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MASTER'S THESIS

Influence of the processes of resonant scattering on critical Josephson current through Anderson insulator

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МАГИСТЕРСКАЯ ДИССЕРТАЦИЯ

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Abstract

We calculate Josephson critical current I_c of a SAS structure, where A is a thick Anderson insulator, with thickness L much larger than localization length ξ. Level spacing Δ_ξ in localization volume is considered to be much larger than superconducting gap Δ_0. For a one-dimensional as well as quasi-1D models of Anderson insulator we find asymptotic expressions for the average magnitude of Josephson current $\langle I_c(L)\rangle$ for cases $L \ll L_M$, $L_M \ll L \ll L_M^2/\xi$ and $L \gg L_M^2/\xi$ where $L_M = 2\xi \ln \frac{\Delta_\xi}{\Delta}$.

Contents

Chapter 1

Introduction

1.1 Motivation

Transmission of Cooper pairs through insulating barriers has been studied since the discovery of the Josephson effect in superconductor-insulator-superconductor (SIS) structures. In band insulators probability of electron transmission decays exponentially with barrier thickness, with a short decay length of atomic scale. The situation exists for the cases in which the band insulator is replaced by the Anderson insulator. In this case, the density of electron states at the Fermi-level is relatively high and due to strong disorder the electron states are localized.

Particularly interesting results were reported in Ref. [9] where Josephson critical current was detected up to the thickness of insulating barrier $L \leq 60$nm, while localization length ξ is just few nanometers long; In addition, large subgap conductance G_{NS} was measured in similar structures of the SAN type [9], together with a rich voltage-dependent structure at subgap voltages $V < \Delta/e$, where Δ is superconducting gap in the S terminal. These results demonstrate the persistence of a superconducting proximity effect (and, therefore, existence of quantum coherence) on distances much above localization length.

Usually, Josephson critical current via tunnel barrier $I_c(L) = I_c^{AB}(L) = \# \frac{e\Delta}{\hbar}\sigma(L)$ according to classical Ambegaokar-Baratoff relation (here and below we consider zero-temperature limit for the critical current magnitude). Similar kind of relation is known for short SNS junctions, according to Beenakker theory [5]. The key feature of electron transport leading to such a relation is that tunnelling transmission amplitudes $t(L, E)$ are energy-independent within the relevant energy scale $E \sim \Delta_0$. For transport via Anderson insulator, it works for relatively short junctions only; relevant condition will be shown to be $L \leq L_M(\Delta_0) = 2\xi \ln \frac{\Delta_\xi}{\Delta_0}$. Here ξ is typical localization length defined via relation

$$\left\langle \ln \left| \frac{\psi^2(x)}{\psi^2(0)} \right| \right\rangle \approx -\frac{x}{\xi} \tag{1.1}$$

for the asymptotic decay of wave-function intensity at long distances $x \gg \xi$ from its maximum. $\Delta_\xi = (\pi\nu A\xi)^{-1}$ is typical level spacing between single-electron levels within localization volume $A\xi$ (A is the cross-section), and ν is the density of states. The ratio $\frac{\Delta_\xi}{\Delta_0}$ is very large, $\sim 10^2 - 10^3$ for relevant experimental conditions like in Ref. [9]. The length $L_M(\Delta_0) \sim (10-15)\xi$ corresponds to a typical size of Mott resonant pair [16] of localized eigenstates with energy separation $\Delta_0 \ll \Delta_\xi$.

We will demonstrate the existence of three different types of behavior of $I_c(L)$ for the lengths L which belong to the regions of short junctions $L \le L_M(\Delta_0)$, intermediate junctions with $L_M(\Delta_0) \le L \le L_M^2(\Delta)/\xi$, and longest junctions with $L \ge L_M^2(\Delta_0)/\xi$. The major exponential dependence of the average critical current $I_c(L)$ in all three regions follows the one of $I_c^{AB}(L)$ and $\sigma(L)$, as they are proportional to $\exp(-L/4\xi)$ for a quasi-one-dimensional model of the insulator. However, the ratio $I_c(L)/I_c^{AB}(L)$ grows $\propto L$ in a broad intermediate region $L_M(\Delta) \le L \le L_M^2(\Delta_0)/\xi$. For longest junctions with $L \gg L_M^2(\Delta_0)/\xi$, the ratio $I_c(L)/I_c^{AB}(L)$ does not depend on L anymore, but it is much larger than unity; however, this surprising result is valid for extremely low temperatures only.

1.2 Wave function statistics in the Anderson insulator

Strong fluctuations of localized wave-function's amplitudes were first found by V.Melnikov [14] who studied statistics via a finite one-dimensional wire. One of immediate results of these fluctuations is that average tunneling conductance through an insulating wire of length $L \gg \xi$ decays $\sigma(L) \propto \exp(-L/4\xi)$ instead of naively expected $\exp(-L/\xi)$, with ξ defined as (1.1). From now on the we will measure lengths in the units of the localization length ξ if not stated otherwise.
Here we follow Ref. [11] in which the phenomenological approach to the problem of probability distribution of localized wave function was proposed. We will describe the statistics of the tails of the localized wave functions in terms of the following logarithm:

$$\chi(x) = -\ln|\psi(x)|^2. \tag{1.2}$$

An additional minus sign is added so that $\chi(x) \ge 0$. It was shown in [15] that the statistics of χ are given by the functional integral. To avoid an accurate analytical treatment of the integral, it can be shown that for the description of the tails, one can fix the point x_0 and the value $\chi(x_0)$ so that $|\psi(x_0)|^2$ is the global maximum. The normalization delta function can be replaced with an approximate condition $\chi(0) > \chi(x_0)$. This condition will guarantee that the normalization of wave functions will be of order one. This allows us to write down an approximate measure for the functional integral, describing the statistics of $\chi(x)$:

$$d\mu_{x_0}[\chi(x)] \propto \exp\left\{-\frac{1}{4}\int\left[\frac{d\chi}{dx} - \text{sign}\,(x-x_0)\right]^2\right\}\mathcal{D}[\chi(x)]. \tag{1.3}$$

We can already conclude that the tails on the different sides of the position of the maximum x_0 are distributed independently. The action in the exponent of (1.3) leads to the stationary trajectory described be the Fokker-Planck equation:

$$\frac{\partial P}{\partial r} = \frac{\partial^2 P}{\partial \chi^2} - \frac{\partial P}{\partial \chi}, \tag{1.4}$$

where $r = |x - x_0|$. Since we are interested in the solutions for $r \gg 1$, we can write down the asymptotic for of the solution as

$$P(\chi, r) = \frac{1}{2\sqrt{\pi r}} f\left(\frac{\chi}{r}\right) \exp\left[-\frac{(\chi - r)^2}{4r}\right]. \tag{1.5}$$

Here $f(\chi/r)$ is the cutoff factor which takes into account the condition $P(\chi < 0, r) = 0$. The exact form of the factor f cannot be determined within the discussed approach. Nevertheless, for the purpose of numerical analysis we will need a better understanding of its shape. The first observation is that the maximum value of the distribution function (1.5) at $\chi = r$ should not be affected by the cutoff factor. Furthermore, the function f should not change the shape of the distribution $P(\chi, r)$ at $\chi > r$. All of the mention properties combine in the following expressions:

$$f(0) = 0, \quad f(x > 1) = 1. \tag{1.6}$$

This, however, is not enough as the incline near the zero argument is also important. One method to define it is to consider the following correlation function:

$$S(\omega, L) = \nu^{-2} \left\langle \sum_{n \neq m} \delta(E_n - E)\delta(E_m - E - \omega)\psi_n(-L/2)\psi_n^*(L/2)\psi_m^*(-L/2)\psi_m(L/2) \right\rangle. \tag{1.7}$$

In Ref. [12] the asymptotic expressions for this correlation function were studied using methods described in the Section 1.2. For the limit $\omega \to 0$, the correlation function (1.7) was calculated exactly in Ref. [12]. This allows to find an accurate leading term in the limit $L \gg 1$ with the correct coefficient:

$$S(\omega \to 0, \xi \ll L) \approx \frac{\pi^{7/2}}{16} \frac{e^{-L/4}}{L^{3/2}}. \tag{1.8}$$

On the other hand, the average in (1.7) can be calculated by implementing the distribution function (1.5) (see Sec. III of Ref. [12]):

$$S(\omega \to 0, L) \approx \int_0^L dz \int_0^{+\infty} d\chi_1 \int_0^{+\infty} d\chi_2 \, P(\chi_1, z) P(\chi_2, L - z) e^{-(\chi_1 + \chi_2)}. \tag{1.9}$$

