1 + \frac{1}{15}\mathrm{Tr}(\epsilon) = \frac{1}{15}\mathrm{Tr}(\lambda) \tag{3.84}$$

We have for the vector $\boldsymbol{f_n}$:

$$\boldsymbol{f_n} = M^n \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \approx \mu^n \cdot \begin{pmatrix} 1-\epsilon_1 \\ 1-\epsilon_2 \\ 1-\epsilon_3 \end{pmatrix} \tag{3.85}$$

Where small numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ represent the correction to the first eigenvector. We are not interested in them because they would result in the boundary contributions A, B, with the exponential precision we may state:

$$P(n) = A + B\mu^n \tag{3.86}$$

So, in general case the rate of the exponent seen on the fidelity from the length plot matches the rate obtained with complete averaging over the two qubit Clifford group C_2 up to the second order corrections.

Chapter 4

Conclusion

In the second chapter we reviewed main ideas of interleaved randomized benchmarking, such as the averaged over the full Clifford group sequence acts as dilatation of the initial state density matrix.

In the third chapter we studied partial RB of two qubit gates, where we averaged over single qubit rotations, to be precise over the group $C_1 \times C_1$ which is a subgroup of the two qubit Clifford group C_2. Its main difference from the complete averaging is that we are not able to describe the action of the sequence with a single parameter. We showed that in case of $C_1 \times C_1$ we have three independent parameters. Thus, one should expect three exponents in the fidelity dependence $P(n)$. For the close to unity interleaved gate, we explained how one can extract these parameters from the measurable probabilities of the final state.

For the arbitrary interleaved gate we showed that the parameters on the step n are linearly connected with the parameters on the step $n+1$, this connection is described by the iteration matrix M. We expressed the iteration matrix M through the error superoperator and the interleaved gate W. The spectrum of the iteration matrix M determines the rates of the exponents in the fidelity dependence.

For the zero order matrix $M^{(0)}$ (in the absence of any errors) we showed that there are only two independent elements in it. We expressed these elements through the local invariants G_1, G_2 of the interleaved two qubit gate W. Then we showed that for the major part of the gates W (except for the unity gate and the SWAP gate) only single exponent base is close to one, while the others two are clearly smaller than one. Therefore, one should expect a single exponent in the fidelity dependence $P(n)$ for the long enough sequences. Then we studied the eigenvalues of the iteration matrix M via perturbation theory, where the error was the small parameter. We showed that the rate of this exponent is close to that of the complete averaging. The rates coincide up to the second order corrections, however they differ in the higher orders. For the experiment, it means that by more physically accessible method of partial RB, one may obtain the decay rate, which is close to the decay rate of the complete averaging over the two qubit Clifford group. However, the corrections become even larger when the interleaved gate gets closer to the unity gate or SWAP gate. For the latter gates there are three independent exponents with the bases close to one. Thus in the experiment with the unity or SWAP interleaved gate, one should expect three exponents in the fidelity dependence $P(n)$, all three exponents should be accounted for in the data analysis.

Appendix A

Common gates two qubit gates

In the basis of $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ two-qubits gates mentioned in this work are written as:

$$\text{CNOT (CX)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{CZ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{A.1}$$

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{iSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \sqrt{\text{SWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & \frac{1-i}{2} & \frac{1+i}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.2}$$

Appendix B

Clifford group

B.1 2-design

The Clifford group is closely related to RB, because of its property: averaging over it gives the same results as averaging over whole unitary group. This property is called unitary 2-design (20) and means:

$$\frac{1}{|\boldsymbol{C_m}|}\sum_{V_k\in\boldsymbol{C_m}} V_k\Lambda[V_k^\dagger\rho V_k]V_k^\dagger = \int_{\mathrm{SU}(2^m)} d\eta(U)U\Lambda[U^\dagger\rho U]U^\dagger \tag{B.1}$$

Here $\boldsymbol{C_m}$ is the m qubit Clifford group, V_k is a gate from it, U is a gate from $\mathrm{SU}(2^m)$, $d\eta(U)$ is Haar measure of the unitary group, Λ is some superoperator and ρ is a density matrix. It is very useful and important property of the Clifford gruop. We used it to show that M^0 coincides for the gates W and U_1WU_2, where U_1, U_2 are single qubit rotations.

