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MASTER'S THESIS

Advection of Passive Scalar in the Coherent Columnar Vortex Flow

Master's Educational Program: Mathematical and Theoretical Physics

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МАГИСТЕРСКАЯ ДИССЕРТАЦИЯ

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Advection of Passive Scalar

in the Coherent Columnar Vortex Flow

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Abstract

A coherent columnar vortex is a large-scale statistically stable structure that emerges in hydrodynamic turbulence of the three-dimensional fast-rotating systems. It is characterized by strong shear with weak turbulent fluctuations in its background. We study the statistical properties of a passive scalar field, which is carried by such type of flow.

We show that in this system, turbulent fluctuations can be considered as white noise added to the strong mean shear. To study the statistics of scalar evolution, we solve a simplified model of scalar blob mixing in two dimensions, which quantitatively describes the picture in the cross-section of the vortex. As a result, it has been found how the distribution behaves at large times for different regions. We show a change in the dynamics regime to exponential decay for the scalar quantity at the origin and anisotropic algebraic spatial dependence for the long distances.

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Chapter 1

Introduction

1.1 Coherent columnar vortices

The properties of three-dimensional turbulent flows change significantly when rotation comes into play, like it was shown in a recent review of experimental and numerical results [12]. In case of fast rotating system, following the Taylor-Proudman theorem [22], the velocity field along the rotation axis uncouples from the perpendicular planar flow, thus becomes effectively two-dimensional. This fact makes it possible to form an inverse energy cascade - a distinctive feature of hydrodynamic turbulence in 2D[18]. The kinetic energy transfers to larger scales up to the termination point, where the cascade flux is compensated by system's energy damping mechanisms. If the latter is small enough, then the inverse cascade can lead to the formation of large-scale columnar vortices, which velocity remains statistically stable over time over multiple turnover periods. One can establish a parallel between these coherent structures spontaneously forming from small turblulent pulsations and the processes of a vapor tending to gather into a larger droplet on a surface, which explains usage of such term as "condensate".

The columnar vortices formation has been observed both in numerical simulations (see, e.g. [4, 25]) and experiments [27]. There is a novel analytical theory [16] describing such flow as a coherent stationary solution for fast-rotating system, which is characterised by the limit of small Rossby number $\mathrm{Ro}_R = \sqrt{\epsilon/\nu}/\left(2\Omega\right) \ll 1$ and small-scale energy pumping on such k_f^{-1} that $k_f L \gg (\mathrm{Ro}_R)^{-1}$, where Ω is the angular velocity of the external rotation, ν is the kinematic viscosity of medium, ϵ is the flux of energy injection into the system per unit mass and L is a characteristic scale of the columnar vortex size. One of theory's important results is the log-linear radial profile for a formed vortex's velocity. That can be achieved in two-dimensional flow as well, in regime of viscous dissipation as a dominant energy damping mechanism. The latter was confirmed in direct numerical simulation [10].

There is a recent extension of the theory in a work [21] applied to the cyclones and anticyclones. It was shown that symmetry between them is present only for the fast rotation case; as the external Ω decreases, the asymetry appears: in cyclones the maximum velocity value is greater and its radial position is closer to the axis of the vortex in comparison with anticyclones.

1.2 Passive scalar

Advection of scalar field in smooth flow (so-called Batchelor regime) was first studied in [3] for some dynamically passive (considering no feedback on a flow itself), quantity θ like temperature or concentration of tracer. R. Kraichnan extended it to the limit of short-correlated flow in [19], and after that passive scalar model was developed into theory describing θ, which is an observable quantity in experiments, for any statistics and temporal correlations of the random hydrodynamic flow [11]. An advection in velocity field and diffusion mechanisms are both considered for its dynamics according to the equation:

$$\partial_t \theta + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \theta - \varkappa \boldsymbol{\nabla}^2 \theta = \phi \tag{1.1}$$

where ϕ is the source injecting tracer into the system and $\varkappa$ is the diffusion coefficient.

Theoretical analysis of the so-called decay problem, when $\phi = 0$, for Batchelor-regime turbulence were presented in [6, 9]. There was used Kraichnan's model to examine the passive scalar pair correlation function asymptotic behaviour for different regions of the flow. For scales smaller than characteristic viscous length, the corresponding passive scalar correlation length is an exponentially increasing function of time with characteristic Lyapunov exponent λ.

The initial stage at which the scalar correlation length l is smaller than the viscous scale can be qualitatively characterized as follows [29]. Consider an evolution in distribution of blobs of size l. When the velocity field has a nonzero gradient, the blob expands and contracts exponentially in time along certain asymptotes. Diffusion counterbalances contraction as the blob size reduces to diffusive scale $r \sim r_d = \sqrt{\varkappa/\lambda}$ in the contracting direction, starting to smear it out, whereas the blob size in other directions is not affected by diffusion. At the same time, the overall initial scalar distribution $\int d\boldsymbol{r}\, \theta_0$ is conserved in mixing governed by (1.1).

In [1], mean values of a scalar field in isotropic turbulence were calculated by finding the optimal velocity fluctuation intensity. That optimal scalar fluctuation turned out to be different for moments of different orders, which implies strong intermittency of the scalar field. In [29] it is demonstrated that the pair correlation functions calculated in [9] in case of the delta-correlated $\boldsymbol{v}$ field remain qualitatively correct for a velocity, which is arbitrarily correlated in time. There were also analyzed correlation functions of higher order, because knowledge of the pair correlation function is not sufficient for characterizing the scalar distribution in space due to its intermittency.

The passive scalar theory may be generalized, allowing to study such passive vectors in incompressible flows as gradient of a tracer $\boldsymbol{\omega} = \boldsymbol{\nabla}\theta$ and the divergenceless magnetic field evolving in incompressible flow, which are governed by the following equations [11]:

$$\partial_t \boldsymbol{\omega} + \boldsymbol{\nabla}\left(\boldsymbol{v}\cdot\boldsymbol{\omega}\right) = \varkappa \boldsymbol{\nabla}^2 \boldsymbol{\omega} \tag{1.2}$$

$$\partial_t \boldsymbol{B} + \left(\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{B} - \left(\boldsymbol{B}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v} = \varkappa \boldsymbol{\nabla}^2 \boldsymbol{B} \tag{1.3}$$

For the latter one, there was analyzed the case of the kinematic dynamo in a conducting fluid (plasma) where the stationary shear flow is accompanied by relatively weak random velocity fluctuations [13][14].

1.3 The statement of the problem

In this work, we consider scalar field θ described by (1.1) which is transferred inside the coherent vortex. Such flow has strong constant shear component from azimuthal vortex rotation, which should be compared with the stochastic contribution from the velocity turbulent pulsations. We study the so-called decay problem, desribing the evolution of initial scalar distribution in absence of external sources.

Our goal is to study quantitatively the behavior of the solution averaged over the pulsations. First we make general conclusions about their statistics within the framework of the existing theoretical model [16], which allows us to move on to passive scalar consideration in the general shear flow with weak white noise. We aim to obtain analytical expressions for $\langle\theta\rangle$, indicating how the presence of noise influence the dynamics in contrast to constant shear flow case. In order to do that we utilize the ansatz of the solution, which reduces our problem to study the statistics of spatial distribution parameters. Adhering the treatment of similar dynamic system that describes the divergence of the Lagrangian trajectiories in that flow [8], we derive in our model asymptotics at the large times.

1.4 Passive scalar evolution in the shear flow

Consider the equation (1.1) in a steady coherent vortex flow. The characteristic scale of the scalar distribution in space is assumed to be much smaller than the characteristic scale of the turbulent pulsations. This means, first, that the statistics of the scalar is homogeneous in space on scales less than the vortex radius. Second, the velocity field of the turbulent pulsations $\boldsymbol{u}$ can be considered to be smooth so one can approximate the spatial velocity profile to be linear in space, $u_i = \sigma_{ij} r_j$. In this case the dynamics of advection in the reference frame associated with fluid element moving with the vortex is governed by equation:

$$\partial_t\theta\left(\boldsymbol{r},t\right)+\hat{\Sigma}(t)\boldsymbol{r}\cdot\partial_{\boldsymbol{r}}\theta\left(\boldsymbol{r},t\right)-\varkappa\partial_{\boldsymbol{r}}^2\theta\left(\boldsymbol{r},t\right)=\phi\left(\boldsymbol{r},t\right) \tag{1.4}$$

where elements of traceless matrix $\hat{\Sigma}$ describe the gradients of the flow: $\Sigma_{ij}=\partial_j v_i$.

Retarded Green function for the equation (1.4) can be found formally in a spatial Fourier domain as a solution of the following problem:

$$G\left(t,\boldsymbol{q}|\boldsymbol{r}_0\right)=\mathcal{F}\left[G\left(t,\boldsymbol{r}\right)\right]\left(\boldsymbol{q}\right),\ \begin{cases}\partial_t G\left(t,\boldsymbol{q}\right)-\boldsymbol{q}^T\hat{\Sigma}\left(t\right)\partial_{\boldsymbol{q}}G+\varkappa\boldsymbol{q}^2G\left(t,\boldsymbol{q}\right)=0,\ t>0\\ G\left(0,\boldsymbol{q}\right)=e^{-i\boldsymbol{q}\cdot\boldsymbol{r}_0}\end{cases} \tag{1.5}$$

This can be treated with changing the wave vector to the characteristics $\boldsymbol{k}$ by a special transformation

$$k_i=W\left(t,t_0\right)_{ji}q_j, \tag{1.6}$$

where we introduce chronologically ordered matrix exponent

$$\hat{W}\left(t,t_0\right)=\mathrm{T}\exp\left[\int_{t_0}^{t}dt'\hat{\Sigma}\left(t'\right)\right]. \tag{1.7}$$

In this way, the equation on characteristics takes the following expression:

$$\frac{d}{dt}G\left(t,\boldsymbol{k}|\boldsymbol{r}_0\right)+\varkappa\left(\boldsymbol{k}^T\hat{W}^{-1}\left(t,t_0\right)\right)^2G\left(t,\boldsymbol{k}|\boldsymbol{r}_0\right)=0,$$

that one arrives that the following answer for $G\left(t,\boldsymbol{q}\right)$:

$$G\left(t,\boldsymbol{q}|\boldsymbol{r}_0\right)=\exp\left[-i\boldsymbol{q}\hat{W}\left(t,0\right)\boldsymbol{r}_0-\varkappa\int_0^t d\tau\left(\boldsymbol{q}^T\hat{W}\left(t,\tau\right)\right)^2\right] \tag{1.8}$$