The exponential part of the dependence of (1.9) is not influenced by the form of the cutoff factor and is $e^{-L/4\xi}$. The pre-exponent, however, is dependent on the exact form of the cutoff factor. The same pre-exponent as the one in (1.8) can be achieved if the cutoff factor satisfies the condition $f(x \to 0) \propto x$. Under this condition, the main contribution to the integral over z comes from the ends of the domain of integration. This leads to the estimation

$$S(\omega \to 0, L) \approx f'(0) \frac{\pi^{3/2}}{2\sqrt{2}} \frac{e^{-L/4}}{L^{3/2}}. \tag{1.10}$$

By comparing (1.10) to (1.8) we find the estimation for the incline of the function $f(x)$ near $x = 0$:

$$f'(0) \approx \frac{\pi^2}{4\sqrt{2}} \approx 1.7 \tag{1.11}$$

Finally, for the numerical analysis we will use the simplest form of f possible, which satisfies all conditions mentioned above:

$$f(x) \approx \begin{cases} f'(0)\, x, & f'(0)x < 1; \\ 1, & f'(0)x \geq 1. \end{cases} \tag{1.12}$$

1.3 Beenakker's formula

To compute the current we apply similar methods to those which were first applied for the analysis of the fluctuations in the critical Josephson current of a short disordered SNS junction (Ref. [5]). The problem was considered in the dirty limit $l \ll L \ll \xi_c$, where L is the junction length, l is the elastic mean free path, and $\xi_c = \hbar v_F / \pi \Delta_0$ is the superconducting coherence length. The condition $l \ll L$ and the general form of the equations derived by the means of a transmission matrix formalism makes them applicable to the case of the insulator discussed in the present work. From now on we consider a Q1D problem of single-channel scattering.

For the derivation of the convenient expression, we will start by writing the expression for the critical current, which results from the Bogoliubov-de Gennes Hamiltonian. As it was shown in Ref. [5],

$$I(T) = \frac{2e}{\hbar} T \int\limits_{-\infty}^{+\infty} d\varepsilon \ln\left(\cosh \frac{\varepsilon}{2T}\right) \partial_\varphi \rho(\varepsilon, \varphi), \tag{1.13}$$

where $\varphi = \varphi_2 - \varphi_1$ is the difference of the superconducting phases and $\rho(\varepsilon, \varphi)$ is the density of states, which can be expressed as

$$\rho = -\frac{1}{\pi i} \partial_\varepsilon \ln g(\varepsilon, \varphi). \tag{1.14}$$

After introducing the notations

$$\alpha(\varepsilon) = \exp\left(-i \arccos \frac{\varepsilon}{\Delta_0}\right), \quad r_A(\varphi) = \begin{Vmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{Vmatrix}, \tag{1.15}$$

we can write the function $g(\varepsilon, \varphi)$ as the following determinant:

$$g(E, \varphi) = \det\left[1 - \alpha^2(E) r_A^* \hat{S}(E) r_A \hat{S}^*(-E)\right] \tag{1.16}$$

and $\hat{S}(E)$ is the scattering matrix. Then, we can deform the contour of integration in (1.13) and, instead of the integral, we get a formula with the summation over the Matsubara frequencies:

$$I(T) = \frac{4e}{\hbar} T \sum_{\omega > 0} \partial_\varphi \ln g(i\omega, \varphi), \quad \omega = 2\pi T(n + 1/2). \tag{1.17}$$

In our work, we will discuss only the case of extremely small temperatures. Taking the limit $T \to 0$, we come to the following equation:

$$I(T \to 0) = \frac{2e}{\pi \hbar} \int\limits_0^{+\infty} d\omega \, \partial_\varphi \ln g(i\omega, \varphi). \tag{1.18}$$

Next, we consider the case of the single-channel scattering, which gives us the ability to consider only a 2×2 scattering matrix and analyze the expression (1.18). Introducing the elements of the scattering matrix

$$\hat{S}(E) = \begin{Vmatrix} S_{11}(E) & S_{12}(E) \\ S_{21}(E) & S_{22}(E) \end{Vmatrix}, \tag{1.19}$$

we can find the general expression for $g(E,\varphi)$ by substituting (1.15) and (1.19) into (1.16):

$$g(E,\varphi) = \\ = 1-\alpha^2\Big\{S_{11}(E)S_{11}^*(-E)+2\cos\varphi\, S_{12}(E)S_{12}^*(-E)+S_{22}(E)S_{22}^*(-E)-\alpha^2\det[S(E)]\det[S^*(-E)]\Big\}. \tag{1.20}$$

Here we have used that $S_{12} = S_{21}$ due to time-reversal symmetry. Substituting into (1.18), gives the following:

$$I(T\to 0) = \frac{4e}{\pi\hbar}\sin\varphi \int\limits_0^{+\infty} d\omega\, \alpha^2(i\omega)\frac{|S_{12}(i\omega)|^2}{g(i\omega,\varphi)}. \tag{1.21}$$

In this work we will analyze the critical current for different values of length of the insulating junction L. Firstly, to determine the regimes stemming from the different values of L, we note that the function

$$\alpha^2(i\omega) = \left[\sqrt{1+(\omega/\Delta_0)^2} - (\omega/\Delta_0)\right]^2 \tag{1.22}$$

is decaying with characteristic scale of Δ_0. We take into account the fact that the localized states in the bulk of the insulator have finite lifetime and, subsequently, the complex term in their energies $E - i\Gamma$, where Γ is the level width. The exact value of Γ is dependent on the position of the localized state. However, to approximate the expression (1.21) averaged over the insulator statistics, Δ_0 should be compared to the value of Γ_{typ} which gives the largest contribution to the average value of the current. It will be shown later in the text that if the average over the realizations of disorder is taken, the relevant values of the width are $\Gamma_{typ} \sim \Delta_\xi e^{-L}$. Here we emphasize that the comparison between Δ_0 and $\Delta_\xi e^{-L}$ is equivalent to comparison between L and L_M to underline the importance of the Mott scale. Now, we can proceed with the approximations for both cases:

1.3.1 Approximation for $\Delta_0 \ll \Gamma_{typ}$

Here since Δ_0 and, as the result, ω is negligible in comparison to the typical values of the level width, we approximate expression (1.16) by saying that $\hat{S}(i\omega) \approx \hat{S}(0)$.

$$g(i\omega,\varphi)\Bigg|_{\Delta_0\ll\Gamma_{typ}} \approx \left[1-\alpha^2(i\omega)\right]^2 + 4\alpha^2(i\omega)\,|S_{12}(0)|^2\sin^2(\varphi/2). \tag{1.23}$$

Here we have used the unitary property to get $\det\left[\hat{S}(0)\right]\det\left[\hat{S}^*(0)\right] = 1$. Using (1.23) in (1.21), we find

$$I(T\to 0)\Big|_{\Delta_0\ll\Gamma_{typ}} \approx \frac{e}{\pi\hbar}\sin\varphi\int\limits_0^{+\infty} d\omega\frac{|S_{12}(0)|^2}{1+\omega^2/\Delta_0^2-|S_{12}(0)|^2\sin^2(\varphi/2)}. \tag{1.24}$$

Calculation of the integral over ω in (1.24) yields

$$I(T\to 0)\Big|_{\Delta_0\ll\Gamma_{typ}} \approx \frac{e\Delta_0}{2\hbar}\frac{|S_{12}(0)|^2\sin\varphi}{\sqrt{1-|S_{12}(0)|^2\sin^2(\varphi/2)}}, \tag{1.25}$$

which is the classical form of the Beenakker's formula. In this limit, the transmission amplitudes $t = S_{12}$ are calculated at zero energy and, therefore, are not dependent on it.

1.3.2 Approximation for $\Delta_0 \gg \Gamma_{typ}$

In this case the superconducting gap is significantly larger than the typical level width. If the level width is neglected, however, the resonant scattering on the localized states will be absent and

$$|S_{11}(\Gamma=0)| = |S_{22}(\Gamma=0)| = 1, \quad |S_{12}(\Gamma=0)| = 0. \tag{1.26}$$

Since in the numerator of (1.21) a small value $|S_{12}(i\omega)|^2$ is already present, for the denominator we can use expressions (1.26). Calculating $g(i\omega,\varphi)$ in this limit, we get

$$g(i\omega,\varphi)\Big|_{\Delta_0\gg\Gamma_{typ}} \approx \left[1-\alpha^2(i\omega)\right]^2. \tag{1.27}$$

Finally, we substitute (1.27) into (1.21):

$$I(T\to 0)\Big|_{\Delta_0\gg\Gamma_{typ}} \approx \frac{e}{\pi\hbar}\sin\varphi\int\limits_0^{+\infty} d\omega\frac{|S_{12}(i\omega)|^2}{1+\omega^2/\Delta_0^2}. \tag{1.28}$$

We wish to use formulas (1.24) and (1.28) for the calculation of the average critical current. However, averaging in the case $\Delta_0 \ll \Gamma_{typ}$ presents a problem because of the presence of the term containing $|S_{12}(0)|^2$ in the denominator of (1.24). To make analytical analysis of these two regions of the parameters possible, we will neglect this term during the calculation in the region $\Delta_0 \ll \Gamma_{typ}$. Analyzing (1.24), we see that in the limit $\Delta_0 \ll \Gamma_{typ}$ this approach effectively provides the lower bound for the first Fourier component of the current. We expect this approximation to give a result which differs from the accurately evaluated asymptotic by the factor of the order of one.