B.2 m qubit Clifford group

Now we will discuss the m qubit Clifford group and its order. Firstly, let us introduce m qubit Pauli set $\boldsymbol{P_m}$ defined as:

$$\boldsymbol{P_m} = \{\sigma_{j_1}\otimes ... \otimes \sigma_{j_m} | j_k = 0, x, y, z\} \tag{B.2}$$

And this set without unity $\boldsymbol{P^*_m}$:

$$\boldsymbol{P^*_m} = \{\sigma_{j_1}\otimes ... \otimes \sigma_{j_m} | j_k = 0, x, y, z\}/\sigma_0^{\otimes m} \tag{B.3}$$

Clifford group of m qubits can then be described as:

$$\boldsymbol{C_m} = \{V \in \mathrm{SU}(2^m) | \sigma \in \pm\boldsymbol{P^*_m} \Rightarrow V\sigma V^\dagger \in \pm\boldsymbol{P^*_m}\} \tag{B.4}$$

In other words it is a group of all possible unitary operators that normalizes Pauli set via conjugation.

B.2.1 Order of $\boldsymbol{C_1}$

Notice that the eigenvalues of each $\sigma \in \pm\boldsymbol{P^*_m}$ has eigenvalues of ± 1 with equal multiplicities. Indeed, consider induction logic, for $\pm\boldsymbol{P^*_1}$ it is evident. Then, if one tensor multiplying by any Pauli matrix (identity or X, Y, Z), the multiplicities are not changed.

To count the order of $\boldsymbol{C_1}$, it is enough to specify where X and Z go under the conjugation by $V \in \boldsymbol{C_1}$. Because the transformation of Y is fixed after we choose the images of X and Z due to the property $Y = iXZ$. Also, the image of σ_x must anti-commute with the image of Z. To sum up, X may go to any element of $\pm\boldsymbol{P^*_1}$, so 6 possibilities. Z may go to $\pm\boldsymbol{P^*_1}/\{\pm VXV^\dagger\}$, so 4 possibilities. We get the order of 24 for $\boldsymbol{C_1}$.

The single qubit Clifford group has a clear geometrical representation, it is isomorphic to the group of the rotations of a three dimensional cube.

B.2.2 Order of $\boldsymbol{C_m}$

Now let us count the order of $\boldsymbol{C_m}$. Define as X_k element from $\boldsymbol{P^*_m}$ with σ_x at k-th position, while the others components are σ_0. Same for Z_k. Notice that we can construct any element of $\boldsymbol{P_m}$ from X_k and Z_k. Thus if we choose the images of X_k and Z_k for every $k \in \{1, 2...m\}$, we immediately fix the image of any other element from $\boldsymbol{P_m}$. Also, every element σ from $\boldsymbol{P^*_m}$ anti-commutes with exactly half of the elements from $\boldsymbol{P_m}$. Indeed, let k be the position where σ has non-unity entry. Every element which anti-commutes with σ may be constructed as follows: firstly, we fill each position apart from k-th and then choose k-th in such way that our constructed element anti-commutes with σ.

Now we are ready to calculate the order of $\boldsymbol{C_m}$. Firstly, we will decide where X_m goes under conjugation by some $U \in \boldsymbol{C_m}$, the image is any element of $\pm\boldsymbol{P^*_m}$. The image of Z_m must anti-commute with our first choice, we know it is half of the elements from $\boldsymbol{P_m}$. So,

$$\left|\pm\boldsymbol{P^*_m}\right| \cdot \frac{2}{2}\boldsymbol{P_m} = 2(4^m - 1)4^m \tag{B.5}$$

We wrote $\frac{2}{2}|\boldsymbol{P_m}|$, because the subset from $\pm\boldsymbol{P^*_m}$ which anti-commutes with the image of X_m has half of the elements from $\boldsymbol{P_m}$, but taken with any sign. Proceeding to smaller indices we have for $|\boldsymbol{C_m}|$:

$$|\boldsymbol{C_m}| = \prod_{k=1}^{m} 2(4^k - 1)4^k \tag{B.6}$$

Which yields for the two qubit Clifford group $|\boldsymbol{C_2}| = 11520$, and even more for the three qubit Clifford group: $|\boldsymbol{C_3}| = 92897280$.

