Generalized multifractality in the spin quantum Hall symmetry class

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Abstract

Scaling of various local observables with a system size at Anderson transition criticality is characterized by a generalized multifractality. To begin with, we study surface generalized multifractality in the spin quantum Hall symmetry class (class C) in 2D. We further derive a relation of the generalized surface multifractal spectra in 2D and linear combinations of Lyapunov exponents of a strip in quasi-1D geometry under the assumption of conformal invariance. Then, we probe conformal invariance by testing that relation numerically. In addition, we study the bulk generalized multifractality in class C with interaction. We employ Finkel'stein nonlinear sigma model and construct the pure scaling derivativeless operators for class C in the presence of interaction. Within the two-loop renormalization group analysis we compute the anomalous dimensions of the pure scaling operators and demonstrate that they are affected by the interaction. We find that the interaction breaks exact symmetry relations between generalized multifractal exponents known for a noninteracting problem.

Contents

Chapter 1

Introduction

A quantum state in a disordered electronic system can be localized or delocalized [1, 2], depending on the strength of the disorder and the energy of the state. The transition between these localized and delocalized phases is called the Anderson transition [3]. In a broader sense, the Anderson transition can also occur between two localized phases of different topology. At the critical point of the Anderson transition, wave functions have unusual statistical properties: their moments have a pure-scaling dependence on the size of the system, $L^d\langle|\psi(r)|^{2q}\rangle \propto L^{-\tau_q}$, where τ_q are independent critical exponents, q is integer. This property is called multifractality [4].

Recently, it was discovered that for certain values of q, surface effects have a dominant contribution to the multifractality of the entire system (bulk+surface) [5]. Moreover, experimental studies of the Anderson transition usually require transport measurements, which can be performed by attaching leads to the surfaces of the system, making it possible to study the contribution of surface to multifractality directly. Pure-scaling dependence of moments of wave functions on the surface at Anderson transition is called surface multifractality [5, 6, 7]: $L^{d-1}\langle|\psi(r \in S)|^{2q}\rangle \propto L^{-\tau_q^s}$, here, τ_q^s is a set of surface critical exponents.

It has been recognized that there are many other combinations of wave functions, known as pure-scaling observables, which demonstrate pure-scaling dependence on the size of the system after averaging over disorder; that property is called generalized multifractality [8]. Pure-scaling observables are characterized by a set of independent subleading critical exponents τ_λ, where the multi-index $\lambda = (q_1, q_2, ...)$ labels different observables [9, 10, 11]. It has been shown that the critical exponents τ_λ satisfy exact symmetry relations [12, 13, 14, 15, 16], which can be used as a benchmark for numerical calculations.

The interest in the Anderson transition was further increased when a full classification of 10 Altland-Zirnbauer symmetry classes of disordered fermionic systems was achieved [17, 18, 19]. A way to study these symmetry classes analytically is provided by the nonlinear sigma model (NLσM) [4] description. In $d = 2$ at the MIT point, the NLσM is generically at strong coupling and cannot be straightforwardly applied. A separate critical theory is required to describe the system at the transition. Recently, it was demonstrated that if the critical theory is locally conformal invariant, then the critical exponents τ_λ have a quadratic dependence on the components $q_1, q_2, \ldots$ of multi-index λ, this property is called generalized parabolicity [8]. Recently, generalized multifractality in the bulk was studied numerically in the spin quantum Hall (SQH) transition (class C) [8]. An important feature of Class C is that some of the critical exponents can be calculated exactly through mapping to percolation [20, 21, 22]. This mapping is also applicable to surface multifractality [6].

Multifractal correlations of wave functions result in a variety of interesting physical effects. In particular, they lead to strong enhancement of superconducting transition temperature and the superconducting gap at zero temperature [23, 24, 25, 26, 27, 28], induce log-normal distribution for the superconducting order parameter [24, 29, 30] and local density of states [28, 31] in dirty superconductors, are responsible for instabilities of surface states in topological superconductors [32, 33], result in strong mesoscopic fluctuations of the Kondo temperature [34, 35, 36], enhance depairing effect of magnetic impurities on superconducting state in dirty films [37], make cooling of electrons due to electron-phonon coupling more efficient [38], and affect the Anderson orthogonality catastrophe [39].

The scaling properties at Anderson transition criticality can be modified by electron-electron interaction. In this case the so-called Mott–Anderson transition emerges being controlled by both disorder and interaction strengths (see Refs. [40, 41] for a review). Interaction drives a disordered electron system into a new interacting fixed point corresponding to a metal-insulator transition and, consequently, affects the generalized multifractal exponents. Unfortunately, present numerical computing power is not enough to access generalized multifractal exponents and to check symmetry relations in the presence of interaction [42, 43, 44, 45].