Taking all made approximations into account, we can write the expression for the critical current, which will be used during the calculation of its average value for all regions of the parameters.

$$I_c(T \to 0) \approx \frac{e}{\pi \hbar} \int\limits_0^{+\infty} d\omega \frac{|t(i\omega)|^2}{1 + \omega^2/\Delta_0^2}. \tag{1.29}$$

Here $t(i\omega)$ is the transmission amplitude of the resonant scattering.

Chapter 2

Calculation of the critical current

In this chapter, we discuss the calculation of the critical current through an SAS structure, in which two superconducting bulks are connected by a Q1D Anderson insulator. We will use an approximate expression (1.29) for the calculation of the current. We are interested in the effects caused by the statistics of the localized wave functions. In order to include them, we to average an integrand taking into account the distribution (1.5). Throughout this work we denote values averaged over disorder with angle brackets.

$$\langle I_c \rangle = \frac{e}{\pi\hbar} \int\limits_0^{+\infty} d\omega \left\langle \frac{|t(i\omega)|^2}{1+\omega^2/\Delta_0^2} \right\rangle . \tag{2.1}$$

Since the distribution (1.5) describes the statistics of the tails of the localized wave function, we need to express the integrand through localized states. This can be done using the relation between transmission amplitudes and Green's functions:

$$|t(i\omega)|^2 = \nu_0^{-2} \sum_{n,m} \frac{\psi_n(-L/2)\psi_n^*(L/2)\psi_m^*(-L/2)\psi_m(L/2)}{(E_n - i\Gamma_n - i\omega)(E_m + i\Gamma_m + i\omega)}, \tag{2.2}$$

where $\nu_0 = (\pi\Delta_\xi\xi)^{-1}$. Here we emphasize the presence of the level widths Γ_n and Γ_m. The necessity of their inclusion will be more transparent as the calculation will progress.

From this expression we can discern two distinct contributions to the current. The first on, which we will call the **diagonal contribution** comes from the terms of the sum (2.2) in which $n = m$. The other, **non-diagonal contribution**, is made of the terms of the sum (2.2) over different localized states $n \neq m$.

Calculation of these terms is immensely different. The most straightforward reason for it is that the terms in the diagonal contribution depend only on a single energy of the localized state and in the non-diagonal contribution every term is dependent on two energies.

2.1 Diagonal contribution

For the diagonal contribution we can rewrite the expression as follows:

$$\langle I_{diag}\rangle = \frac{e}{\pi\hbar\nu_0^2}\int\limits_0^{+\infty}\frac{d\omega}{1+\omega^2/\Delta_0^2}\left\langle\sum_n\frac{|\psi_n(-L/2)|^2|\psi_n(L/2)|^2}{E_n^2+(\Gamma_n+\omega)^2}\right\rangle =$$

$$= \frac{e}{\pi\hbar\nu_0^2}\int d\boldsymbol{r}\int\limits_{-\infty}^{+\infty}dE\sum_n\delta(\boldsymbol{r}_n-\boldsymbol{r})\delta(E_n-E)\int\limits_0^{+\infty}\frac{d\omega}{1+\omega^2/\Delta_0^2}\left\langle\frac{|\psi_{\boldsymbol{r}}(-L/2)|^2|\psi_{\boldsymbol{r}}(L/2)|^2}{E^2+(\Gamma+\omega)^2}\right\rangle =$$

$$= \frac{e}{\pi\hbar\nu_0^2}\int d\boldsymbol{r}\int\limits_{-\infty}^{+\infty}dE\,\nu(E,\boldsymbol{r})\int\limits_0^{+\infty}\frac{d\omega}{1+\omega^2/\Delta_0^2}\left\langle\frac{|\psi_{\boldsymbol{r}}(-L/2)|^2|\psi_{\boldsymbol{r}}(L/2)|^2}{E^2+(\Gamma+\omega)^2}\right\rangle. \quad (2.3)$$

Here for ψ_n and $\psi_{\boldsymbol{r})}$ we denote wave functions of the localized states which have their center at the coordinates $\boldsymbol{r}_n$ and $\boldsymbol{r}$ respectively.
From now on we assume the density of states to be uniform in the volume of the insulator and to be not dependent on the energy. Assuming A is the cross-section area of the insulator, substituting $\nu = (\pi A\Delta_\xi\xi)^{-1}$ from the definition of the localization energy, we come to the following expression:

$$\langle I_{diag}\rangle = \frac{e}{\pi\hbar\nu_0}\int\limits_{-L/2}^{L/2}dz\int\limits_{-\infty}^{+\infty}dE\int\limits_0^{+\infty}\frac{d\omega}{1+\omega^2/\Delta_0^2}\left\langle\frac{|\psi_z(-L/2)|^2|\psi_z(L/2)|^2}{E^2+(\Gamma_l+\Gamma_r+\omega)^2}\right\rangle \quad (2.4)$$

Here we have taken into account that total level width consists of two contributions Γ_l and Γ_r, coming from the processes of the decay of the localized state to the left or to the right lead respectively. This means, that the total level width should be written as their sum:

$$\Gamma = \Gamma_l + \Gamma_r. \quad (2.5)$$

The first detail worth noting is that the integral over ω in (2.4) diverges if $\Gamma = 0$. This means that without taking into account the finite level width the critical current is infinite due to the diverging single-state resonant scattering. We emphasize here that this is not the case for the non-diagonal contribution to the critical current due to the finite spacing between levels with different indexes. This particular behavior of the two contributions justifies their separate consideration. Now we can finally implement the distribution function (1.5).
As it was described in the section (1.2), the shape of this function in 1D can be justifiably approximated with the log-normal distribution with an additional cutoff factor. Since the values of the wave

function is taken at two coordinates, it will require two distribution functions: $P(\chi_l, L/2+z)$ and $P(\chi_r, L/2-z)$. Following the definition of χ, wave functions should be replaced as

$$|\psi(-L/2)|^2 \longrightarrow e^{-\chi_l}, \quad |\psi(L/2)|^2 \longrightarrow e^{-\chi_r}. \tag{2.6}$$

We also need to remember that Γ_l and Γ_r are also disorder dependent. To take it into account, we assume level widths to be proportional to the values of the $|\psi|^2$ at the coordinate of the corresponding lead with the proportionality coefficient being the order of Δ_ξ:

$$\Gamma_l \sim \Delta_\xi |\psi(-L/2)|^2, \quad \Gamma_r \sim \Delta_\xi |\psi(L/2)|^2. \tag{2.7}$$

Using all described approximations and taking an integral over E in (2.4), lets us write down the expression which will be used to extract different asymptotic expressions for different values of the insulator length:

$$\langle I_{diag} \rangle = \frac{e}{\hbar\nu_0} \int\limits_{-L/2}^{L/2} dz \int\limits_{0}^{+\infty} \frac{d\omega}{1+\omega^2/\Delta_0^2} \times \\ \times \int\limits_{0}^{+\infty} d\chi_l d\chi_r P(\chi_l, L/2+z) P(\chi_r, L/2-z) \frac{e^{-(\chi_l+\chi_r)}}{\Delta_\xi(e^{-\chi_l}+e^{-\chi_r})+\omega} \tag{2.8}$$

Analyzing this expression, we can extract information about the characteristic scale which can be compared to the length L. Since the characteristic scale of the variable ω is Δ_0, we can roughly compare it with $\Delta_\xi e^{-\chi}$. Furthermore, the distribution function $P(\chi, z)$ has the maximal value at $\chi = z$ and z, in our case, has the characteristic scale L.