Appendix C

Invariants G_1, G_2

Let W be a two qubit gate, multiplication by single qubit rotations $U_1, U_2 \in$ SU(2)×SU(2) transforms it into $U_1 W U_2$. It turns out that a two qubit gate may be transformed into another two qubit gate using only single qubit rotations if and only if the local invariants G_1, G_2 (defined below) of these two qubit gates are the same. In other words, let W_1 and W_2 be some two qubit gates, one can transform them into each other if and only if their local invariants G_1, G_2 coincide. These local invariants are useful if one needs to construct some two qubit gate out of the given two qubit gate and single qubit rotations. Generally, local invariants G_1, G_2 are used to optimize quantum logic circuits and to study entangling properties of unitary operations.

C.1 Definition

In order to find these invariants for a gate $W \in$ SU(4), one should perform the following steps:

1) Transform W into Bell's basis, $W_B = Q^\dagger W Q$, where Q is:

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & i & -1 & 0 \\ 1 & 0 & 0 & -i \end{pmatrix} \tag{C.1}$$

2) Find the following operator:

$$\mathrm{w} = W_B^T W_B \tag{C.2}$$

3) Calculate the invariants:

$$G_1 = \frac{\mathrm{Tr}^2 \mathrm{w}}{16}; \;\; G_2 = \frac{\mathrm{Tr}^2 \mathrm{w} - \mathrm{Tr}(\mathrm{w}^2)}{4} \tag{C.3}$$

C.2 Invariance under qubits transposition

Now we will show that these local invariants coincide before and after transposition of the qubits. Let us compare the parameters G_1, G_2 for the two cases. One should remember that the SWAP gate is real, symmetric, orthogonal matrix. For the gate W and the gate SWAP$^\dagger \cdot W \cdot$ SWAP:

$$\mathrm{Tr}(\mathrm{w}) = \mathrm{Tr}(W_B^T W_B) = \mathrm{Tr}(Q^T W^T (Q^\dagger)^T Q^\dagger W Q) =$$

$$= \mathrm{Tr}(QQ^T W^T (QQ^T)^\dagger W) = \mathrm{Tr}(PW^T PW) \tag{C.4}$$

$$\mathrm{Tr}(\mathrm{w}^2) = \mathrm{Tr}(W_B^T W_B W_B^T W_B) =$$

$$= \mathrm{Tr}(Q^T W^T (Q^\dagger)^T Q^\dagger W Q Q^T W^T (Q^\dagger)^T Q^\dagger W Q) = \mathrm{Tr}(PW^T PW PW^T PW) \tag{C.5}$$

Here we denoted $P = QQ^T$. It can be directly seen that $P \cdot$ SWAP $=$ SWAP $\cdot\, P$. Due to this commutative relation, the replacement of W with SWAP$^\dagger \cdot W \cdot$ SWAP, preserves the values Tr(w) and Tr(w^2). Therefore, the parameters G_1, G_2 remain unchanged.