Considering the case of coherent vortex's steady shear and ignoring the turbulent pulsations in its background, so that

$$\Sigma_{ij}(t)\to S_{ij}\equiv S\delta_{ix}\delta_{jy},$$

one obtains the expression of the diffusion gaussian form in coordinate domain:

$$\begin{gathered}G\left(t,\boldsymbol{r},\boldsymbol{r}_0\right)=\left[\det\left(4\pi\varkappa\int_0^t d\tau e^{\hat{S}(t-\tau)}e^{\hat{S}^T(t-\tau)}\right)\right]^{-1/2}\times\\ \times\exp\left[-\left(\boldsymbol{r}-e^{\hat{S}t}\boldsymbol{r}_0\right)^T\left(4\varkappa\int_0^t d\tau\hat{W}\left(t,\tau\right)\hat{W}^T\left(t,\tau\right)\right)^{-1}\left(\boldsymbol{r}-e^{\hat{S}t}\boldsymbol{r}_0\right)\right]\end{gathered} \tag{1.9}$$

To study how field $\theta(t,\boldsymbol{r})$ is advected in such type of flow, let's consider the evolution of the axisymmetric blob - an initial state of gauss spatial profile with the characteristic scale L:

$$\theta(t,\boldsymbol{r}) = \int d^3\boldsymbol{r}' G(t,\boldsymbol{r},\boldsymbol{r}') \frac{\exp\left[-(r'/L)^2/2\right]}{(2\pi L)^{3/2}} =$$
$$= \sqrt{\frac{3}{(2\pi)^3 (L^2+2\varkappa t)\left[2\varkappa t L^2\left((St)^2+6\right)+3L^4+(\varkappa t)^2\left((St)^2+12\right)\right]}} \times$$
$$\times \exp\left\{-\frac{3L^2\left[(x-Sty)^2+y^2\right]+2\varkappa t\left[3x^2-3Stxy+\left((St)^2+3\right)y^2\right]}{4\varkappa t L^2\left((St)^2+6\right)+6L^4+2(\varkappa t)^2\left((St)^2+12\right)} - \frac{z^2}{2L^2+4\varkappa t}\right\} \tag{1.10}$$

First let us outline that perpendicular to shear component z becomes uncoupled in this model. To study how flow stretches the blob, one need to find dependence of eigenvalues of the quadratic form matrix. Let's introduce them dimensionless terms: $(Ll_+)^{-2}$, $(Ll_-)^{-2}$ as functions of $\tau = t/S$. The weakness of thermal diffusion mechanism is manifested in parameter $\alpha = SL^2/2\varkappa \gg 1$. The corresponding characteristic equation on $l = l_\pm$:

$$l^4 - \left(2+\frac{\tau^3}{3\alpha}\right)l^2 + 1 + \frac{\tau^3}{3\alpha} + \frac{\tau^4}{12\alpha^2} + \frac{\tau^2}{\alpha^2} + \frac{2\tau}{\alpha}\left(1-l^2\right) = 0 \tag{1.11}$$

Neglecting the last small terms, one can find approximate asymptotic behavior:

$$\begin{aligned} &1 \ll \tau \ll \alpha^{1/3}: \quad l_+ \sim t, \quad l_- \sim t^{-1}; \\ &\alpha^{1/3} \ll t \ll \alpha: \quad l_+ \sim t, \quad l_- \sim \sqrt{\tau/\alpha}; \\ &\tau \gg \alpha: \quad l_+ \sim \tau^{3/2}\alpha^{-1/2}, \quad l_- \sim \sqrt{\tau/\alpha}. \end{aligned} \tag{1.12}$$

We see that presence of weak $\varkappa$ mechanism at first stops the contraction in specified direction and then begins to expand the blob diffusively.

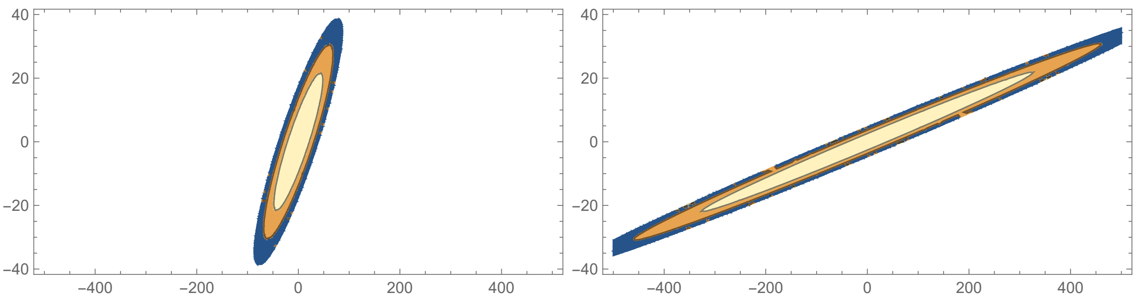

Figure 1.1: Illustration of the initial blob mixing in a constant shear flow. Contours are lines of scalar constant quantity, axes units are L. Evolution time is $2S^{-1}$ for the left picture and $15S^{-1}$ for the right one. Parameter: $\alpha = 1000$.

1.5 Reduction of the pair correlation function problem in space homogeneous case

Consider the equation on the pair correlation function $F(t, \boldsymbol{r}_1, \boldsymbol{r}_2) = \langle \theta(t, \boldsymbol{r}_1)\, \theta(t, \boldsymbol{r}_2) \rangle$ which is governed by the following differential equation in absence of sources:

$$\left[\partial_t + \left(\hat{\Sigma}(t)\boldsymbol{r}_1 \cdot \boldsymbol{\nabla}_{(1)}\right) + \left(\hat{\Sigma}(t)\boldsymbol{r}_2 \cdot \boldsymbol{\nabla}_{(2)}\right) - \varkappa \boldsymbol{\nabla}^2_{(1)} - \varkappa \boldsymbol{\nabla}^2_{(2)}\right] F_2\left(\boldsymbol{r}_1, \boldsymbol{r}_2, t\right) = 0, \qquad (1.13)$$

We transform variables by going to the inertia center's coordinate $\boldsymbol{R}$ and radius-vector $\boldsymbol{r}$ connecting these two particles:

$$\begin{cases} \boldsymbol{r}_1 = \boldsymbol{R} + \boldsymbol{r}/2 \\ \boldsymbol{r}_2 = \boldsymbol{R} - \boldsymbol{r}/2 \end{cases} \Leftrightarrow \quad \begin{cases} \boldsymbol{R} = \left(\boldsymbol{r}_1 + \boldsymbol{r}_2\right)/2 \\ \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \end{cases}, \begin{cases} \boldsymbol{\nabla}_{(1)} = \frac{1}{2}\boldsymbol{\nabla}_{\boldsymbol{R}} + \boldsymbol{\nabla}_{\boldsymbol{r}} \\ \boldsymbol{\nabla}_{(2)} = \frac{1}{2}\boldsymbol{\nabla}_{\boldsymbol{R}} - \boldsymbol{\nabla}_{\boldsymbol{r}} \end{cases}$$

Suppose the problem is homogeneous, which could be achieved for the initial distribution via averaging over realizations ensemble of scalar chaotic dispersion in space. Also for example, in theoretical works [29, 7, 17] external sources are assumed to be short-correlated in time and homogeneous in space:

$$\langle \phi\left(\boldsymbol{r}_1, t_1\right) \phi\left(\boldsymbol{r}_2, t_2\right) \rangle = \zeta\left(\boldsymbol{r}_1 - \boldsymbol{r}_2\right) \delta\left(t_1 - t_2\right). \qquad (1.14)$$

In this case, function as a result of averaging over only the statistics of the external sources ϕ, satisfies the following equation:

$$\left[\partial_t + \left(\hat{\Sigma}(t)\boldsymbol{r} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}}\right) - 2\varkappa \boldsymbol{\nabla}^2_{\boldsymbol{r}}\right] F_2\left(\boldsymbol{r}, t\right) = \zeta\left(\boldsymbol{r}\right) \qquad (1.15)$$

Chapter 2

Three-dimensional rotating turbulent flow

In this chapter, we will work within the framework of the theory, which was developed in [16]. Generation of the coherent columnar vortex could be achieved in three-dimensional hydrodynamic systems rotating with angular velocity Ω with external excitation of turbulent regime. One will model forcing by random $\boldsymbol{f}$ with zero mean value, energy pumping rate ϵ per unit mass on the characteristic scale $l_f = 2\pi/k_f$. Theory considers the limit of low Rossby number $\mathrm{Ro} = \left(\epsilon k_f^2\right)^{1/3}/\Omega \ll 1$ and high Reynolds number $\mathrm{Re} = \left(\epsilon/k_f^4\right)^{1/3}/\nu \gg 1$ for excitation eddies, where ν is kinematic viscosity. Such vortices were observed and investigated in recent numerical simulations [25, 4].