In the first part of this work, we extend the construction of subleading multifractal pure scaling observables to the surface and numerically study generalized surface multifractality in the spin quantum Hall (SQH) transition using the SU(2) Chalker-Coddington network model and analyze the resulting generalized multifractal spectrum. We derive the connection (2.20) between surface subleading critical exponents of a 2D system and Lyapunov exponents for a quasi-1D system assuming invariance under the exponential map. This generalizes the work of Ref. [46] to subleading multifractality. Using the transfer matrix method, we study Lyapunov exponents for a long Q1D strip of these models and verify our result (2.20). Thereby we probe for conformal invariance.

In the second part of this work, we develop the theory of the generalized multifractality for the spin quantum Hall symmetry class in the dimension $d = 2 + \epsilon$ in the presence of electron-electron interaction [47]. Using Finkel'stein NLσM for class C, we demonstrate that the pure scaling derivativeless operators can be constructed by straightforward generalization of the pure scaling operators without derivatives known in the absence of interaction. Within the two-loop approximation we compute how the anomalous dimensions of the pure scaling derivativeless operators are affected by the presence of interaction. Also for a reader's convenience, within the Finkel'stein NLσM we rederive the results known in the literature for the one-loop renormalization of the spin conductance, dimensionless interaction, the Finkel'stein frequency renormalization parameter, and the averaged LDoS.

Chapter 2

Generalized surface multifractality without interaction

2.1 Analytical framework

2.1.1 Generalized surface multifractality

An essential feature of the critical points of Anderson transition in 2D disordered systems is generalized multifractality: certain combinations of wave functions demonstrate pure scaling dependence on the size of the system after averaging over disorder:

$$L^d \langle P_\lambda[\psi] \rangle \sim L^{-\tau_\lambda}, \tag{2.1}$$

here $P_\lambda[\psi]$ is a composite object of wave functions ψ that are close in energy and evaluated at positions in a neighborhood $\mathcal{N}$ much smaller than the linear system size L. The brackets denote average over disorder configurations. The multi-index $\lambda = (q_1, q_2, ..., q_n)$ serves as label for the different pure scaling operators. For positive integer q_i that are ordered ascendingly.

We can generalize these operators defined in the bulk of the d-dimensional systems to the $d-1$ dimensional surface:

$$L^{d-1} \langle P_\lambda[\psi] \rangle \sim L^{-\tau_\lambda^{(s)}}. \tag{2.2}$$

The coordinates of the wavefunctions are restricted to the surface. The superscript s indicates the corresponding set of critical exponents $\tau_\lambda^{(s)}$. These exponents are related to the corresponding scaling dimensions $x_\lambda^{(s)}$ and anomalous multifractal exponents $\Delta_\lambda^{(s)}$ in the usual way:

$$\Delta^{(s)}_{(q_1,q_2,\ldots)} = \tau^{(s)}_{(q_1,q_2,\ldots)} - 1 - d\left(|\lambda| - 1\right) - |\lambda|\mu \tag{2.3}$$

$$x^{(s)}_{(q_1,q_2,\ldots)} = \Delta^{(s)}_{(q_1,q_2,\ldots)} + |\lambda| x^{(s)}_{(1)} \tag{2.4}$$

here, $|\lambda| = q_1 + q_2 + ...$, constants μ and $x^s_{(1)}$ depend on a symmetry class. The pure-scaling operators $\mathcal{P}_\lambda[\psi]$ can be written explicitly in terms of proper combinations of wave functions. Let us start with an operator with all $q_i = 1$ i.e. $\lambda = (1, 1, ...)$. For classes without (pseudo)spin degree of freedom, it has the form resembling a Slater determinant [48]:

$$P_{(1,1,\ldots)}[\psi] = \left| \det \begin{pmatrix} \psi_{\alpha,r_1} & \psi_{\beta,r_1} & \cdots \\ \psi_{\alpha,r_2} & \psi_{\beta,r_2} & \cdots \\ \cdots & \cdots & \ddots \end{pmatrix} \right|^2 . \tag{2.5}$$

For classes with (pseudo)spin degree of freedom it has similar form:

$$P^{\text{sp}}_{(1,1,\ldots)}[\psi] = \det \left(\begin{array}{ccc|ccc} \psi_{\alpha\uparrow r_1} & \psi_{\beta\uparrow r_1} & \cdots & -\psi^*_{\alpha\downarrow r_1} & -\psi^*_{\beta\downarrow r_1} & \cdots \\ \psi_{\alpha\uparrow r_2} & \psi_{\beta\uparrow r_2} & \cdots & -\psi^*_{\alpha\downarrow r_2} & -\psi^*_{\beta\downarrow r_2} & \cdots \\ \cdots & \cdots & \ddots & \cdots & \cdots & \ddots \\ \hline \psi_{\alpha\downarrow r_1} & \psi_{\beta\downarrow r_1} & \cdots & \psi^*_{\alpha\uparrow r_1} & \psi^*_{\beta\uparrow r_1} & \cdots \\ \psi_{\alpha\downarrow r_2} & \psi_{\beta\downarrow r_2} & \cdots & \psi^*_{\alpha\uparrow r_2} & \psi^*_{\beta\uparrow r_2} & \cdots \\ \cdots & \cdots & \ddots & \cdots & \cdots & \ddots \end{array} \right) \tag{2.6}$$