C.3 Set of gates with different local invariants

Now we will guide which values the local invariants G_1, G_2 may have. Notice that the parameters G_1, G_2 are strictly determined by the spectrum of w. Considering that det(w) $= 1$, define its spectrum as:

$$\{e^{i\phi_1},\ e^{i\phi_2},\ e^{i\phi_3},\ e^{-i\phi_1 - i\phi_2 - i\phi_3}\} \tag{C.6}$$

The phases ϕ_1, ϕ_2, ϕ_3 are independent and by varying them over the period, we would get any possible G_1, G_2. Which are expressed in terms of the phases as follows:

$$G_1 = \frac{1}{16}(e^{i\phi_1} + e^{i\phi_2} + e^{i\phi_3} + e^{-i\phi_1 - i\phi_2 - i\phi_3})^2 \tag{C.7}$$

$$G_2 = \cos(\phi_1 + \phi_2) + \cos(\phi_2 + \phi_3) + \cos(\phi_3 + \phi_1) \tag{C.8}$$

Now let us compare it with the G_1, G_2 for the gates of the form:

$$W = \exp[\frac{i}{4}(c_1 \cdot \mathbf{XX} + c_2 \cdot \mathbf{YY} + c_3 \cdot \mathbf{ZZ})] \tag{C.9}$$

We have for the invariants:

$$G_1 = \frac{1}{16} e^{-i(c_1+c_2+c_3)} (e^{i(c_1+c_2)} + e^{i(c_2+c_3)} + e^{i(c_3+c_1)} + 1)^2 \tag{C.10}$$

$$G_2 = \cos(c_1) + \cos(c_2) + \cos(c_3) \tag{C.11}$$

It can be straightforwardly checked that by the injective correspondence: $c_1 = \phi_1 + \phi_2, \;\; c_2 = \phi_2 + \phi_3, \;\; c_3 = \phi_3 + \phi_1$, the expressions of G_1, G_2 through c_1, c_2, c_3 transform into the expressions through the phases and vice versa. Which shows that by varying numbers c_1, c_2, c_3, one may obtain any possible G_1, G_2.

C.4 The upper boundary

The inequality for the upper boundary on the (m_1, m_2) plot (Figure 3.2):

$$\sqrt{m_1} + \sqrt{m_2} \leq 1 \Leftrightarrow (G_2)^2 + 3 \geq 12|G_1|\,, \tag{C.12}$$

rewrites in terms of c_1, c_2, c_3 as:

$$\cos^2 c_1 + \cos^2 c_2 + \cos^2 c_3 - \cos c_1 \cos c_2 - \cos c_1 \cos c_3 - \cos c_2 \cos c_3 \geq 0 \tag{C.13}$$

The quadratic form corresponding to this polynomial has the eigenvalues $\{3/2, 3/2, 0\}$. The zero mode $c_1 = c_2 = c_3$ defines the upper boundary.

Appendix D

Zero order $M^{(0)}$ for common gates

The gate	The matrix $M^{(0)}$ or m_1, m_2	The spectrum of $M^{(0)}$
Unity	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\{1,\ 1,\ 1\}$
SWAP	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\{1,\ 1,\ -1\}$
CNOT, CZ	$\begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1/3 & 2/3 \\ 2/9 & 2/9 & 5/9 \end{pmatrix}$	$\{1,\ \frac{1}{3},\ -\frac{1}{9}\}$
iSWAP	$\begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/3 & 0 & 2/3 \\ 2/9 & 2/9 & 5/9 \end{pmatrix}$	$\{1,\ -\frac{1}{3},\ -\frac{1}{9}\}$
$\sqrt{\text{SWAP}}$	$\begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/6 & 1/6 & 2/9 \end{pmatrix}$	$\{1,\ \frac{1}{6},\ 0\}$
$\exp\left[\frac{i}{4}(\sigma_y^1\sigma_y^2)t\right]$	$\frac{2+\cos t}{3},\ 0$	$\{1,\ \frac{2+\cos t}{3},\ \frac{4+5\cos t}{9}\}$
$\exp\left[\frac{i}{4}(\sigma_x^1\sigma_x^2+\sigma_y^1\sigma_y^2)t\right]$	$\frac{1}{6}\cos^2\frac{t}{2}\cdot(5+\cos t),\ \frac{1}{3}\sin^4\frac{t}{2}$	$\{1,\ \frac{1+2\cos t}{3},\ \frac{11+20\cos t+5\cos(2t)}{36}\}$
$\exp\left[\frac{i}{4}(\sigma_x^1\sigma_x^2+\sigma_y^1\sigma_y^2+\sigma_z^1\sigma_z^2)t\right]$	$\cos^4\frac{t}{2},\ \sin^4\frac{t}{2}$	$\{1,\ \cos t,\ \frac{7+5\cos(2t)}{12}\}$