2.1 The model

In the reference frame rotating with $\boldsymbol{\Omega}$ velocity field is governed by the Navier-Stokes equation for incompressible fluid $\mathrm{div}\boldsymbol{v} = 0$ in presence of Coriolis force:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}, \boldsymbol{\nabla})\, \boldsymbol{v} + 2\left[\boldsymbol{\Omega} \times \boldsymbol{v}\right] = -\rho^{-1}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{f} \tag{2.1}$$

where p/ρ is the effective pressure per unit mass, which has an addition of centrifugal forces potential to the physical pressure. We decompose velocity into sum the large scale steady coherent component $\boldsymbol{U}$ and turbulent pulsations $\boldsymbol{u}$, thus $\boldsymbol{v} = \boldsymbol{U} + \boldsymbol{u}$. Assuming mean flow to be azimuthal and symmetrical around Oz axis in cylindrical coordinates $\{r, \varphi, z\}$), we can write time-averaged equation projected on $\boldsymbol{e}_\varphi$:

$$\partial_t U = -\left(\partial_r + \frac{2}{r}\right)\left(\langle u^r u^\varphi \rangle - \nu S\right), \tag{2.2}$$

where $S = rd\left(U(r)/r\right)/dr$ is the shear rate of the coherent flow. We consider it is strong enough, that suppresses fluctuations in its background, which can be justified by a relation on the velocity gradient [15]:

$$U/r \gg \epsilon^{1/3} k_f^{2/3},$$

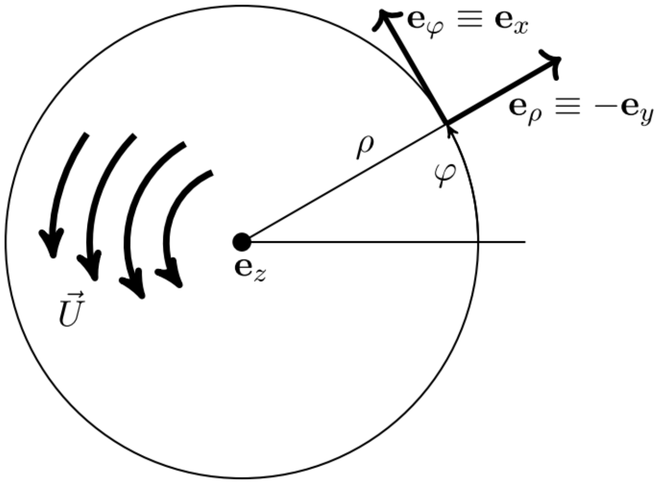

Figure 2.1: Adopted from work [16]: coordinates of fluid element moving in a vortex system.

which allows to apply so-called quasilinear approximation to Navier-Stokes equation with respect to $\boldsymbol{u}$, neglecting smaller $(\boldsymbol{u}, \boldsymbol{\nabla})\,\boldsymbol{u}$ term:

$$\partial_t \boldsymbol{u} + (\boldsymbol{U}, \boldsymbol{\nabla})\,\boldsymbol{u} + (\boldsymbol{u}, \boldsymbol{\nabla})\,\boldsymbol{U} + 2\,[\boldsymbol{\Omega} \times \boldsymbol{u}] = -\rho^{-1}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{u} + \boldsymbol{f} \tag{2.3}$$

where p is a fluctuating part of the pressure. Because pulsations $\boldsymbol{u}$ are of small-scale l_f, one can consider them around some Lagrangian particle moving with the vortex's mean flow. Moving to the corresponding reference frame one can introduce local Cartesian coordinates $\{x, y, z\}$, see Fig.2.1. The mean velocity profile in the neighborhood can be approximated by a linear shear flow $U_x = -Sy$, which allows to rewrite (2.3) in Fourier space as:

$$(\partial_t + Sk_x\partial_{k_y})\boldsymbol{u_k} = -2\Omega[\boldsymbol{e}_z \times \boldsymbol{u_k}] + Su^y_{\boldsymbol{k}}\boldsymbol{e}_x - i\boldsymbol{k}p_{\boldsymbol{k}} - \nu\boldsymbol{k}^2\boldsymbol{u_k} + \boldsymbol{f_k}, \tag{2.4}$$

where we have employed $\rho = 1$. Energy pumping with flux ϵ is modelled via random force $\boldsymbol{f}$ with gaussian statistics, characterized by the pair correlation function which is assumed to be short-correlated in time and homogeneous in space:

$$\left\langle f^i_{\boldsymbol{k}}(t_1) f^j_{\boldsymbol{q}}(t_2) \right\rangle = (2\pi)^3 \left(\delta^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2}\right) \epsilon\delta\left(\boldsymbol{k} + \boldsymbol{q}\right)\delta(t_1 - t_2)\chi\left(\boldsymbol{k}\right). \tag{2.5}$$

Looking for the solution on the characteristics:

$$\boldsymbol{k}'(t) = \left(k_x, k'_y, k_z\right), \quad k'_y(t) = k_y - Stk_x,$$

one can exclude $p_{\boldsymbol{k}'}$ from (2.4) by dot product with k' and using the relation obtained from incompressibility condition:

$$\partial_t\left(\boldsymbol{k}'(t)\cdot u_{\boldsymbol{k}'}\right)=\left(\boldsymbol{k}'(t)\cdot\partial_t u_{\boldsymbol{k}'}\right)-Sk_x u^y_{\boldsymbol{k}'}=0,$$

thus, substituting into (2.4) obtained relation:

$$p=\frac{2i}{(\boldsymbol{k}')^2}\left[Sk_x u^y_{\boldsymbol{k}'}+\Omega\left(k'_y u^x-k_x u^y\right)\right].$$

2.1.1 Basis of the circular polarization vectors

We consider the fluctuations dynamics in the basis of two circular polarizations $\{\boldsymbol{h}^s_{\boldsymbol{k}}\}$, $s=\pm1$:

$$\boldsymbol{u}_{\boldsymbol{k}}=\sum_{s=\pm1}a_{\boldsymbol{k}s}\boldsymbol{h}^s_{\boldsymbol{k}},\qquad \boldsymbol{h}^s_{\boldsymbol{k}}=\frac{[\boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{e}_z]]-isk\,[\boldsymbol{k}\times\boldsymbol{e}_z]}{\sqrt{2}\,kk_\perp},\tag{2.6}$$

introducing $k_\perp=\sqrt{k_x^2+k_y^2}$. One can show they satisfy relations:

$$\left(\boldsymbol{h}^{-s}_{\boldsymbol{k}},\boldsymbol{h}^s_{\boldsymbol{k}}\right)=1,\qquad \left(\boldsymbol{h}^s_{\boldsymbol{k}},\boldsymbol{h}^s_{\boldsymbol{k}}\right)=0,\qquad \boldsymbol{h}^{*,s}_{\boldsymbol{k}}=\boldsymbol{h}^{-s}_{\boldsymbol{k}}=\boldsymbol{h}^s_{-\boldsymbol{k}}\tag{2.7}$$

Rewriting the fluctuations dynamics (2.4) in terms of circular wave amplitudes $a^s_{\boldsymbol{k}'}$, one comes to the system [16]:

$$\frac{d}{dt}a^s_{\boldsymbol{k}'(t)}(t)=\sum_{\sigma=\pm1}H^{s\sigma}\left(\boldsymbol{k}'(t)\right)a^\sigma_{\boldsymbol{k}'}-\nu\boldsymbol{k}'^2(t)a^s_{\boldsymbol{k}'}+f^s_{\boldsymbol{k}'},\tag{2.8}$$

where we have introduced in polarizations basis space matrix $\hat{H}$ with elements:

$$H^{ss}_{\boldsymbol{k}}=is\big(\omega_{\boldsymbol{k}}+\delta\omega_{\boldsymbol{k}}\big)+\ell_{\boldsymbol{k}},\qquad \ell_{\boldsymbol{k}}=-S\frac{k_xk_y}{2k^2},\qquad \omega_{\boldsymbol{k}}=\frac{2\Omega k_z}{k},\qquad \delta\omega_{\boldsymbol{k}}=S\frac{k_z(3k_x^2+k_y^2)}{2kk_\perp^2},$$
$$H^{-s,s}_{\boldsymbol{k}}=Sh^{s,x}_{\boldsymbol{k}}h^{s,y}_{\boldsymbol{k}}=S\frac{k_xk_y(k^2+k_z^2)+iskk_z(k_x^2-k_y^2)}{2k_\perp^2k^2}\tag{2.9}$$

and mapped forcing with the transformed expression (2.5) for pair correlation function:

$$\left\langle f^s_{\boldsymbol{k}}(t_1)f^\sigma_{\boldsymbol{q}}(t_2)\right\rangle=(2\pi)^3\delta(\boldsymbol{k}+\boldsymbol{q})\,\delta(t_1-t_2)\,\delta_{s\sigma}\epsilon\,\chi(\boldsymbol{k})\tag{2.10}$$

where $\chi(k)$ is a correlation function of pumping that is assumed to be isotropic, concentrated on a scale k_f and normalised in Fourier space:

$$\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\chi(k)=1.$$

The solution of (2.8) can be written with the notation of chronologically ordered matrix exponent $\hat{Q}$:

$$a_{\boldsymbol{k}'(t)\,s}(t) = \int\limits_{-\infty}^{t} \mathrm{d}\tau\ \exp\left(-\nu \int\limits_{\tau}^{t} dt_1 k'^2(t_1)\right) \sum_{\sigma} Q^{s\sigma}(t,\tau) f^{\sigma}_{\boldsymbol{k}'(\tau)}(\tau),$$
$$\hat{Q}\left(t,\tau\right) = \mathrm{T}\exp\left(\int\limits_{\tau}^{t} \mathrm{d}t_1\, \hat{H}_{\boldsymbol{k}'(t_1)}\right) = \sqrt{\frac{k'\left(\tau\right)}{k'\left(t\right)}} \exp\left(isG_{\boldsymbol{k}'}\left(t,\tau\right)\right) \hat{P}_{\boldsymbol{k}'}\left(t,\tau\right) \tag{2.11}$$
$$G_{\boldsymbol{k}'}(t,\tau) = \int_{\tau}^{t} dt_1 \left(\omega_{\boldsymbol{k}'(t_1)} + \delta\omega_{\boldsymbol{k}'(t_1)}\right).$$

which expression (see [24]) simplifies in the limit of fast rotation $\Omega \to \infty$, which is justified by a relation $Sk/\Omega k_z \ll 1$ and corresponds to the low Rossby number regime in our system: $\mathrm{Ro}_R \sim \Sigma/\Omega \ll 1$:

$$P^{s\sigma}_{\boldsymbol{k}'}\left(t,\tau\right) \approx \delta^{s\sigma},$$
$$a^{s}_{\boldsymbol{k}'(t)}(t) = \int\limits_{-\infty}^{t} d\tau \sqrt{\frac{k'(\tau)}{k'(t)}} \exp\left(isG_{\boldsymbol{k}'}(t,\tau) - \nu \int\limits_{\tau}^{t} dt_1 k'^2(t_1)\right) f^{s}_{\boldsymbol{k}'(\tau)}(\tau). \tag{2.12}$$