2.2 Non-diagonal contribution

For the non-diagonal contribution, similar actions lead to the following result:

$$\langle I_{non-diag} \rangle = \frac{2e}{\pi\hbar\nu_0^2} \int\limits_{-L/2}^{L/2} dz_A dz_B \int\limits_{-\infty}^{+\infty} dE_A dE_B \int\limits_{0}^{+\infty} \frac{d\omega}{1+\omega^2/\Delta_0^2} \times \\ \times \operatorname{Re} \left\langle \frac{\psi_{z_B}(-L/2)\psi^*_{z_B}(L/2)\psi_{z_A}(L/2)\psi^*_{z_A}(-L/2)}{(E_B - i\Gamma_B - i\omega)(E_A + i\Gamma_A + i\omega)} \right\rangle. \tag{2.9}$$

Here ψ_{z_A} and ψ_{z_B} are the wave functions of the bound states with the localization center having the coordinate z_A and z_B respectively. In this case, however, we need to take into account the hybridization of the wave functions. It can be easily done by treating these two states as the Mott pair with the following Hamiltonian:

$$H_{Mott} = \begin{Vmatrix} E_A & J \\ J & E_B \end{Vmatrix}. \tag{2.10}$$

The real wave functions of the Hamiltonian (2.10) are the following linear combinations:

$$\psi_+ = u_+\psi_{z_A} + u_-\psi_{z_B}, \quad \psi_- = u_-^*\psi_{z_A} - u_+^*\psi_{z_B}, \tag{2.11}$$

where

$$|u_\pm|^2 = \frac{1}{2}\left(1 \mp \frac{\varepsilon_-}{\Delta}\right), \quad \Delta = \sqrt{\varepsilon_-^2 + 4|J|^2}, \quad \varepsilon_\pm = E_B \pm E_A. \tag{2.12}$$

Eigenvalues of the Hamiltonian are expressed through the energies of the localized states as

$$E_\pm = \frac{1}{2}(\varepsilon_+ \pm \Delta). \tag{2.13}$$

To write down the complete expression for the non-diagonal term, we need to take into account the fact that the hybridization matrix element J is also dependent on the realization of the disorder. Since it can be estimated as the product of the overlapping wave functions

$$J \approx \Delta_\xi \psi_{z_A}(x)\psi_{z_B}(x). \tag{2.14}$$

Here x is some point which lies on the tails of the localized wave functions ψ_{z_A} and ψ_{z_B}. Due to the known property of the log-normal distribution that the product of the independent random variables, having the log-normal distribution, also has the log-normal distribution. Furthermore, the probability distribution of the right-hand side of (2.14) is independent on the exact value of the coordinate x at which we look at the value of the product.

As the result, we can safely assume that the hybridization matrix element J can be parameterized in the same manner as it was done before, when we had only one localized state. The distribution function for J, however, will differ. Since, as it was already stated, hybridization can be viewed as the product of two tails of the wave functions, the cutoff factor in the distribution P_J for J should be squared. So the parameterization and the distribution function are the following:

$$P_J(\chi_J, r) = \frac{1}{2\sqrt{\pi r}} f^2\left(\frac{\chi_J}{r}\right) \exp\left[-\frac{(\chi_J - r)^2}{4r}\right], \quad J = \Delta_\xi e^{-\chi_J/2}. \tag{2.15}$$

For the localized wave functions the parameterization will have the usual form:

$$|\psi_{z_A}(-L/2)|^2 = e^{-\chi_A^{(l)}}, \quad |\psi_{z_A}(L/2)|^2 = e^{-\chi_A^{(r)}}, \quad |\psi_{z_B}(-L/2)|^2 = e^{-\chi_B^{(l)}}, \quad |\psi_{z_B}(L/2)|^2 = e^{-\chi_B^{(r)}}. \tag{2.16}$$

So the correct way to write down the average for the non-diagonal contribution is the following:

$$\langle I_{non-diag}\rangle = \frac{2e}{\pi\hbar}\int\limits_0^{+\infty}\frac{d\omega}{1+\omega^2/\Delta_0^2}\int\limits_{-\infty}^{+\infty}d\varepsilon_+ d\varepsilon_- \int\limits_0^{+\infty}\frac{d\chi_A^{(l)}d\chi_A^{(r)}d\chi_B^{(l)}d\chi_B^{(r)}d\chi_J}{32\pi^{5/2}}\times$$
$$\times\int\limits_{-L/2}^{L/2} dz_A dz_B P(\chi_A^{(l)}, L/2+z_A)P(\chi_A^{(r)}, L/2-z_A)P(\chi_B^{(l)}, L/2+z_B)P(\chi_B^{(r)}, L/2-z_B)\times$$
$$\times P_J(\chi_J, |z_B - z_A|)\,\mathrm{Re}\,\frac{\psi_-(-L/2)\psi_-^*(L/2)\psi_+(L/2)\psi_+^*(-L/2)}{(E_- - i\Gamma_- - i\omega)(E_+ + i\Gamma_+ + i\omega)}, \tag{2.17}$$

where we also need to use the expressions from (2.11) to (2.16).

2.3 Short contact ($L \ll L_M$)

In this section we start the calculation of the Josephson current and consider the case of the contact much shorter than the Mott scale.

2.3.1 Diagonal term

Let us calculate the **diagonal contribution** for the lengths much shorter than the Mott scale. The first step is to substitute (1.5) into (2.8):

$$\langle I_{diag}(L)\rangle = \frac{e}{\hbar\nu_0}\int\limits_0^{+\infty}\frac{d\omega}{1+\omega^2/\Delta_0^2}\int\limits_{-L/2}^{+L/2}\frac{dz}{\sqrt{(L/2-z)(L/2+z)}}\int\limits_0^{+\infty}\frac{d\chi_l d\chi_r}{4\pi}\times \tag{2.18}$$

$$\times f\left(\frac{\chi_l}{L/2+z}\right) f\left(\frac{\chi_r}{L/2-z}\right)\frac{e^{-\chi_l-\chi_r}}{\omega+\Delta_\xi[e^{-\chi_l}+e^{-\chi_r}]}\exp\left[-\frac{(L/2+z-\chi_l)^2}{4(L/2+z)} - \frac{(L/2-z-\chi_r)^2}{4(L/2-z)}\right]. \tag{2.19}$$

Condition $L \ll L_M$ in terms of the integral means $\omega \ll \Delta_\xi e^{-\chi_{l,r}}$. After we neglect ω in the denominator, we can easily calculate an integral over this variable:

$$\langle I_{diag}(L \ll L_M)\rangle \approx \frac{\Delta_0}{\Delta_\xi}\frac{\pi}{2}\frac{e}{\hbar\nu_0}\int\limits_0^{+\infty}\frac{d\omega}{1+\omega^2/\Delta_0^2}\int\limits_{-L/2}^{+L/2}\frac{dz}{\sqrt{(L/2-z)(L/2+z)}}\int\limits_0^{+\infty}\frac{d\chi_l d\chi_r}{4\pi}\times$$
$$\times f\left(\frac{\chi_l}{L/2+z}\right) f\left(\frac{\chi_r}{L/2-z}\right)\frac{e^{-\chi_l-\chi_r}}{e^{-\chi_l}+e^{-\chi_r}}\exp\left[-\frac{(L/2+z-\chi_l)^2}{4(L/2+z)}-\frac{(L/2-z-\chi_r)^2}{4(L/2-z)}\right]. \quad (2.20)$$

The main contribution to the initial integral over z comes from the ends of the integration limits. We can estimate the integral over z by cutting out the ends of the integration interval at $z \sim -L/2+\delta$ and $z \sim L/2-\delta$, where $\delta \sim 1$. Due to the symmetry of the problem, contributions from the two ends are equal. For this reason, we will take the integrand at $z \sim -L/2+1$ (left lead) twice.

$$\langle I_{diag}(L \ll L_M)\rangle \approx \frac{\Delta_0}{\Delta_\xi}\pi\frac{e}{\hbar\nu_0}\frac{1}{\sqrt{L}}\int\limits_0^{+\infty}\frac{d\chi_l d\chi_r}{4\pi}\times$$
$$\times f\left(\frac{\chi_l}{1}\right) f\left(\frac{\chi_r}{L}\right)\frac{e^{-\chi_l-\chi_r}}{e^{-\chi_l}+e^{-\chi_r}}\exp\left[-\frac{\chi_l^2}{4}-\frac{(L-\chi_r)^2}{4L}\right]. \quad (2.21)$$

It is important to note that the relevant value for the variables are $\chi_l \le 1$ and $\chi_r \le L$. Thus, we can expand the cutoff factor $f(\chi_r/L)$ near zero:

$$f\left(\frac{\chi_r}{L}\right) \approx f'(0)\frac{\chi_r}{L} \quad (2.22)$$

and $f'(0)$ can be approximated as (1.11).

Since χ_l determines the value of Γ it may seem that in the case of the short contact, the level width is of the order of Δ_ξ as there are no other limiting conditions for Γ. This, however, is not the case. As we will see in the Section 2.4.2, the correct approximation for the longer contact requires the assumption $L \gg L_M$. This is due to the fact that for the long contact the relevant values of z are of the order of L, but not too close to the superconducting leads.