Table D.1: Information about the matrices $M^{(0)}$ for the common gates

Appendix E

Some calculations

E.1 Examples of unity interleaved gate

E.1.1 Qubit relaxation

In Bloch representation action on the density matrix reads as:

$$W[\rho] : \boldsymbol{n} \to W\boldsymbol{n} \tag{E.1}$$

Here W is diagonal matrix with $(e_2,\ e_2,\ e_1)$ on the diagonal correspondingly[2]. According to the isomorphism of the groups $SU(2)$ and $SO(3)$ action on the density matrix by $V^{\dagger}\rho\, V$ means the rotation $\boldsymbol{n} \to O\boldsymbol{n}$, where O is an orthogonal matrix. Then:

$$\overline{W}[\rho] : \boldsymbol{n} \to \int O^T W O\boldsymbol{n}\, dO \tag{E.2}$$

As we showed earlier:

$$\overline{W}[\rho] = a \cdot \rho + \frac{1-a}{2} \cdot \hat{I} \tag{E.3}$$

For $\boldsymbol{n}$ it means $\boldsymbol{n} \to a \cdot \boldsymbol{n}$. It yields:

$$\int O^T W O\boldsymbol{n}\, dO = a \cdot \boldsymbol{n} \tag{E.4}$$

Multiple by $\boldsymbol{n}$ (scalar product):

$$\int (O\boldsymbol{n}, WO\boldsymbol{n})\, dO = a \cdot |\boldsymbol{n}|^2 \tag{E.5}$$

Define $\boldsymbol{k} = \boldsymbol{n}/|\boldsymbol{n}|$:

$$\int (O\boldsymbol{k}, WO\boldsymbol{k})\, dO = a \tag{E.6}$$

Under the integral O runs through the whole $SO(3)$ group, therefore vector $O\boldsymbol{k}$ follows the

[2]We are working in interaction picture, there is not trivial phase evolution, but only evolution connected with the interaction with the bath.

unity sphere. We can rewrite our integral:

$$\int (\boldsymbol{k}, W\boldsymbol{k})\,\frac{d\Omega}{4\pi} = a \tag{E.7}$$

$$\int (e_2(k_x^2 + k_y^2) + e_1 k_z^2)\,\frac{d\Omega}{4\pi} = a \tag{E.8}$$

$$\frac{2}{3}e_2 + \frac{1}{3}e_1 = a \tag{E.9}$$

E.1.2 Noise along Z axis

To find a, let us look at $\overline{W}[\sigma_x] = a \cdot \sigma_x$:

$$\overline{W}[\sigma_x] = \langle (VWV^\dagger)[\sigma_x] \rangle_V =$$

$$\langle \mathcal{V}_{\alpha x}(WV^\dagger)[\sigma_\alpha] \rangle_V = \langle \mathcal{V}_{\alpha x} V^\dagger \exp{(i\frac{\phi}{2}\sigma_z)} \sigma_\alpha \exp{(-i\frac{\phi}{2}\sigma_z)} V \rangle_V =$$

$$= \langle V^\dagger (\mathcal{V}_{xx}(\cos\phi\,\sigma_x + \sin\phi\,\sigma_y) + \mathcal{V}_{yx}(\cos\phi\,\sigma_y - \sin\phi\,\sigma_x) + \mathcal{V}_{zx}\sigma_z) V \rangle_V =$$

$$= \langle \mathcal{V}_{xx}(\mathcal{V}^{-1}_{\gamma x}\cos\phi\,\sigma_\gamma + \mathcal{V}^{-1}_{\gamma y}\sin\phi\,\sigma_\gamma) + \mathcal{V}_{yx}(\mathcal{V}^{-1}_{\gamma y}\cos\phi\,\sigma_\gamma - \mathcal{V}^{-1}_{\gamma x}\sin\phi\,\sigma_\gamma) +$$

$$+ \mathcal{V}_{zx}\mathcal{V}^{-1}_{\gamma z}\sigma_\gamma) \rangle_V = \tag{E.10}$$