2.1.2 Amplitudes pair correlation function

We are looking for an expression on $\left\langle a^{s}_{\boldsymbol{k}'}(T_1) a^{*,\sigma}_{\boldsymbol{q}'}(T_2)\right\rangle$ correlation function in different time moments. Let's assume first the case case $T_1 > T_2$ and denote $T = T_1 - T_2 > 0$. One should calculate expression With a substitution of (2.10) in (2.11) and performing integration over time:

$$\begin{aligned}
\left\langle a^{s}_{\boldsymbol{k}'}\left(T\right) a^{*,\sigma}_{\boldsymbol{q}'}(0)\right\rangle &= \int\limits_{-\infty}^{T} d\tau_1 \int\limits_{-\infty}^{0} d\tau_2 \exp\left[-\nu\left(\int\limits_{\tau_1}^{T_1} dt_1 k'^2\left(t_1\right) + \int\limits_{\tau_2}^{0} dt_2 q'^2(t_2)\right)\right] \times \\
&\times \sum_{\lambda\rho} Q^{s\lambda}_{\boldsymbol{k}'(\tau_1)}\left(T,\tau_1\right) Q^{*\sigma\rho}_{\boldsymbol{q}'(\tau_2)}\left(0,\tau_2\right) \left\langle f^{\lambda}_{\boldsymbol{k}'(\tau_1)}(\tau_1) f^{\rho}_{-\boldsymbol{q}'(\tau_2)}(\tau_2)\right\rangle = \\
&= \epsilon(2\pi)^3 \delta^{(3)}\left(\boldsymbol{k}-\boldsymbol{q}\right) \int\limits_{-\infty}^{T} d\tau_1 \exp\left[-\nu\left(\int\limits_{\tau_1}^{T_1} dt_1 k'^2\left(t_1\right) + \int\limits_{\tau_2}^{0} dt_2 k'^2(t_2)\right)\right] \times \\
&\times \chi\left(k'\left(\tau_1\right)\right) \sum_{\rho} Q^{s\rho}_{\boldsymbol{k}'(\tau_1)}\left(T,\tau_1\right) Q^{*\sigma\rho}_{\boldsymbol{k}'(\tau_1)}\left(0,\tau_2\right)
\end{aligned} \tag{2.13}$$

Substituting $\hat{Q}$ from (2.12), we will proceed with approximation of matrix in the leading order: $P^{s\sigma} \approx \delta^{s\sigma}$, arriving to the following expression:

$$\left\langle a^{s}_{\boldsymbol{k}'}\left(T\right)a^{*,\sigma}_{\boldsymbol{q}'}(0)\right\rangle = \epsilon(2\pi)^3\delta^{(3)}\left(\boldsymbol{k}-\boldsymbol{q}\right)\delta^{s\sigma}\int\limits_{-\infty}^{T}d\tau_1 \exp\left[-\nu\left(\int\limits_{\tau_1}^{T_1}dt_1 k'^2\left(t_1\right)+\int\limits_{\tau_2}^{0}dt_2 q'^2(t_2)\right)\right]\times$$
$$\times\frac{k'\left(\tau_1\right)}{\sqrt{kk'\left(T\right)}}\chi\left(k'\left(\tau_1\right)\right)\exp\left[is\int_0^T dt_1\left(\omega_{\boldsymbol{k}'(t_1)}+\delta\omega_{\boldsymbol{k}'(t_1)}\right)\right]$$

In the opposite case when $T_1 < T_2$, one can use $a^s_{\boldsymbol{k}} = a^{*,s}_{-\boldsymbol{k}}$ and rewrite answer above with a change of signs: $\boldsymbol{k} \to -\boldsymbol{k}$ ($\boldsymbol{k}'(\tau) \mapsto -\boldsymbol{k}'(\tau)$ (meanwhile $\omega_{\boldsymbol{k}'}$ changes its sign too under this transformation) and obtain:

$$\left\langle a^s_{\boldsymbol{k}}(T_1)a^{*,\sigma}_{\boldsymbol{q}}(T_2)\right\rangle = \left\langle a^{\sigma}_{-\boldsymbol{q}}(T_2)a^{*,s}_{-\boldsymbol{k}}(T_1)\right\rangle = \delta^{\sigma s}\delta^{(3)}\left(\boldsymbol{k}-\boldsymbol{q}\right)\dots$$

Finally, the general expression for a pair correlation function has the following form:

$$\begin{gathered}\left\langle a^s_{\boldsymbol{k}}(t+T)a^{*,\sigma}_{\boldsymbol{q}}(t)\right\rangle = (2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{q})\delta^{s\sigma}A\left(\boldsymbol{k},T=T_1-T_2\right),\\ A\left(\boldsymbol{k},T\right)=\epsilon\exp\left(i\sigma\text{sgn}\,T\int\limits_0^{|T|}dt'dt_1\left(\omega_{\boldsymbol{k}'(t_1)}+\delta\omega_{\boldsymbol{k}'(t_1)}\right)\right)\times\\ \times\int\limits_{-\infty}^{T}d\tau\,\frac{k'\left(\tau\right)}{\sqrt{kk'\left(|T|\right)}}\chi\left(k'\left(\tau\right)\right)\exp\left[-\nu\left(\int\limits_{\tau}^{|T|}+\int\limits_{\tau}^{0}\right)dt_1 k'^2(t_1)\right]\end{gathered} \tag{2.14}$$

2.2 The columnar vortex steady state

The result (2.14) brings us to the expression for one-moment velocity correlation function $\langle u^r u^\varphi\rangle$, as it was found in [16].

$$\langle u^\rho u^\varphi\rangle = -\langle u^x u^y\rangle = -\sum_{s=\pm1}\left\langle\langle a^s_{\boldsymbol{k}}(0)a^{s\,*}_{\boldsymbol{k}}(0)\rangle_f\,h^x_{\boldsymbol{k},s}h^y_{-\boldsymbol{k},s}\right\rangle_{\boldsymbol{k}} =$$
$$=\sum_{s=\pm1}\langle A(\boldsymbol{k},0)h^{s,\,x}_{\boldsymbol{k}}h^{s,\,y}_{-\boldsymbol{k}}\rangle_{\boldsymbol{k}} = \left\langle A(\boldsymbol{k},0)\frac{k_x k_y}{k^2}\right\rangle_{\boldsymbol{k}} =$$

$$=\epsilon\int\left(d^3\boldsymbol{k}\right)\int\limits_{-\infty}^{0}d\tau\left(\frac{k^x k^y}{k^2}\right)\frac{k'(\tau)}{k}\chi(\boldsymbol{k}'(\tau))\exp\left[-2\nu\int\limits_{\tau}^{0}dt'k'^2(t')\right], \tag{2.15}$$

because of relation for corresponding basis vectors components:

$$h^x_{\boldsymbol{k},s}h^y_{-\boldsymbol{k},s} = \frac{k_x k_y(k^2-k_z^2)+iskk_z(k_x^2+k_y^2)}{2k_\perp^2 k^2} = \frac{k_x k_y + iskk_z}{2k^2} \tag{2.16}$$

After transformation of the integration wavevector to characteristics $\boldsymbol{k} \to \boldsymbol{k}'(\tau)$, one can find answer in the leading order, performing integration by parts and neglecting the remaining integral, which is small as a power of parameter $\nu k_f^2/S \ll 1$:

$$\langle u^\rho u^\varphi \rangle = \frac{\epsilon}{S} \int \left(d^3k\right) \chi(k) k \int\limits_{-\infty}^{0} d\tau \frac{\partial}{\partial \tau} \left(k_x^2 + k_z^2 + (k_y - S k_x \tau)^2\right)^{-1/2} \exp\left[-2\nu \int\limits_{\tau}^{0} dt' k'^2\left(t'\right)\right] =$$
$$= \frac{\epsilon}{S} \int \left(d^3k\right) \chi(k) \cdot 1 + O\left[\left(\frac{\nu k_f^2}{S}\right)^{1/3}\right] \approx \frac{\epsilon}{S} \tag{2.17}$$

Substitution of the result in (2.2) bring us to the stationary solution with constant shear rate $S = \sqrt{\epsilon/\nu}$, which yields the characteristic logarithmic-linear spatial profile for the large-scale coherent velocity $U(r)$:

$$\langle u^r u^\varphi \rangle - \nu S(r) = 0$$
$$U(r) = \pm r \int\limits_{R_u}^{r} d\rho \sqrt{\frac{\epsilon}{\nu}}/\rho = \mp r \sqrt{\frac{\epsilon}{\nu}} \ln \frac{R_u}{r}, \tag{2.18}$$

where U becomes 0 at the vortex boundary $r = R_u$. Different signs in (2.18) correspond to anti-cyclone and cyclone case respectively.

Chapter 3

Fluctuations in the three-dimensional coherent flow

3.1 Velocity gradients pair correlation function

For further study of the advection in the shear flow, one needs to work with velocity gradients matrix $\hat{\Sigma}$, written in local coordinate system from Fig.2.1. $\hat{\Sigma}$ has strong constant component $S = \sqrt{\epsilon/\nu}$ from the coherent vortex flow and random fluctuating part of turbulent pulsations $\hat{\sigma}$:

$$\Sigma^{ij}(t) = S\delta^{i,x}\delta^{j,y} + \sigma^{ij}(t), \quad \sigma^{ij} = \partial^j u^i \to ik^j u^i_{\boldsymbol{k}} \tag{3.1}$$

Their statistics is described by pair correlation function $\langle \hat{\sigma}(t)\hat{\sigma}(0) \rangle$. Let's obtain an expression on it within the theoretical model framework, considering expression for pair correlation function $\left\langle u^m_{\boldsymbol{k}} u^j_{\boldsymbol{k}} \right\rangle$ of velocities, which we decompose into a sum of circular polarizations, leaving only nonzero terms due to $\left\langle a^s_{\boldsymbol{k}}(t+T) a^{*,\sigma}_{\boldsymbol{q}}(t) \right\rangle \sim \delta^{\sigma s}\delta\left(\boldsymbol{k}-\boldsymbol{q}\right)$ dependence in (2.14)[1]:

$$\begin{aligned} \left\langle u^m_{\boldsymbol{k}}(t) u^j_{-\boldsymbol{k}}(0) \right\rangle = \left\langle \left(a_{\boldsymbol{k},+}(t) h^m_{\boldsymbol{k},+} + a_{\boldsymbol{k},-}(t) h^m_{\boldsymbol{k},-}\right) \left(a^*_{\boldsymbol{k},+}(0) h^j_{-\boldsymbol{k},+} + a^*_{\boldsymbol{k},-}(0) h^j_{-\boldsymbol{k},-}\right) \right\rangle \approx \\ \approx \left\langle a_{\boldsymbol{k},+}(t) a^*_{\boldsymbol{k},+}(0) \right\rangle h^m_{\boldsymbol{k},+} h^j_{-\boldsymbol{k},+} + \left\langle a_{\boldsymbol{k},-}(t) a^*_{\boldsymbol{k},-}(0) \right\rangle h^m_{\boldsymbol{k},-} h^j_{-\boldsymbol{k},-} \end{aligned} \tag{3.2}$$