Since the convergence of the integral over χ_l is mainly determined by the exponent $e^{-\chi_l}$ and $f(x \geq 1) = 1$, we put $f(\chi_l) = 1$. An integral over χ_r is convergent in the limit of $L \to +\infty$, meaning that the main order approximation is

$$\langle I_{diag}(L \ll L_M)\rangle \approx f'(0)C_1\Delta_0\frac{e}{\hbar}\frac{e^{-L/4\xi}}{(L/\xi)^{3/2}} \approx 7.8\Delta_0\frac{e}{\hbar}\frac{e^{-L/4\xi}}{(L/\xi)^{3/2}}. \quad (2.23)$$

Here

$$C_1 = \frac{\pi}{4} \int\limits_0^{+\infty} d\chi_l d\chi_r \, \chi_r \frac{e^{-\chi_l - \chi_r/2}}{e^{-\chi_l} + e^{-\chi_r}} \exp\left[-\frac{\chi_l^2}{4}\right] \approx 4.5 \tag{2.24}$$

Since we had to cut the integration limits at arbitrary value, this approximation does not provide an exact coefficient.

Nonetheless, from (2.23) we can see the effect of the insulator statistics: the exponential factor $e^{-L/4}$, which serves as evidence of the fact that the main contribution to the average critical current comes from rare events.

2.3.2 Non-diagonal term

One option to evaluate the non-diagonal contribution is to study the integral (2.17). In this case, however, this process can be avoided.

As it was mentioned before, the non-diagonal contribution is finite even for the case of $\Gamma = 0$. This means that for the estimation of the non-diagonal contribution in the case of the short contact we can put level width equal to zero. This can only be implemented in the case $L \ll L_M$: for the long contact the energy dependence of the electron transmission amplitudes is strong due to the presence of Mott resonant pairs. Now we have all required tools to calculate the non-diagonal contribution for the short contact. The average in (2.9) can be easily expressed through the correlation function (1.7) as follows:

$$\nu_0^{-2} \left\langle \sum_{n \neq m} \frac{\psi_n(-L/2)\psi_n^*(L/2)\psi_m^*(L/2)\psi_m(-L/2)}{(E_n - i\omega)(E_m + i\omega)} \right\rangle =$$

$$= \int \frac{dE_A dE_B}{(E_A + E_B - i\omega)(E_B + i\omega)} S(E_A, L) = -\int dE_A \frac{2\pi i}{2i\omega - E_A} S(E_A, L) = \int\limits_{-\infty}^{+\infty} dy \frac{2\pi\omega}{\omega^2 + y^2} S(2y, L). \tag{2.25}$$

As the result, the non-diagonal contribution can be expressed as the following integral:

$$\langle I_{non-diag}(L \ll L_M) \rangle \approx \frac{e}{\pi\hbar} \int\limits_0^{+\infty} \frac{d\omega}{1 + \omega^2/\Delta_0^2} \int\limits_{-\infty}^{+\infty} \frac{2\pi\omega \, dy}{y^2 + \omega^2} S(2y, L) \tag{2.26}$$

In the limit $L \ll L_M$ the energy scale $\omega \sim \Delta_0$ which gives the main contribution to the integral is much smaller than the relevant values of y. Thus, we implement the limit representation of the delta function

$$\lim_{\omega\to 0}\frac{2\pi\omega}{\omega^2+y^2}=2\pi^2\delta(y). \tag{2.27}$$

By inserting the asymptotic expression (1.8) and performing the trivial remaining integration over ω, we come to the following result:

$$I_{non-diag}(L \ll L_M) \approx \frac{\pi^{11/2}}{16}\Delta_0\frac{e}{\hbar}\frac{e^{-L/4}}{L^{3/2}} \approx 33.9\Delta_0\frac{e}{\hbar}\frac{e^{-L/4}}{L^{3/2}}. \tag{2.28}$$

The non-diagonal contribution happens to be of the same order for the case of the short contact.

2.3.3 Final result

To find the resulting critical current, we need to sum the two contributions (2.23) and (2.28):

$$\langle I_c(L \ll L_M)\rangle \approx 41.7\Delta_0\frac{e}{\hbar}\frac{\exp(-L/4\xi)}{(L/\xi)^{3/2}}. \tag{2.29}$$

Here we, once again, emphasize that the methods used to find this approximation do not provide the correct coefficient. Nevertheless, we can compare with the exact result calculated using the sigma-model approach [20] in the limit $\xi \ll L \ll L_M$. The exact result for the current has the following form:

$$\langle I_{exact}(\xi \ll L \ll L_M)\rangle = \frac{e}{\hbar}\Delta_0\frac{\pi^{3/2}}{2}\frac{\exp(-L/4\xi)}{(L/\xi)^{3/2}}K^2\left[\sin(\varphi/2)\right]\sin\varphi. \tag{2.30}$$

Here

$$K(x)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-x^2\sin^2\theta}}. \tag{2.31}$$

The sigma-model approach also allows to find the normal conductivity for $L \gg \xi$ [20]:

$$\langle \sigma_{exact}(\xi \ll L)\rangle = \frac{\pi^{5/2}}{4}\frac{e^2}{\hbar}\frac{\exp(-L/4\xi)}{(L/\xi)^{3/2}}. \tag{2.32}$$

Direct comparison of (2.32) and (2.30) shows that for $L \ll L_M$ the Ambegaokar-Baratoff relation takes holds. The comparison of (2.30) and (2.29), however, shows that the coefficient in (2.29) is ~ 6 times larger than the one in the exact result.

2.4 Long contact ($L_M \ll L \ll L_M^2$)

In this section, we carry out the calculation for the critical current in for the case when length is much larger than the Mott Scale. However, as we will see, an upper bound $L \ll L_M^2$ exists.

2.4.1 Diagonal term

Let us now consider the case of the lengths larger than the Mott scale. We start by calculating an integral over ω in (2.8):

$$\int_0^{+\infty} \frac{d\omega}{1+\omega^2/\Delta_0^2}\frac{1}{\omega+\Gamma} = \frac{\Delta_0}{2}\frac{\pi\Gamma + 2\Delta_0 \ln(\Delta_0/\Gamma)}{\Gamma^2+\Delta_0^2}. \tag{2.33}$$

The integration results in two terms in the numerator: $\pi\Gamma$ and $2\Delta_0 \ln(\Delta_0/\Gamma)$. As we will see in this section, in the limit of $L \gg L_M$ the main contribution to the integral comes from the integration domain which corresponds the values of level width $\Gamma \sim \Delta_\xi e^{-\sqrt{L}}$.

This fact allows us to compare the two terms in the numerator of the right-hand side of (2.33). Doing so leads to the conclusion that the term $\pi\Gamma$ dominates for the values of L which satisfy the condition $L_M \ll L \ll L_M^2$, which we will call the case of the **long contact**. On the other hand, the term $2\Delta_0 \ln(\Delta_0/\Gamma)$ dominates in the case of $L \gg L_M^2$, which we will call the case of the **very long contact**.

To calculate the integral (2.8) in the limit $L_M \ll L \ll L_M^2$, as it was mentioned, we leave only the term $\pi\Gamma$ in the numerator of the right-hand side of (2.33). The integral now looks as follows:

$$\begin{aligned}\left\langle I_{diag}(L_M \ll L \ll L_M^2)\right\rangle = \frac{\pi^2}{2}\frac{e}{\pi\hbar\nu_0}\int_{-L/2}^{+L/2}\frac{dz}{\sqrt{(L/2-z)(L/2+z)}}\int_0^{+\infty}\frac{d\chi_l d\chi_r}{4\pi}\times \\ \times\frac{\Delta_0\Delta_\xi[e^{-\chi_l}+e^{-\chi_r}]e^{-\chi_l-\chi_r}}{\Delta_0^2+\Delta_\xi^2[e^{-\chi_l}+e^{-\chi_r}]^2} f\left(\frac{\chi_l}{L/2+z}\right) f\left(\frac{\chi_r}{L/2-z}\right)\exp\left[-\frac{(L/2+z-\chi_l)^2}{4(L/2+z)}-\frac{(L/2-z-\chi_r)^2}{4(L/2-z)}\right].\end{aligned} \tag{2.34}$$

The main complication arising in the calculation of the integral is that the dependence on χ_l and χ_r is present in the denominator, making integrals over these variables non-Gaussian. One method to circumvent this difficulty involves the calculation of the integral over z first. Following the steps described in Appendix A, we come to (A.9), which gives an expression for the critical current in the form of the integral:

$$\left\langle I_{diag}(L_M \ll L \ll L_M^2)\right\rangle = 2\pi^{5/2}[f'(0)]^2\Delta_0 \frac{e}{\hbar}\frac{e^{-L/4}}{L^{3/2}} \int\limits_0^{+\infty} d\chi_+ \frac{\chi_+ \, e^{-\chi_+^2/L}}{\sqrt{4+e^{2\chi_+}(\Delta_0/\Delta_\xi)^2}}. \qquad (2.35)$$

Here $\chi_+ = (\chi_l + \chi_r)/2$. From (2.35) it is clear that main contribution to the integral comes from $\chi_+ \sim \sqrt{L}$. This fact confirms the statement made before about typical values of the width satisfying the condition $\Gamma \sim \Delta_\xi e^{-\sqrt{L}}$.