Using the averages:

$$\langle \mathcal{V}_{\alpha\beta}\mathcal{V}^{-1}_{\gamma\delta} \rangle = \frac{1}{3}\delta_{\alpha\delta}\delta_{\beta\gamma} \tag{E.11}$$

We get:

$$= \frac{2\cos\phi + 1}{3}\sigma_x \tag{E.12}$$

It yields:

$$a = \int d\phi P(\phi) \frac{2\cos\phi + 1}{3} = \frac{2e^{-\frac{A}{2}} + 1}{3} \tag{E.13}$$

E.2 $C_1 \times C_1$ averaging

To show the presence of three parameters we will pick different gates U from $C_1 \times C_1$ and by solving the equations which follow from isotropy condition, we will limit the amount of parameters describing the action of $\overline{W}$. Also, we will use linear independence of $\sigma_i \otimes \sigma_j$, where $\{i, j\} \in \{0, x, y, z\}$ and shorter notation by omitting $\sigma_i \otimes \sigma_j$ in the equations. Firstly, let us look at:

$$\overline{W}[\sigma_x \otimes \sigma_0] = s_\alpha \cdot \sigma_\alpha \otimes \sigma_0 + p_\beta \cdot \sigma_0 \otimes \sigma_\beta + t_{\alpha\beta} \cdot \sigma_\alpha \otimes \sigma_\beta \tag{E.14}$$

The equations representing the isotropy condition for various U:

1) $U = \sigma_x \otimes \sigma_0$:

$$s_\alpha + p_\beta + t_{\alpha\beta} = s_x - s_y - s_z + p_\beta + t_{x\beta} - t_{y\beta} - t_{z\beta} \Rightarrow s_y = s_z = t_{y\beta} = t_{z\beta} = 0 \tag{E.15}$$

2) $U = \sigma_0 \otimes \sigma_x$:

$$s_x + p_\beta + t_{x\beta} = s_x + p_x - p_y - p_z + t_{xx} - t_{xy} - t_{xz} \Rightarrow p_y = p_z = t_{xy} = t_{xz} = 0 \tag{E.16}$$

3) $U = \sigma_y \otimes \sigma_0$:

$$s_x - p_x - t_{xx} = s_x + p_x - t_{xx} \Rightarrow p_x = 0 \tag{E.17}$$

4) $U = \sigma_0 \otimes \sigma_y$:

$$s_x + t_{xx} = s_x - t_{xx} \Rightarrow t_{xx} = 0 \tag{E.18}$$

So, we showed:

$$\overline{W}[\sigma_x \otimes \sigma_0] = a_x \cdot \sigma_x \otimes \sigma_0 \tag{E.19}$$

By the same logic, we can get:

$$\overline{W}[\sigma_y \otimes \sigma_0] = a_y \cdot \sigma_y \otimes \sigma_0, \quad \overline{W}[\sigma_z \otimes \sigma_0] = a_z \cdot \sigma_z \otimes \sigma_0 \tag{E.20}$$

It can be seen that $a_x = a_y = a_z$ by considering $U = \exp\left(-i\sigma_y \frac{\pi}{4}\right) \otimes \sigma_0$ and $U = \exp\left(-i\sigma_z \frac{\pi}{4}\right) \otimes \sigma_0$. For example, take a look at:

$$\overline{W}[n_x\sigma_x \otimes \sigma_0 + n_y\sigma_y \otimes \sigma_0] = a_x n_x \cdot \sigma_x \otimes \sigma_0 + a_y n_y \cdot \sigma_y \otimes \sigma_0 \tag{E.21}$$