After taking coordinate derivatives, we consider the following expression in Fourier space:

$$\left\langle \sigma^{mn}(t)\sigma^{jl}(0) \right\rangle = \left\langle k^l k^n u^m_{\boldsymbol{k}}(t) u^j_{-\boldsymbol{k}}(0) \right\rangle = \sum_{s=\pm 1} \left\langle k^l k^n \left\langle a_{\boldsymbol{k},s}(t) a^*_{\boldsymbol{k},s}(0) \right\rangle_f h^j_{\boldsymbol{k},s} h^m_{-\boldsymbol{k},s} \right\rangle_{\boldsymbol{k}} \tag{3.3}$$

Substitution of the $\langle a_{\boldsymbol{k}} a^*_{\boldsymbol{k}} \rangle$ (2.14) yields to an expression for a pair correlation function of fluctuations' gradients, which we denote as tensor D:

$$D^{jlmn}(t) = \left\langle \sigma^{jl}(\boldsymbol{r},t)\sigma^{mn}(\boldsymbol{r},0) \right\rangle_f = \sum_{s=\pm 1} \left\langle k^l k^n A\left(\boldsymbol{k},t\right) h^j_{\boldsymbol{k},s} h^m_{\boldsymbol{k},-s} \right\rangle_{\boldsymbol{k}} \tag{3.4}$$

[1] In this section for conveniency purposes polarisation indices are placed in subscript and spatial ones are in superscript.

For simplicity of further calculations we consider δ-dependent expression for correlation function $\chi(k)$ in Fourier space, so that integration should be performed for characteristics vector $\boldsymbol{k}'$ over sphere with radius k_f:

$$\chi(k) = \frac{4\pi^2}{k_f}\delta\left(k^2 - k_f^2\right), \tag{3.5}$$

that integration of (2.14) over τ for can be taken explicitly - for the case $k_x > 0$ it gives the expression:

$$\int_{-\infty}^{0} d\tau\, k'(\tau)\,\chi\left(k'(\tau)\right)\exp\left[-2\nu\int_{\tau}^{0} dt' k'^2(t')\right] = \frac{2\pi^2}{S}\frac{\theta\left(k_f^2 - k_x^2 - k_z^2\right)}{k_x\sqrt{k_f^2 - k_x^2 - k_z^2}}$$
$$\left[\theta\left(k_y + \sqrt{k_f^2 - k_x^2 - k_z^2}\right)e^{-2\nu\int_{\tau_+}^{0} dt' k'^2(t')} + \theta\left(k_y - \sqrt{k_f^2 - k_x^2 - k_z^2}\right)e^{-2\nu\int_{\tau_-}^{0} dt' k'^2(t')}\right] \tag{3.6}$$

where was introduced:

$$\tau_{\pm} = \frac{1}{Sk_x}\left(k_y \pm \sqrt{k_f^2 - k_x^2 - k_z^2}\right).$$

3.1.1 Correlation time

Next, for study of time correlations, we will need to calculate the integral of $D^{jlmn}(t)$ (3.4) or equivalent $A(\boldsymbol{k}, t)$ (2.14) over time t, considering

$$\int_{0}^{\infty}\frac{dt}{\sqrt{k'(t)}}\exp\left[\int_{0}^{t} dt'\left(i\sigma\left(\omega_{\boldsymbol{k}'(t_1)} + \delta\omega_{\boldsymbol{k}'(t_1)}\right) - \nu k'^2(t')\right)\right]$$

in the limit: $\Omega \gg S \gg \nu k'^2$. Because of highly oscillating function in exponent, the main contribution comes from the narrow vicinity near the integration limit $t = 0$, giving in the leading order the following:

$$\int_{0}^{\infty}\frac{dt}{\sqrt{k'(t)}}\exp\left[\int_{0}^{t} dt'\left(i\sigma\left(\omega_{\boldsymbol{k}'(t_1)} + \delta\omega_{\boldsymbol{k}'(t_1)}\right) - \nu k'^2(t')\right)\right] \approx \int_{0}^{\infty}\frac{dt}{\sqrt{k}}\exp\left[\left(\frac{2i\sigma\Omega k_z}{k} - \nu k^2\right)t\right] =$$
$$= \frac{1}{\sqrt{k}}\frac{1}{\nu k^2 - 2i\sigma\Omega k_z/k} = \frac{1}{\sqrt{k}}\frac{\nu k^4 + 2i\sigma\Omega k_z k}{\left(\nu k^3\right)^2 + \left(2\Omega k_z\right)^2}, \tag{3.7}$$

and also one can see characteristic integration range scale gives estimate that fluctuations' correlation time is small as reciprocal of system rotation frequency:

$$\tau_{\text{corr.}} \equiv \int_{0}^{\infty} dt D^{jlmn}(t)/D^{jlmn}(0) \sim \Omega^{-1}$$

Now one can write averaging of an ambiguous function $F(\boldsymbol{k})$ with $A(\boldsymbol{k},t)$ over wavevector by using isotropy of $\chi(k)$ in the following form:

$$\int_0^\infty dt\,\langle F(\boldsymbol{k})A(\boldsymbol{k},t)\rangle_{\boldsymbol{k}} = \frac{\epsilon}{S}\int \frac{d^3\boldsymbol{k}}{4\pi}\left(F(\boldsymbol{k}) + F(-k_x,-k_y,k_z)\right)\frac{\theta(k_x)\,\theta\left(k_f^2-k_x^2-k_z^2\right)}{kk_x\sqrt{k_f^2-k_x^2-k_z^2}}\times$$
$$\times\frac{\nu k^4}{(\nu k^3)^2+(2\Omega k_z)^2}\sum_{\sigma=\pm}\theta\left(k_y+\sigma\sqrt{k_f^2-k_x^2-k_z^2}\right)e^{-2\nu\int_{\tau_\sigma}^0 dt' k'^2(t')} \tag{3.8}$$

Let's use it to evaluate time integral for the velocity $\langle u^\rho u^\varphi\rangle$ pair correlation function in the same limits of the model: $\Omega \gg S \gg \nu k_f^2$:

$$\int_0^\infty dt\,\langle u^\rho(t)u^\varphi(0)\rangle \approx \frac{\epsilon}{2\pi S}\int dk_x\int dk_y\int dk_z \frac{\theta(k_x)\,\theta\left(k_f^2-k_x^2-k_z^2\right)}{k_x k\sqrt{k_f^2-k_x^2-k_z^2}}\frac{k_x k_y}{k^2}\frac{\nu k^4}{(\nu k^3)^2+(2\Omega k_z)^2}\times$$
$$\times\left[\theta\left(k_y+\sqrt{k_f^2-k_x^2-k_z^2}\right)e^{-2\nu\int_{\tau_+}^0 dt'k'^2(t')}+\theta\left(k_y-\sqrt{k_f^2-k_x^2-k_z^2}\right)e^{-2\nu\int_{\tau_-}^0 dt'k'^2(t')}\right]$$

One can rescale the integration wavevector: $\boldsymbol{k}\to k_f\boldsymbol{k}$ and employ the approximation by crossing out small $\left(\nu k_f^2 k^3\right)^2$ term:

$$\int_0^\infty dt\,\langle u^\rho(t)u^\varphi(0)\rangle = \frac{\epsilon}{2\pi S}\int dk_x\int dk_y\fint dk_z\frac{\theta(k_x)\,\theta\left(1-k_x^2-k_z^2\right)}{\sqrt{1-k_x^2-k_z^2}}\frac{\nu k_f^2 k k_y}{(2\Omega k_z)^2}\times$$
$$\times\left[\theta\left(k_y+\sqrt{1-k_x^2-k_z^2}\right)e^{-2\nu k_f^2\int_{\tau_+}^0 dt'k'^2(t')}+\theta\left(k_y-\sqrt{1-k_x^2-k_z^2}\right)e^{-2\nu k_f^2\int_{\tau_-}^0 dt'k'^2(t')}\right]\approx$$
$$\approx\frac{\epsilon\nu k_f^2}{8\pi S\Omega^2}\int_0^1 dk_x\fint dk_z\frac{\theta\left(1-k_x^2-k_z^2\right)}{\sqrt{1-k_x^2-k_z^2}}\frac{1}{k_z^2}$$

Integration over k_z contributes from the small cut-off scale $k_z^*\sim\nu k_f^2/\Omega$. Finally, the estimate for $\int dt\,\langle u^\rho u^\varphi\rangle$ scale

$$\int_0^\infty dt\,\langle u^\rho(t)u^\varphi(0)\rangle\sim\frac{\epsilon\nu k_f^2}{S\Omega^2}\frac{1}{k_z^*}\int_0^1\frac{dk_x}{\sqrt{1-k_x^2}}\sim\frac{\epsilon}{S\Omega} \tag{3.9}$$

gives the expected scale for correlation time $\tau_{\text{corr.}}\sim\Omega^{-1}$.

3.2 Calculation of σ_{yx} statistics.

As it will reveal in the next chapter of our work, due to fluctuations are weak compared to the coherent flow, σ^{yx} is of particular interest to us among other components, whose statistics based on $\langle\sigma^{yx}(t)\sigma^{yx}(0)\rangle$, we will study in this section.