One final step is to calculate the integral over χ_+ in the limit of $L_M \ll L \ll L_M^2$, which yields the following result:

$$\left\langle I_{diag}(L_M \ll L \ll L_M^2)\right\rangle = \frac{\pi^{5/2}}{4}[f'(0)]^2\Delta_0 \frac{e}{\hbar}\frac{\exp\left(-L/4\xi\right)}{\sqrt{L/\xi}}\left[1-e^{-L_M^2/(4\xi L)}\right] \approx$$

$$\approx 12.6\frac{\exp\left(-L/4\xi\right)}{\sqrt{L/\xi}}\left[1-e^{-L_M^2/(4\xi L)}\right]. \qquad (2.36)$$

Here we once again emphasize the exponential factor $e^{-L/4\xi}$, which is a notable effect of the insulator statistics. Factors ξ were restored where needed. Comparing this result to (2.23) we can see that these formulas share the same exponential factor, but the pre-exponential factors are different.

2.4.2 Non-diagonal term

This time, level widths Γ cannot be neglected, so we have to calculate the integral (2.17). This is due to the fact that for $L \gg L_M$ the dependence of the transmission amplitudes on energy is important. Let the localization center B be located to the right of the localization center A, or simply $z_B > z_A$. Due to the invariance of the right-hand side of (2.17) to change of indexes $A \leftrightarrow B$, imposing the constraint $z_B > z_A$ gives exactly one half of the initial value.

After that, we neglect the terms $e^{-\chi_A^{(r)}}$ and $e^{-\chi_B^{(l)}}$ as they are exponentially smaller than the terms $e^{-\chi_A^{(l)}}$ and $e^{-\chi_B^{(r)}}$. In the numerator of the (2.17) after $\psi_\pm$ are expressed through ψ_{z_A} and ψ_{z_B} only one term does not have exponentially small factors in this case. This approximation, when applied to the numerator, has the form

$$\psi_-(-L/2)\psi_-^*(L/2)\psi_+(L/2)\psi_+^*(-L/2) \approx -|u_+|^2|u_-|^2|\psi_{z_A}(-L/2)|^2|\psi_{z_B}(L/2)|^2. \qquad (2.37)$$

At this point it is clear that the non-diagonal term for the case of the long contact is negative. For the next step, we can calculate an integral over ε_+:

$$\mathrm{Re}\int\limits_{-\infty}^{+\infty}\frac{d\varepsilon_+}{\left\{\frac{1}{2}(\varepsilon_+-\Delta)-i\Gamma_--i\omega\right\}\left\{\frac{1}{2}(\varepsilon_++\Delta)+i\Gamma_++i\omega\right\}}=\frac{4\pi(\Gamma_++\Gamma_-+2\omega)}{\Delta^2+(\Gamma_++\Gamma_-+2\omega)^2}. \quad (2.38)$$

The sum $\Gamma_++\Gamma_-$ has a very simple form:

$$\Gamma_++\Gamma_-=(|u_+|^2+|u_-|^2)\left[|\psi_{z_A}(-L/2)|^2+|\psi_{z_A}(L/2)|^2+|\psi_{z_B}(-L/2)|^2+|\psi_{z_B}(L/2)|^2\right]\approx$$
$$\approx|\psi_{z_A}(-L/2)|^2+|\psi_{z_B}(L/2)|^2. \quad (2.39)$$

The integrals over variables $\chi_A^{(r)}$ and $\chi_B^{(l)}$ can be trivially taken using the normalization of the probability distribution. Since, as the result, these variables will be excluded from the equation, without any ambiguity we can omit indexes l and r for the remaining variables $\chi_A^{(l)}$ and $\chi_B^{(r)}$. The result can be simplified

$$\langle I_{non-diag}\rangle=-\frac{4e}{\pi\hbar}\int\limits_0^{+\infty}\frac{d\omega}{1+\omega^2/\Delta_0^2}\int\limits_{-\infty}^{+\infty}d\varepsilon_-\int\limits_0^{+\infty}\frac{d\chi_Ad\chi_Bd\chi_J}{8\pi^{3/2}}\int\limits_{-L/2}^{L/2}dz_A\int\limits_{z_A}^{L/2}dz_B\times$$
$$\times P(\chi_A,z_A+L/2)P(\chi_B,z_B-L/2)P_J(\chi_J,z_B-z_A)\times$$
$$\times 4\pi\left(1-\frac{\varepsilon_-^2}{\varepsilon_-^2+4\Delta_\xi^2e^{-\chi_J}}\right)\frac{e^{-(\chi_A+\chi_B)}\left[e^{-\chi_A}+e^{-\chi_B}+2\omega/\Delta_\xi\right]}{(\Delta/\Delta_\xi)^2+\left[e^{-\chi_A}+e^{-\chi_B}+2\omega/\Delta_\xi\right]^2}. \quad (2.40)$$

The further analytical treatment of the integral (2.40) is difficult. However, the performed approximations make it possible to study these integrals numerically with a reasonable precision. Numerical calculations of the integrals (2.40) and (2.34) shows that for $L\gg L_M$ the diagonal contribution dominates over the non-diagonal one. An example of the numerical calculation for $\Delta_0=10^{-3}\Delta_\xi$ is shown in the Fig. (2.1). In the region $L_M\ll L\ll L_M^2/\xi$ the error caused by neglecting the non-diagonal term does not exceed 10%.

2.4.3 Final result

The resulting expression for the critical current with a good accuracy is just equal to $\langle I_{diag}(L)\rangle$:

$$\left\langle I(L_M\ll L\ll L_M^2/\xi)\right\rangle=\frac{\pi^{5/2}}{4}[f'(0)]^2\Delta_0\frac{e}{\hbar}\frac{\exp\left(-L/4\xi\right)}{\sqrt{L/\xi}}\left[1-e^{-L_M^2/(4\xi L)}\right]\approx$$
$$\approx 12.6\frac{e}{\hbar}\Delta_0\frac{\exp\left(-L/4\xi\right)}{\sqrt{L/\xi}}\left[1-e^{-L_M^2/(4\xi L)}\right]. \quad (2.41)$$

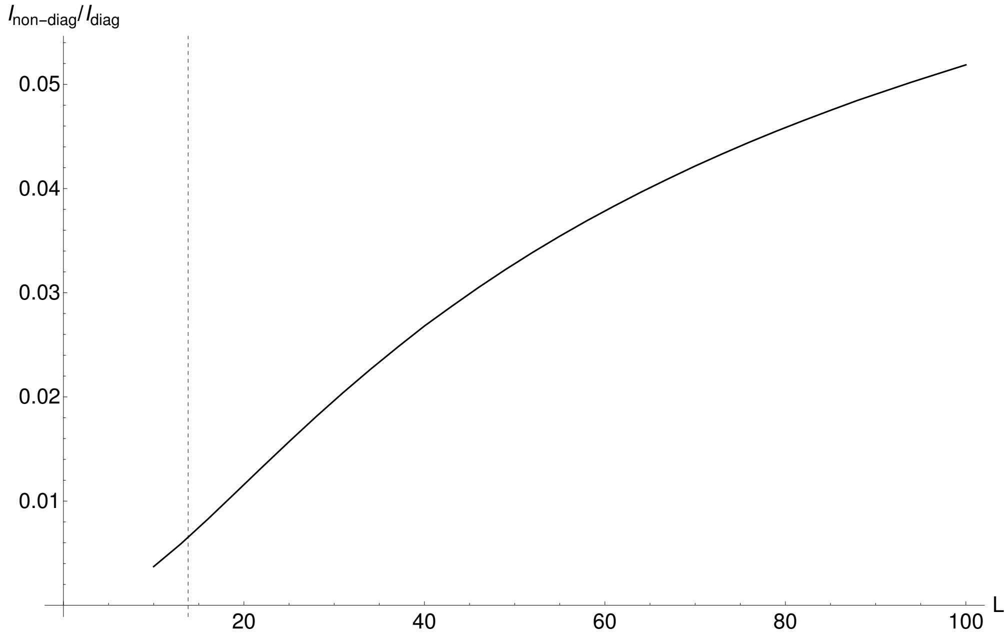

Figure 2.1: Solid black line: Numerical calculation for the ratio of the contributions to the critical current: Numerically evaluated (2.40) divided by the numerically evaluated (2.34). The values are taken for $\Delta_0 = 10^{-3}\Delta_\xi$. Length L is measured in ξ. The dashed vertical black line is drawn at L corresponding Mott Scale $L_M \approx 13.8$.