Expanding for $U = \exp\left(-i\sigma_z \frac{\pi}{4}\right) \otimes \sigma_0$, we have:

$$a_y \cdot n_x\sigma_y \otimes \sigma_0 - a_x \cdot n_y\sigma_x \otimes \sigma_0 = a_x \cdot n_x\sigma_y \otimes \sigma_0 - a_y \cdot n_y\sigma_y \otimes \sigma_0 \Rightarrow a_x = a_y \tag{E.22}$$

So, we get:

$$\overline{W}[n_\alpha \cdot \sigma_\alpha \otimes \sigma_0] = a \cdot n_\alpha \cdot \sigma_\alpha \otimes \sigma_0 \tag{E.23}$$

Since, our group $C_1 \times C_1$ is symmetric under transposition of the qubits, we can immediately conclude:

$$\overline{W}[m_\beta \cdot \sigma_0 \otimes \sigma_\beta] = b \cdot m_\beta \cdot \sigma_0 \otimes \sigma_\beta \tag{E.24}$$

Now, let us manage with $\overline{W}[\sigma_x \otimes \sigma_y]$, again the general form:

$$\overline{W}[\sigma_x \otimes \sigma_y] = s_\alpha \cdot \sigma_\alpha \otimes \sigma_0 + p_\beta \cdot \sigma_0 \otimes \sigma_\beta + t_{\alpha\beta} \cdot \sigma_\alpha \otimes \sigma_\beta \tag{E.25}$$

1) $U = \sigma_x \otimes \sigma_0$:

$$s_\alpha + p_\beta + t_{\alpha\beta} = s_x - s_y - s_z + p_\beta + t_{x\beta} - t_{y\beta} - t_{z\beta} \Rightarrow s_y = s_z = t_{y\beta} = t_{z\beta} = 0 \tag{E.26}$$

2) $U = \sigma_0 \otimes \sigma_y$:

$$s_x + p_\beta + t_{x\beta} = s_x - p_x + p_y - p_z - t_{xx} + t_{xy} - t_{xz} \Rightarrow p_x = p_z = t_{xx} = t_{xz} = 0 \tag{E.27}$$

3) $U = \sigma_y \otimes \sigma_0$:

$$-s_x - p_y - t_{xy} = -s_x + p_y - t_{xy} \Rightarrow m_y = 0 \tag{E.28}$$

4) $U = \sigma_0 \otimes \sigma_x$:

$$-s_x - t_{xy} = s_x - t_{xy} \Rightarrow n_x = 0 \tag{E.29}$$

To sum up:

$$\overline{W}[\sigma_x \otimes \sigma_y] = c \cdot \sigma_x \otimes \sigma_y \tag{E.30}$$

If we take $U = \exp\left(-i\sigma_z \frac{\pi}{4}\right) \otimes \sigma_0$, we will show that c is the same for $\sigma_x \otimes \sigma_y$ and $\sigma_y \otimes \sigma_y$. Also, we can take $U = \sigma_0 \otimes \exp\left(-i\sigma_z \frac{\pi}{4}\right)$ and get that c is the same for $\sigma_x \otimes \sigma_x$ and $\sigma_x \otimes \sigma_y$. Finally, by taking $U = \exp\left(-i\sigma_y \frac{\pi}{4}\right) \otimes \sigma_0$, we will mix $\sigma_x \otimes \sigma_y$ and $\sigma_z \otimes \sigma_y$. Same can be done to mix $\sigma_x \otimes \sigma_x$ and $\sigma_x \otimes \sigma_z$, therefore:

$$\overline{W}[k_{\alpha\beta} \cdot \sigma_\alpha \otimes \sigma_\beta] = c \cdot k_{\alpha\beta} \cdot \sigma_\alpha \otimes \sigma_\beta \tag{E.31}$$

And that is all we could obtain, the group $C_1 \times C_1$ does not allow us to further mix the sets $\sigma_\alpha \otimes \sigma_0,\ \sigma_0 \otimes \sigma_\beta$ and $\sigma_\alpha \otimes \sigma_\beta$, in order to do that, we need a bigger group - C_2 as we already know.

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