First, let's calculate the mean square $D^{yxyx}(0)$, performing integration over pumping time τ

with use of (3.5) expression for χ.

$$D^{yxyx}(0)=\frac{\epsilon}{2\pi S}\int dk_x\int dk_y\int dk_z\frac{\theta\left(k_x\right)\theta\left(k_f^2-k_x^2-k_z^2\right)}{k_xk\sqrt{k_f^2-k_x^2-k_z^2}}\frac{k_x^2\left(k_y^2k_z^2+k^2k_x^2\right)}{k^2\left(k_x^2+k_y^2\right)}\times$$
$$\times\left[\theta\left(k_y+\sqrt{k_f^2-k_x^2-k_z^2}\right)e^{-2\nu\int_{\tau_+}^0 dt'k'^2(t')}+\theta\left(k_y-\sqrt{k_f^2-k_x^2-k_z^2}\right)e^{-2\nu\int_{\tau_-}^0 dt'k'^2(t')}\right]=$$
$$=\frac{\epsilon k_f^2}{2\pi S}\int dk_x\int dk_y\int dk_z\frac{\theta\left(k_x\right)\theta\left(1-k_x^2-k_z^2\right)}{\sqrt{1-k_x^2-k_z^2}}\frac{d}{dk_y}\left(\frac{k_x\left(k_x^2+k_z^2\right)}{k^3}\right)\times$$
$$\times\left[\theta\left(k_y+\sqrt{1-k_x^2-k_z^2}\right)e^{-2\nu k_f^2\int_{\tau_+}^0 dt'k'^2(t')}+\theta\left(k_y-\sqrt{1-k_x^2-k_z^2}\right)e^{-2\nu k_f^2\int_{\tau_-}^0 dt'k'^2(t')}\right]$$

In the leading order on small parameter $\nu k_f^2/S$ one comes to result

$$D^{yxyx}(0)=\frac{\epsilon k_f^2}{\pi S}\left[\int_0^1 dk_x\int_{-\sqrt{1-k_x^2}}^{\sqrt{1-k_x^2}} dk_z\frac{k_x\left(k_x^2+k_z^2\right)}{\sqrt{1-k_x^2-k_z^2}}+o\left(1\right)\right]\approx\frac{3\epsilon k_f^2}{8S},\tag{3.10}$$

where numerical coefficient $3/8$ is not important and depend on the specific expression for $\chi(k)$. One can see that turbulent pulsations are less than a coherent shear rate by a factor $\nu k_f^2/S$.

Next, one needs to evaluate the value of $\int dtD^{yxyx}(t)$. According to (3.8):

$$\int_0^\infty dtD^{yxyx}(t)\approx\frac{\epsilon k_f^2}{2\pi S}\frac{\nu k_f^2}{4\Omega^2}\int dk_x\fint dk_z\int dk_y\frac{\theta\left(k_x\right)\theta\left(1-k_x^2-k_z^2\right)}{\sqrt{1-k_x^2-k_z^2}}\frac{k_xk\left(k_y^2k_z^2+k^2k_x^2\right)}{k_z^2\left(k_x^2+k_y^2\right)}\times$$
$$\times\left[\theta\left(k_y+\sqrt{1-k_x^2-k_z^2}\right)e^{-2\nu k_f^2\int_{\tau_+}^0 dt'k'^2(t')}+\theta\left(k_y-\sqrt{1-k_x^2-k_z^2}\right)e^{-2\nu k_f^2\int_{\tau_-}^0 dt'k'^2(t')}\right]$$

By evaluating integral over dk_z near cut-off scale k_z^*, one can see that answer is proportional to $\epsilon k_f^2/S\Omega$ with some coefficient of order unity:

$$\int_0^\infty dtD^{yxyx}(t)\propto\frac{\epsilon k_f^2}{S\Omega}\int_0^1 dk_x\int dk_y\frac{k_x^3k_\perp}{\sqrt{1-k_x^2}}\sum_{s=\pm1}\theta\left(k_y+s\sqrt{1-k_x^2}\right)e^{-2\nu k_f^2\int_{\tau_s}^0 dt'k'^2(t')}\tag{3.11}$$

Thus, we have shown that for σ^{yx} correlation time is also small as: $\tau_{\text{corr.}}\sim\Omega^{-1}$.

Due to various combinations of $\boldsymbol{k}$ components in the integrand of (3.4) with different characteristic scales, one could conclude that tensor $D(t)$ is short-correlated in time, though possesses high degree of anisotropy, which was a subject of study in recent work [20].

Chapter 4

Dynamics of passive scalar's spatial distribution parameters

Following the results from previous chapter, one can say that the coherent columnar vortex flow is characterised with the presence of strong steady shear in Oxy plane with short-correlated noise addition. Equation (1.4) averaged over the statistics of $\hat{\sigma}$ was solved analytically [1, 26] in the case when the mean flow is absent, $\hat{S} = 0$, and the statistics of the turbulent pulsations are isotropic. The opposite case, when the random component of the flow is absent so equation (1.4) is deterministic, was considered in Chapter 1. However, simultaneous presence of the constant shear, the stochasticity and the molecular diffusion makes the direct analysis of equation on averaged over σ function cumbersome. Equation (1.1) on a pair correlation function averaged over generally non-smooth random component of the flow was considered in [5] via numerical computation with no analytical progress.

Let us outline the idea of our analytical solution described below. We fix the spatial form of decay problem solution $\theta(t, \boldsymbol{r})$, solving the (1.4) for it in terms of parameters set. In this way we solve equation on θ for at some random realisation of $\hat{\sigma}$. One can consider a question of about the form of optimal, most probable fluctuation of the stochastic process for each parameter, which yields the uniform in time evolution of θ. Therefore, finding the optimal fluctuation regime of the parameters will lead to the asymptotics of the solution θ at large times.

In this chapter we consider simplified model of scalar evolution in reduced two-dimensional problem, as if in the vortex's cross-section.

4.1 Ansatz: evolution of a two-dimensional gaussian blob

We consider the equation (1.4) with initial condition function as Gauss spatial distribution, since such profile of the solution is preserved by the equation, see Chapter 1. While for the average $\langle\theta\rangle$ problem it simply describes the evolution of a blob of such profile, then in case of the equivalent problem for pair correlation function $F(t, \boldsymbol{r})$ in homogeneous space such initial condition can be achieved as an averaging of superposition of N gaussian blobs of same size l over their centers' positions $\boldsymbol{r}_i$ in space, assuming that total scalar quantity equals zero [29]:

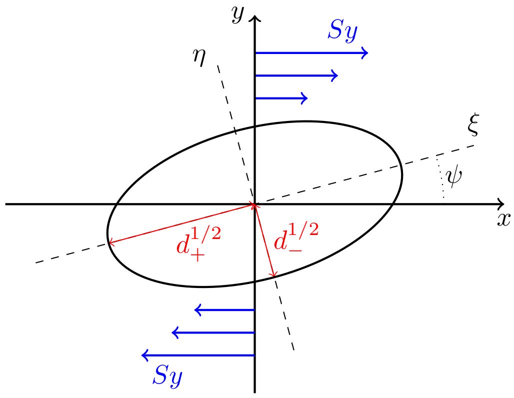

Figure 4.1: The ellipse corresponding to quadratic equation in exponent, which is determined by parameters d_+, d_-, ψ. Blue arrows indicate the direction of strong steady shear Sy.

$$\theta_0(\boldsymbol{r}) = \sum_{i=1}^{N} c_i \vartheta\left(\left|\boldsymbol{r} - \boldsymbol{r}_i\right|/l\right), \quad \sum_{i=1}^{N} c_i = 0, \quad \int_{\mathcal{A}} d^2\boldsymbol{r}\vartheta\left(r/l\right) = l^2, \tag{4.1}$$

where $\mathcal{A}$ is the area of the system. In the limit of large $\mathcal{A}$ averaging over $\boldsymbol{r}_i$ for pair correlation function gives the Gauss spatial distribution with different variance:

$$F_0(r) = \left(\sum_{i=1}^{N} c_i^2\right) \cdot \int \frac{d^2\boldsymbol{r}'}{\mathcal{A}} \vartheta\left(\left|\boldsymbol{r}' - \boldsymbol{r}/2\right|/l\right) \vartheta\left(\left|\boldsymbol{r}' + \boldsymbol{r}/2\right|/l\right). \tag{4.2}$$

We introduce the following parameterization of the ansatz: its exponent function should be a quadratic form on a symmetric positive definite matrix, which can be represented as a transformation of a diagonal one $\hat{D}^{-1}$ with elements $d_+^{-1}(t), d_-^{-1}(t)$ by the orthogonal rotation matrix $\hat{N}$ described with angle $\psi(t)$. Normalized by unity expression has the form:

$$\theta_g(\boldsymbol{r}, t) = \frac{1}{2\pi\sqrt{d_+(t)d_-(t)}} \exp\left[-\frac{1}{2}\boldsymbol{r} \underbrace{\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}}_{\hat{N}} \underbrace{\begin{pmatrix} d_+^{-1} & 0 \\ 0 & d_-^{-1} \end{pmatrix}}_{\hat{D}^{-1}} \underbrace{\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}}_{\hat{N}^T} \boldsymbol{r}\right]$$

$$= \frac{1}{2\pi\sqrt{d_+(t)d_-(t)}} \exp\left[-\frac{1}{2}\boldsymbol{r} \begin{pmatrix} \cos^2\psi/d_+ + \sin^2\psi/d_- & -\frac{d_+ - d_-}{2d_+d_-}\sin 2\psi \\ -\frac{d_+ - d_-}{2d_+d_-}\sin 2\psi & \cos^2\psi/d_- + \sin^2\psi/d_+ \end{pmatrix} \boldsymbol{r}\right] \tag{4.3}$$

The parameters $d_+(t), d_-(t) > 0$ indicate the dispersion of the blur in the directions specified by the angle $-\pi/2 < \psi(t) < \pi/2$. Substituting it in the equation, one will obtain the following autonomous system of differential equations for these three functions:

$$\begin{cases} \psi'(t) = \frac{1}{2}\left(\sigma_{yx} - \sigma_{xy} - S\right) + \frac{1}{2}\frac{d_+ + d_-}{d_+ - d_-}\left[(S + \sigma_{xy} + \sigma_{yx})\cos 2\psi - 2\sigma_{xx}\sin 2\psi\right] \\ d'_+(t) = 2\varkappa + d_+\left[2\sigma_{xx}\cos 2\psi + (S + \sigma_{xy} + \sigma_{yx})\sin 2\psi\right] \\ d'_-(t) = 2\varkappa - d_-\left[2\sigma_{xx}\cos 2\psi + (S + \sigma_{xy} + \sigma_{yx})\sin 2\psi\right] \end{cases} \tag{4.4}$$