Here we have restored the dimensional factors ξ.

2.5 Very long contact ($L \gg L_M^2$)

In this section we finalize the calculation of the critical current by discussing the final case in which the contact length is much larger than the Mott scale squared.

2.5.1 Diagonal term

Finally, we can discuss the results in the region when the length of the insulator length is much larger than the Mott scale squared. As in the previous cases, we need to evaluate the integral (2.8) in the limit $L \gg L_M^2$. As in the case of the long contact, we will start by taking an integral over ω as in (2.33). However, in the currently discussed limit, we use that $2\Delta_0 \ln(\Delta_0/\Gamma) \gg \pi\Gamma$ and leave only the second term in the numerator of the right-hand side of (2.33). This yields the following expression:

$$\left\langle I_{diag}(L \gg L_M^2)\right\rangle = \frac{e}{\hbar\nu_0} \int\limits_{-L/2}^{+L/2} \frac{dz}{\sqrt{(L/2-z)(L/2+z)}} \int\limits_{0}^{+\infty} \frac{d\chi_l d\chi_r}{4\pi} \ln\left[\frac{\Delta_0/\Delta_\xi}{e^{-\chi_l}+e^{-\chi_r}}\right] e^{-\chi_l-\chi_r} \times$$
$$\times f\left(\frac{\chi_l}{L/2+z}\right) f\left(\frac{\chi_r}{L/2-z}\right) \exp\left[-\frac{(L/2+z-\chi_l)^2}{4(L/2+z)} - \frac{(L/2-z-\chi_r)^2}{4(L/2-z)}\right]. \quad (2.42)$$

To evaluate the integral, first, we calculate an integral over z by repeating the steps from (A.1) to (A.6) as it was done for the case $L_M \ll L \ll L_M^2$. This leads to the expression which contains only integrals over χ_l and χ_r:

$$\left\langle I_{diag}(L \gg L_M^2)\right\rangle = \frac{1}{2\sqrt{\pi}}\left[f'(0)\right]^2 \frac{e}{\hbar\nu_0}\sin\varphi\frac{e^{-L/4}}{L^{3/2}}\times \\ \times\int_0^{+\infty} d\chi_l d\chi_r \ln\left[\frac{\Delta_0/\Delta_\xi}{e^{-\chi_l}+e^{-\chi_r}}\right](\chi_l+\chi_r)e^{-(\chi_l+\chi_r)/2}\exp\left[-\frac{(\chi_l+\chi_r)^2}{4L}\right]. \quad (2.43)$$

The expression can be simplified by switching to the variables χ_+ and χ_-:

$$\left\langle I_{diag}(L \gg L_M^2)\right\rangle = \frac{4}{\sqrt{\pi}}\left[f'(0)\right]^2 \frac{e}{\hbar\nu_0}\frac{e^{-L/4}}{L^{3/2}}\int_0^{+\infty} d\chi_+\chi_+\int_{-\chi_+}^{+\chi_+} e^{-\chi_+(L+\chi_+)/L}d\chi_-\ln\left[\frac{e^{\chi_+}}{2\cosh(\chi_-)}\right]. \quad (2.44)$$

The integral over χ_+ and χ_- is convergent in the limit $L \longrightarrow \infty$, which means that the leading term has the following form:

$$\left\langle I_{diag}(L \gg L_M^2/\xi)\right\rangle = \left[f'(0)\right]^2 C_2\Delta_\xi\frac{e}{\hbar}\frac{e^{-L/4\xi}}{(L/\xi)^{3/2}}, \quad (2.45)$$

where

$$C_2 = 2\sqrt{\pi}\int_0^{+\infty} d\chi_+\chi_+\int_{-\chi_+}^{+\chi_+} e^{-\chi_+}d\chi_-\ln\left[\frac{e^{\chi_+}}{2\cosh(\chi_-)}\right] \approx 18.7 \quad (2.46)$$

and the approximation for $f'(0)$ was derived in (1.11). In (2.45) the dimensional factors ξ have been restored.

2.5.2 Non-diagonal term

In case of the very long contact the situation with the non-diagonal contribution is similar to the one described in the Section 2.4.2. Although, in the limit $L \gg L_M^2$ conducting precise numerical analysis (2.40) is complicated, it is certain that the non-diagonal contribution is not larger than the diagonal one. Indeed, the initial expression (2.1) is positive, while the expression (2.40), which is applicable for $L \gg L_M$, (and, subsequently, $L \gg L_M^2$) is negative.

2.5.3 Final result

As the result, we conclude that in the limit $L \gg L_M^2$ the main contribution is the diagonal one:

$$\left\langle I_c(L \gg L_M^2/\xi)\right\rangle = \left[f'(0)\right]^2 C_2 \Delta_\xi \frac{e}{\hbar}\frac{\exp\left(-L/4\xi\right)}{(L/\xi)^{3/2}} \approx 54.0 \Delta_\xi \frac{e}{\hbar}\frac{\exp\left(-L/4\xi\right)}{(L/\xi)^{3/2}}. \quad (2.47)$$

The constant C_2 was defined in (2.46). Here we emphasize the fact that the resulting expression (2.45) is not proportional to the superconducting gap Δ_0.

Chapter 3

Summary of the results and conclusions

In the present work we have calculated the critical Josephson current through an SIS contact. It was shown for all cases that the exponential decay exp $(-L/4\xi)$ is present. The existence of three different regimes was demonstrated. Values of the critical current in these regimes only differ by the pre-exponential function.

The crossover between $\langle I_c(\xi \ll L \ll L_M)\rangle$ and $\langle I_c(L_M \ll L)\rangle$ can be studied numerically. The comparison of the approximations (2.41) and (2.29) to the numerical integration of the diagonal contribution with added non-diagonal one is shown on the Fig. 3.1. From the figure it can be concluded that both approximations work reasonably well for the regions in which they were derived. However, (2.41) and (2.29) are not of the same order at $L = L_M$. In fact, at $L = L_M$ (2.41) is much larger than (2.29).

As it was mentioned in the Sections 2.3 and 2.4.2, for different values of length, different parts of the integration over the coordinate of the localized state z are relevant. Here we remind that for $L \ll L_M$ the main contribution was given by the ends of the interval and for $L \gg L_M$ it was given by large intervals of z. Such qualitative shift in the behaviour of the integral, caused by the resonant scattering, may cause such abrupt shift in behavior of the current.

For $L \gg L_M^2$ the numerical analysis is complicated. Also, the limit of the very long contact presents less interest as it involves experimentally unreachable values of length.

The fact that the coefficient in (2.29) is several times larger that the exact answer (2.30) also requires discussion. The most probable explanation is the fact that used evaluation methods for the short contact did not allow to find an exact coefficient. Furthermore, the largest contribution to the current through the short contact comes from the configurations in which the localization center is close to one of the superconducting leads. Such situations are beyond the scope of the approximation discussed in the Section 1.2.

The final detail worth mentioning is that for $L_M \ll L \ll L_M^2$ and $L \gg L_M^2$, we observe breaking of the Ambegaokar-Baratoff relation.

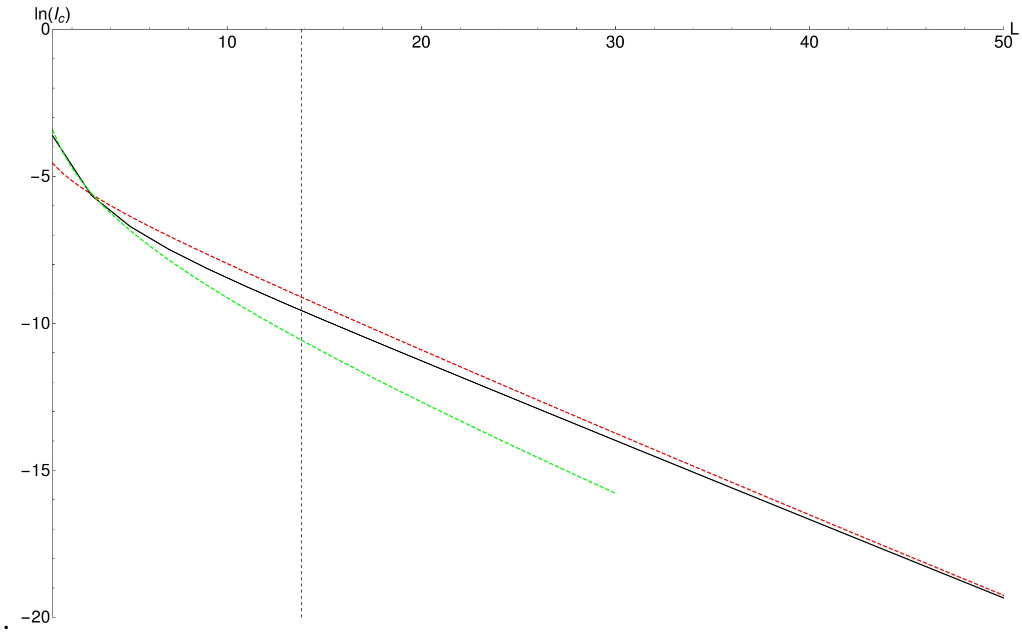

Figure 3.1: Solid black line: Sum of (2.28) and numerically calculated (2.34). Dashed red line: approximation (2.41). Dashed green line: approximation (2.29). The values are taken for $\Delta_0 = 10^{-3}\Delta_\xi$. Length L is measured in ξ. The vertical dashed line corresponds to the Mott scale $L_M \approx 13.8$

Appendix A

Calculation of the integrals for the critical current

Here we will discuss the first steps to calculate the integral (2.34). The first one is to move from the integration of one variable z to integration over two variables $z_l = L/2 + z$ and $L/2 - z$. The constraint $z_l + z_r = L$ is take into account by insertion of the corresponding delta function. This brings us to the following expression:

$$\left\langle I_{diag}(L_M \ll L \ll L_M^2)\right\rangle = \frac{\pi^2}{2}\frac{e}{\pi\hbar\nu_0}\int\limits_0^L \frac{dz}{\sqrt{z_l z_r}}\int\limits_0^{+\infty}\frac{d\chi_l d\chi_r}{4\pi}\delta(z_l+z_r-L)\times$$
$$\times\frac{\Delta_0\Delta_\xi[e^{-\chi_l}+e^{-\chi_r}]e^{-\chi_l-\chi_r}}{\Delta_0^2+\Delta_\xi^2[e^{-\chi_l}+e^{-\chi_r}]^2}f\left(\frac{\chi_l}{z_l}\right)f\left(\frac{\chi_r}{z_r}\right)\exp\left[-\frac{(z_l-\chi_l)^2}{4z_l}-\frac{(z_r-\chi_r)^2}{4z_r}\right]. \quad (A.1)$$

Next, we use the integral representation for the delta function, introducing the new variable of integration p. As the result, (A.1) takes the form

$$\left\langle I_{diag}(L_M \ll L \ll L_M^2)\right\rangle =$$
$$= \frac{\pi}{8}\frac{e}{\pi\hbar\nu_0}\int\limits_{-\infty}^{+\infty}\frac{dp}{2\pi}\left\{\prod_{j=l,r}\int\limits_0^L\frac{dz_j}{\sqrt{z_j}}\int\limits_0^{+\infty}d\chi_j\, f\left(\frac{\chi_j}{z_j}\right)\exp\left[-\frac{(z_j-\chi_j)^2}{4z_j}+ip(z_j-L/2)\right]\right\}\times$$
$$\times\frac{\Delta_0\Delta_\xi[e^{-\chi_l}+e^{-\chi_r}]e^{-\chi_l-\chi_r}}{\Delta_0^2+\Delta_\xi^2[e^{-\chi_l}+e^{-\chi_r}]^2}. \quad (A.2)$$

These two actions lead to the separation of the large exponents. In order to proceed, we need to take into account the condition $L \gg L_M$, which is valid for both long and very long contacts. It allows us to change the upper limit of the integration over z_j to $+\infty$. This is due to the fact that $\chi_j \sim L_M$ and the integral over z_j is convergent on large $z_j \sim L$. This means that the integral over z converges on the broad scale of z. This situation is opposite to the one discussed in the Section 2.3. In this limit we also can expand the cutoff factor near zero. After these actions are performed, the integrals over z_l and z_r can be calculated as

$$\int_0^L \frac{dz_j}{\sqrt{z_j}} f\left(\frac{\chi_j}{z_j}\right) \exp\left[-\frac{(z_j-\chi_j)^2}{4z_j} + ip(z_j - L/2)\right] \approx$$

$$\approx f'(0)\chi_j \int_0^{+\infty} \frac{dz_j}{z_j^{3/2}} \exp\left[-\frac{(z_j-\chi_j)^2}{4z_j} + ip(z_j - L/2)\right] =$$

$$= 2\sqrt{\pi} f'(0) e^{-ipL/2} \exp\left[\frac{\chi_j}{2}\left(1-\sqrt{1-4pi}\right)\right]. \quad \text{(A.3)}$$

Substitution of this result into $(A.2)$ yields the following expression for the critical current:

$$\left\langle I_{diag}(L_M \ll L \ll L_M^2)\right\rangle = \frac{\pi^2}{2}[f'(0)]^2 \frac{e}{\pi\hbar\nu_0} \int_0^{+\infty} d\chi_l d\chi_r \frac{\Delta_0 \Delta_\xi [e^{-\chi_l} + e^{-\chi_r}] e^{-\chi_l - \chi_r}}{\Delta_0^2 + \Delta_\xi^2 [e^{-\chi_l} + e^{-\chi_r}]^2} \times$$

$$\times \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{-ipL} \exp\left\{\frac{1}{2}(\chi_l + \chi_r)\left(1 - \sqrt{1-4pi}\right)\right\}. \quad \text{(A.4)}$$

The integral over p can be easily calculated by deforming an integration contour in the complex plane so that it goes along the branch cut:

$$\int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{-ipL} \exp\left(-\frac{\chi_l + \chi_r}{2}\sqrt{1-4pi}\right) = \frac{(\chi_l + \chi_r)}{2\sqrt{\pi}} \exp\left[-\frac{(\chi_l + \chi_r)^2}{4L}\right] \frac{e^{-L/4}}{L^{3/2}}. \quad \text{(A.5)}$$

Now we are left only with integrals over χ_l and χ_r:

$$\left\langle I_{diag}(L_M \ll L \ll L_M^2)\right\rangle = \frac{\sqrt{\pi}}{4}[f'(0)]^2 \frac{\Delta_0}{\Delta_\xi} \frac{e}{\hbar\nu_0} \frac{e^{-L/4}}{L^{3/2}} \times$$

$$\times \int_0^{+\infty} d\chi_r d\chi_r \frac{[e^{-\chi_l} + e^{-\chi_r}] e^{-(\chi_l + \chi_r)/2}}{\alpha^2 + [e^{-\chi_l} + e^{-\chi_r}]^2} (\chi_l + \chi_r) \exp\left[-\frac{(\chi_l + \chi_r)^2}{4L}\right] \quad \text{(A.6)}$$

Here we have introduced $\alpha = \Delta_0/\Delta_\xi$. Let us now consider the remaining double integral. The natural change of variables is $\chi_\pm = (\chi_l \pm \chi_r)/2$ and leads to

$$\int_0^{+\infty} d\chi_r d\chi_l \frac{(e^{-\chi_l} + e^{-\chi_r}) e^{-\chi_l/2 - \chi_r/2}}{\alpha^2 + [e^{-\chi_l} + e^{-\chi_r}]^2} (\chi_l + \chi_r) \exp\left[-\frac{(\chi_l + \chi_r)^2}{4L}\right] = \quad \text{(A.7)}$$

$$= 8 \int_0^{+\infty} d\chi_+ \int_{-\chi_+}^{+\chi_+} d\chi_- \frac{\chi_+ \cosh(\chi_-)}{\alpha^2 e^{2\chi_+} + 4\cosh^2(\chi_-)} e^{-\chi_+^2/L} \approx 4\pi \int_0^{+\infty} d\chi_+ \frac{\chi_+}{\sqrt{4 + e^{2\chi_+}\alpha^2}} e^{-\chi_+^2/L}. \quad \text{(A.8)}$$

Assembling the total expression for the critical current in the case of the long contact, we come the following formula:

$$\left\langle I_{diag}(L_M \ll L \ll L_M^2)\right\rangle = \frac{\pi^{5/2}}{4}[f'(0)]^2 \frac{e}{\hbar}\Delta_0 \frac{e^{-L/4}}{L^{3/2}} \int\limits_0^{+\infty} d\chi_+ \frac{\chi_+ \, e^{-\chi_+^2/L}}{\sqrt{4 + e^{2\chi_+}(\Delta_0/\Delta_\xi)^2}}. \tag{A.9}$$

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