4.1.1 Absent noise case

Omitting $\hat{\sigma}$, the system above will be simplified to the following:

$$\begin{cases} \psi'(t) = -\frac{S}{2} + \frac{S}{2}\frac{d_+(t) + d_-(t)}{d_+(t) - d_-(t)}\cos 2\psi(t) \\ d'_+(t) = 2\varkappa + Sd_+(t)\sin 2\psi(t) \\ d'_-(t) = 2\varkappa - Sd_-(t)\sin 2\psi(t) \end{cases} \tag{4.5}$$

and one can check straightforwardly it has the following solution, that describes problem from Chapter 1:

$$\begin{aligned} d_+(t) &= L^2\left[1 + \frac{(St)^2}{2}\right] + \varkappa t\left[2 + \frac{(St)^2}{3}\right] + St\sqrt{(L^2 + \varkappa t)^2 + \left(\frac{L^2 St}{2} + \frac{\varkappa St^2}{3}\right)^2} \\ d_-(t) &= L^2\left[1 + \frac{(St)^2}{2}\right] + \varkappa t\left[2 + \frac{(St)^2}{3}\right] - St\sqrt{(L^2 + \varkappa t)^2 + \left(\frac{L^2 St}{2} + \frac{\varkappa St^2}{3}\right)^2} \end{aligned} \tag{4.6}$$

$$\tan\psi = \frac{\sqrt{(L^2 + \varkappa t)^2 + \left(\frac{L^2 St}{2} + \frac{\varkappa St^2}{3}\right)^2} - St\left(L^2/2 + \varkappa t/3\right)}{L^2 + \varkappa t} \tag{4.7}$$

4.2 Effective system and Fokker-Planck equation

In the limit $S \to \infty$, after initial time $\sim S^{-1}$ passes, (4.4) reaches the following dynamics: strong shear stretches the blob in the designated direction, making variances differ significantly: $d_+ \gg d_-$. That allows to consider stochastic equation on $\psi(t)$ decoupled from others; taking into account that shear fluctuations are significantly weaker $\sigma_{ij} \ll S$, one needs to retain only the term $\sigma_{yx}(t)\cos^2\psi$, which is able to interfere the steady shear $-S\sin^2\psi$ on small angles $\psi \ll 1$:

$$\psi'(t) = \sigma_{yx}(t)\cos^2\psi(t) - S\sin^2\psi(t).$$

The equation above describes the process of angle rotation in a preferred direction to negative values, which takes diffusion behavior at small angles. The act of periodic continuation at the boundaries $\psi = -\pi/2 \to \pi/2$ we will further call tumbling [8, 28], and consider the evolution of ψ as sequence of passages between them. Assuming blob size is large enough that one can neglect weak thermal diffusion ($\varkappa \to 0$) effects for the shear-elongated direction, we arrive to the effective form of parameters system, where stochasticity of $d_\pm(t)$ dynamics caused by ψ. Introducing the logarithmic variable change $d_+(t) = L^2 e^\rho(t)$:

$$\begin{cases} \psi'(t) = \sigma_{yx}(t)\cos^2\psi(t) - S\sin^2\psi(t) \\ \rho'(t) = S\sin 2\psi(t) \\ d'_-(t) = 2\varkappa - Sd_-(t)\sin 2\psi(t) \end{cases} \tag{4.8}$$

where $\sigma_{yx}(t)$ is considered to be a white noise with magnitude D, described by a pair correlation function: $\langle \sigma_{yx}(t)\,\sigma_{yx}(t')\rangle = 2D\delta(t-t')$. The neglection of $\varkappa$ for ρ is justified by the relation $L \gg \sqrt{\varkappa D^{-1/3} S^{-2/3}}$.

While blob size in shear-contracted direction exceeds the thermal diffusion scale too $d_-(t) \gg \varkappa D^{-1/3} S^{-2/3}$, one can consider reduced problem on ρ, ψ only utilizing a relation $d_-(t) = L^2 e^{-\rho(t)}$, which probability distribution function (PDF) $\mathcal{P}(t, \rho, \psi)$ is governed by the following Fokker-Planck equation:

$$\partial_t \mathcal{P} + S\sin 2\psi \partial_\rho \mathcal{P} - S\partial_\psi \sin^2\psi \mathcal{P} - D\partial_\psi \cos^2\psi \partial_\psi \cos^2\psi \mathcal{P} = 0 \tag{4.9}$$

with a periodic boundary conditions on $\psi \in (-\pi/2, \pi/2)$. Let's point out general properties of its solution in our limit $D \ll S$.

First, the body of the function $\mathcal{P}$ lies an small angles, up to $|\psi| \sim (D/S)^{1/3}$, where the equation (4.9) can be approximately written as a:

$$\partial_t \mathcal{P} + 2\psi S\partial_\rho \mathcal{P} - S\partial_\psi \left(\psi^2 \mathcal{P}\right) - D\partial_\psi^2 \mathcal{P} = 0, \tag{4.10}$$

that could drop out any scales under the follow following change of variables:

$$t = \left(S^2 D\right)^{-1/3} \tau \equiv t_* \tau, \qquad \psi = (D/S)^{1/3} \psi \equiv \psi_* \phi,\ \phi \in \mathbb{R}.$$

Second, for the reduced problem on angle ψ PDF in large-time limit $t \gg t_*$ asymptotically becomes stationary $P_{\text{st}}(\psi)$, which is in the limit $D/S \ll 1$ a solution of

$$S\partial_\psi \left(\psi^2 P_{\text{st}}(\psi)\right) + D\partial_\psi^2 P_{\text{st}}(\psi) = 0$$

$$P_{\rm st}(\psi) \approx C \int\limits_{-\infty}^{\psi} d\varphi\, e^{\frac{S\left(\varphi^3-\psi^3\right)}{3D}}, \tag{4.11}$$

where the normalization constant C can be found from integration:

$$C^{-1} = \int\limits_{-\infty}^{\infty} d\psi \int\limits_{0}^{\infty} d\varphi e^{-S\left(3\varphi\psi^2-3\varphi^2\psi+\varphi^3\right)/3D} = \left(\frac{\pi D}{S}\right)^{1/2} \int\limits_{0}^{\infty} d\varphi \frac{e^{-\frac{S\varphi^3}{12D}}}{\sqrt{\varphi}} = \frac{2^{1/3}\pi^{1/2}}{3^{5/6}}\Gamma\left(\frac{1}{6}\right)\left(\frac{D}{S}\right)^{2/3}$$

Third, it needs to note that this distribution is significantly asymmetric with $\langle\psi\rangle \sim \psi_*$ and gives a nonzero probability flux, thus indicating a direction of the angle rotation [28]. However, asymptotic of the solution at $\psi \gtrsim \psi_*$ has slow decay law $P(t,\psi) \sim (\psi/\psi_*)^{-2}$, which, for example, results in large-scale higher moments: $\langle\psi^n\rangle \sim 1, n \geqslant 2$.

On the other hand, for one turnover of ψ between neighboring tumblings, ρ in (4.8) increments by $\Delta\rho \sim 1$ - a random variable dependent on $\psi(t)$. That results in exponential growth of size in shear-elongated direction with a Lyapunov exponent $\lambda \sim (DS^2)^{1/3}$. In large-time limit $t \gg t_*$, one can consider statistics of ρ as a sum of $N \gg 1$ independent random variables $\Delta\rho$, which results in reduced PDF $\mathcal{P}\left(t,\rho\right)$ asymptotically goes to exponential self-similar form[2, 23]:

$$\mathcal{P}\left(t,\rho\right) \propto \frac{1}{\sqrt{t}} \exp\left[-\lambda t \mathcal{S}\left(\frac{\rho}{\lambda t}\right)\right], \tag{4.12}$$

where $\mathcal{S}$ is called entropy or Cramer function and it is known that it is a convex of its argument.

4.3 Decay problem asymptotics

The result of averaging the ansatz is an expression for the solution (4.3), which is averaged with the corresponding PDF:

$$\begin{aligned} \theta_g\left(t,r,\varphi\right) = \int\limits_{0}^{\infty} \mathrm{d}d_- \int \mathrm{d}\rho \int\limits_{-\pi/2}^{\pi/2} \mathrm{d}\psi \mathcal{P}\left(t,\rho,d_-,\psi\right) \frac{1}{2\pi L\sqrt{e^\rho d_-}} \times \\ \times \exp\left[-\frac{r^2}{2}\begin{pmatrix}\cos\left(\psi-\varphi\right)\\ \sin\left(\psi-\varphi\right)\end{pmatrix}\begin{pmatrix}L^{-2}e^{-\rho} & 0\\ 0 & 1/d_-\end{pmatrix}\begin{pmatrix}\cos\left(\psi-\varphi\right)\\ \sin\left(\psi-\varphi\right)\end{pmatrix}\right] \end{aligned} \tag{4.13}$$

In this section we provide analytical study of (4.13) without finding directly PDF for (4.8). The following procedure is valid when polar angle φ is not too small: $|\varphi|, |\pi-\varphi| \gtrsim \psi_*$.

That allows to take into consideration separately only the last stage of dynamics, where angle is large $\pi - \psi_* \gg \psi \gg \psi_*$[1], therefore $\sigma_{yx}\cos^2\psi$ term in (4.8) becomes negligible. In that case,

[1]Here for ψ negative values, corresponding to dynamics preceding last tumbling, I applied transformation $\psi \to \psi + \pi$

system solution obeys the following relations with $\psi^{(f)} = \psi(t_f)$, $\psi^{(i)} = \psi(t_i)$:

$$\begin{aligned}
\cot\psi^{(f)} - \cot\psi^{(i)} &= S(t_f - t_i) \\
\rho^{(f)} - \rho^{(i)} = (\Delta\rho)_{fi} &\equiv 2\ln\left|\frac{\sin\psi^{(i)}}{\sin\psi^{(f)}}\right| \\
d_-^{(f)} - d_-^{(i)} = (\Delta d_-)_{fi} &\equiv C\left(\sin^2\psi^{(f)} - \sin^2\psi^{(i)}\right) + \frac{2\varkappa}{3S}\left(\sin 2\psi + \cot\psi\right)\Big|_{\psi^{(i)}}^{\psi^{(f)}}
\end{aligned} \tag{4.14}$$

One can say that in our dynamics $d_-(t)$ cannot decrease below $\varkappa t_*$ scale because of thermal diffusion, which brings us to evaluation $C \gtrsim \varkappa/D$ and makes second term negligibly small everywhere in our interval.

There are two large-time regimes for C dynamics through tumbling acts. First, at times t : $e^{-\lambda t} \lesssim \sqrt{\varkappa t_*}/L$, while thermal diffusion does not affect d_-, this parameter is not independent: $d_- = L^2 e^{-\rho}$. Then at larger times contraction of d_- goes to scale $\varkappa t_*$ and becomes greater with ψ, then constant in (4.14) of order $\varkappa/D$.

Defining t' as a moment of time when angle was small $\psi = -\psi_*$, as the limit of deterministic dynamics applicability

$$\cot\psi + \cot\psi_* = St', \tag{4.15}$$

one can write PDF $\mathcal{P}(t,\rho,\psi)$ as a convolution of $\mathcal{P}$ at time t' with the Dirac delta functions deterministic propagation by (4.14):

$$\mathcal{P}(t,\rho,\psi) = \int d\rho' d\psi' \mathcal{P}(t-t',\rho',\psi')\,\delta\left(\rho - \rho' - \ln\left|\frac{\sin\psi'}{\sin\psi}\right|^2\right)\frac{\delta\left(\cot\psi - \cot\psi' - St'\right)}{\sin^2\psi}, \tag{4.16}$$

where the factor $\sin^{-2}\psi$ was restored to normalize the solution over ψ. Performing the corresponding integration in (4.13) with the shift of ρ brings us to approximate expression

$$\begin{aligned}
\theta_g(t,r,\varphi) \approx \int d\rho \int\limits_{\mathcal{N}(D/S)^{1/3}}^{\pi-\mathcal{N}(D/S)^{1/3}} \frac{d\psi}{\sin^2\psi}\mathcal{P}(t,\rho,-\psi_*)\times \\
\times\frac{\psi_*^2}{2\pi L^2}\exp\left[-\frac{1}{2}\left(\frac{r}{L}\right)^2\left[\frac{\psi_*^2 e^{\rho}}{\sin^2\psi}\sin^2(\psi-\varphi) + \psi_*^{-2}e^{-\rho}\sin^2\psi\cos^2(\psi-\varphi)\right]\right],
\end{aligned} \tag{4.17}$$

where $\int \mathrm{d}\psi$ has a cutoff with $\mathcal{N}$ of order of unity. Considering that characteristic ρ is large, one can simplify the exponent and perform integration over $d\cot\psi$. Utilizing the evaluation of PDF:

$$\psi_*\mathcal{P}(t,\rho,\psi_*) \sim \psi\mathcal{P}(t,\rho,\psi) = \mathcal{P}(t,\rho) \propto \frac{1}{\sqrt{t}}\exp\left[-\lambda t\mathcal{S}\left(\frac{\rho}{\lambda t}\right)\right]$$

we have obtained formula:

$$\begin{gathered}\theta_g(t,r,\varphi)\sim\frac{1}{Lr\,|\sin\varphi|}\int d\rho\,\frac{1}{\sqrt{t}}\exp\left(-\lambda t\mathcal{S}\left(\frac{\rho}{\lambda t}\right)-\rho/2\right)\times\\ \times\left[\operatorname{erf}\left(\frac{e^{\rho/2}r}{\sqrt{2}L}\,|\sin\varphi|\,(\psi_*\cot\varphi+\mathcal{N})\right)-\operatorname{erf}\left(\frac{e^{\rho/2}r}{\sqrt{2}L}\,|\sin\varphi|\,(\psi_*\cot\varphi-\mathcal{N})\right)\right]\end{gathered}\tag{4.18}$$

On times $t\gg t_*\ln\varkappa t_*/L^2$, when $d_-\sim\varkappa t_*$ at $\psi\sim\psi_*$, the expression changes to:

$$\begin{gathered}\theta_g(t,r,\varphi)\sim\frac{\psi_*}{Lr\,|\sin\varphi|}\int d\rho\,\frac{1}{\sqrt{t}}\exp\left(-\lambda t\mathcal{S}\left(\frac{\rho}{\lambda t}\right)-\rho/2\right)\times\\ \times\left[\operatorname{erf}\left(\frac{\mathcal{C}r}{\sqrt{2\varkappa t_*}}\,|\sin\varphi|\,(\psi_*\cot\varphi+\mathcal{N})\right)-\operatorname{erf}\left(\frac{\mathcal{C}r}{\sqrt{2\varkappa t_*}}\,|\sin\varphi|\,(\psi_*\cot\varphi-\mathcal{N})\right)\right]\end{gathered}\tag{4.19}$$

where $\mathcal{C}$ is of order of unity. Let us point out here that according to (4.18,4.19) anisotropy of our solution is achieved effectively by stretching along the Ox axis by a factor of order ψ_*^{-1}.

Recalling that ansatz θ_g was used to describe decay problem for pair correlation function $F(t,\boldsymbol{r})$, we point out further our main findings for it.

4.3.1 One-point mean

While time is limited by $t\lesssim t_*\ln\varkappa t_*/L^2$, there is a region near the origin $r=0$ described by a relation on y from (4.18):

$$L\psi_*e^{-\rho/2}\gtrsim r\,|\sin\varphi|\gtrsim r\psi_*,$$

where answer in the first approximation remains constant both in time and space:

$$F(t,\boldsymbol{r})\sim\frac{1}{L^2}\int d\rho\mathcal{P}(t,\rho)\to F(0,0)\tag{4.20}$$

After that at $t\gg t_*\ln\varkappa t_*/L^2$ in a region of applicability on y bounded from above by a constant:

$$\sqrt{\varkappa t_*}\psi_*\gtrsim r\,|\sin\varphi|\gtrsim r\psi_*,$$

one can arrive to exponential decrease of F with a rate of order t_*^{-1} by considering (4.19) there:

$$F\sim\frac{1}{L}\sqrt{\frac{D}{\varkappa}}\int d\rho\,\frac{1}{\sqrt{t}}\exp\left(-\lambda t\mathcal{S}\left(\frac{\rho}{\lambda t}\right)-\rho/2\right)\sim\sqrt{\frac{DL^2}{\varkappa}}F(0,0)\exp(-\lambda_1 t)\tag{4.21}$$

where integration over ρ in the limit of large t was performed in saddle-point approximation near extremum point $\rho_1=\lambda t\xi_1$ [1]:

$$\mathcal{S}'(\xi_1)=1/2;\quad\lambda_1=\lambda\left[\mathcal{S}(\xi_1)-\xi_1/2\right].\tag{4.22}$$

4.3.2 Asymptotics beyond the diffusive scale

Outside of this neighbourhood of the origin, we study a region where (4.18,4.19) are applicable:

$$|\sin\psi| \gtrsim \psi_* |x|, \quad t \gtrsim t_* \ln|x/L|$$

and also the following relation is fulfilled:

$$r\left|\sin\left(\varphi\pm\psi_*\right)\right| \gtrsim L\psi_* e^{-\rho_1/2}, \quad \text{if } t \lesssim t_* \ln\left(\varkappa t_* L^{-2}\right)$$
$$r\left|\sin\left(\varphi\pm\psi_*\right)\right| \gtrsim \sqrt{\varkappa t_*}, \quad \text{if } t \gg t_* \ln\left(\varkappa t_* L^{-2}\right)$$

Then solution asymptotic becomes independent of $\varkappa$ regime:

$$F\left(t,x,y\right) \sim \frac{\psi_*}{L\left|y\right|}\int d\rho\,\frac{1}{\sqrt{t}}\exp\left(-\lambda t\mathcal{S}\left(\frac{\rho}{\lambda t}\right)-\rho/2\right) \sim \frac{\psi_* e^{-\lambda_1 t}}{L\left|y\right|} \tag{4.23}$$

The dependence of the correlation function on the distance as $1/r$ was established in [29] for statistically isotropic random flow. The equation above (4.23) is applicable if one can neglect the second small term in exponent (4.17): $r\cos\left(\varphi-\psi_*\right) < L\exp(\xi_1 t/t_*)$.

Chapter 5

Conclusions

We have considered coherent fluid flow in a columnar vortex, that emerges in a three-dimensional fast-rotating hydrodynamic turbulent system. Conducting a statistical examination of small-scale turbulent pulsations of velocity via quasilinear approach developed in a theory [16], we moved on to a study of passive scalar field θ statistics in such type of flow. Within that we have analyzed the evolution of initial distribution θ_0 (so-called decay problem) in the cross-section of a vortex for the mean field $\langle\theta\rangle$ and pair correlation function.

Columnar vortex turbulent flow is of strong shear type with a rate $S = \sqrt{\epsilon/\nu}$, in which background velocity random deviations could also be analyzed in terms of their gradients' statistics σ_{ij}. Utilizing the approach of small-scale wave analysis, we show that they are small compared to the coherent flow, possess high anisotropy and characteristic correlation time of order Ω^{-1} - reciprocal angular speed of system rotation. That allows us to consider $\hat{\sigma}$ as weak short-correlated white noise addition to S.

It has been shown that for passive scalar advection in presence of only constant shear $S\delta_{ix}\delta_{jy}$ the dependence on the latter Oz axis effectively uncouples. This was a motive for us to solve for a start a simplified problem in Oxy plane. Considering the stochastic dynamics of parameters, describing the spatial distribution of θ, we looked for the optimal fluctuation - the most probable realization of the parameters stochastic dynamics which yields into the large-time evolution regime for θ. Generalization of the analysis to the three-dimensional case with the consideration of additional spatial parameters is planned to be carried out in the further research.

As it turns out, only one noise component σ_{yx} effectively governs the mixing of scalar by counteracting steady shear near the $y = 0$. This results in exponential divergence of the Lagrangian trajectories with a characteristic rate $(DS^2)^{1/3}$ and anisotropy in space anisotropy which effectively is a stretching along the Ox axis by a factor of order ψ_*^{-1}.

We extend this approach to the pair correlation function for the space homogeneous problem, which is the result of averaging both in time over fluctuations and space over random initial distribution realizations. For its decay problem we have obtained asymptotic expressions for different space regions, see (4.20,4.21,4.23). We point out there is a transition to exponential decay for mean square of the scalar quantity.

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