Since we are interested in the surface multifractality, in the definitions (2.5) and (2.6) we take coordinates which lie on the surface, distance between adjacent points is fixed: $r_{i+1} - r_i = r$, indices $\alpha, \beta, ...$ correspond to different eigenvalues, $\uparrow, \downarrow$ – projections of the spin. In the formulas (2.5) and (2.6), the multi-index λ is $(1, 1, ...)$, for other values of λ we use Abelian fusion [49],

$$\begin{aligned} P_\lambda[\psi] = \left(P_{(1^1)}[\psi]\right)^{q_1 - q_2} \left(P_{(1^2)}[\psi]\right)^{q_2 - q_3} \cdot \ldots \cdot \\ \cdot \left(P_{\left(1^{k-1}\right)}[\psi]\right)^{q_{k-1} - q_k} \left(P_{\left(1^k\right)}[\psi]\right)^{q_k}, \quad \lambda = (q_1, q_2, \ldots q_k) \end{aligned} \tag{2.7}$$

here, (1^k) is an abbreviation for $(\underbrace{1, 1, \ldots, 1}_{k})$.

From the assumption of local conformal invariance, we get a quadratic dependence of $x_{(q_1,q_2,\ldots)}$ on the q_i, which is denoted by generalized parabolicity [50]. Weyl symmetry then enforces the following form:

$$x^{\text{para}}_{(q_1,q_2,\ldots)} = -b \sum_i q_i \left(q_i + c_i\right). \tag{2.8}$$

The constants c_i depend on the symmetry classes and b is the single remaining parameter characterizing the transition (in class C, $c_j = 1 - 4j$ and $b = 1/8$).

2.1.2 Connection between subleading exponents in 2D and Lyapunov exponents in quasi-1D

We can consider the conformal mapping of the 2D circle (of radius R) to the quasi 1D strip with width M and length L:

$$w = \frac{M}{\pi} \ln z, \quad z = \exp\left(\frac{\pi}{M} w\right) \tag{2.9}$$

$$w = u + iv, \ u \leq L \equiv \frac{M}{\pi} \ln R, \ 0 < v < M. \tag{2.10}$$

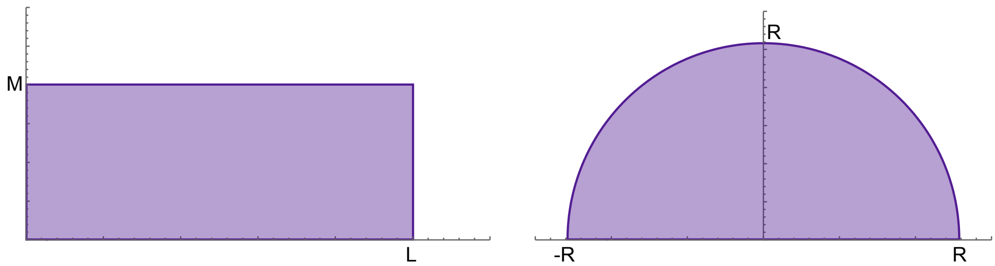

Figure 2.1: Conformal mapping of the 2D circle (of radius R) to the quasi 1D strip with width M and length L with open boundary conditions on horizontal boundaries.

The NLσM for a disordered 2D system defined on a large circle can then be mapped on a NLσM describing a disordered Q1D strip under the assumption of conformal invariance. Their operators are related by the conformal transformation

$$\langle \mathcal{O}(w) \rangle_{Q1D} = \left| \frac{dz}{dw} \right|^{x_\lambda^{(s)}} \langle \mathcal{O}(z) \rangle_{2D}. \tag{2.11}$$

We need to calculate the scaling of both sides in order to relate the Lyapunov exponents to the multifractal spectrum.

The wavefunctions ψ_α of this Q1D strip are determined by their boundary values A_α at $\mathrm{Re}\, w = 0$ and the transfer matrix $T(L)$ relating both ends of the long strip. Here, we made the assumption, that it is sufficient to restrict to a fixed energy E for the transfer matrix. In order to evaluate the scaling of the determinants (2.5), we have to find expressions for the wavefunctions adjacent in energy in the vicinity of the point $W = U + iV$ on the strip. Since we want to take the thermodynamic limit in the end, the wave functions should vanish on the boundary of the disk or at the end of the strip at $u = L$, respectively. This avoids unphysical probability densities at infinity.

With the transfer matrix, we may write:

$$\psi_\alpha(w_i) = B_i T(U) A_\alpha, \tag{2.12}$$

where $B_i = P_{v_i} T(u_i - U)$ selects row v_i of the transfer matrix over $u_i - U$ additional slices and A_α are the initial conditions at $\mathrm{Re}\, w = 0$.

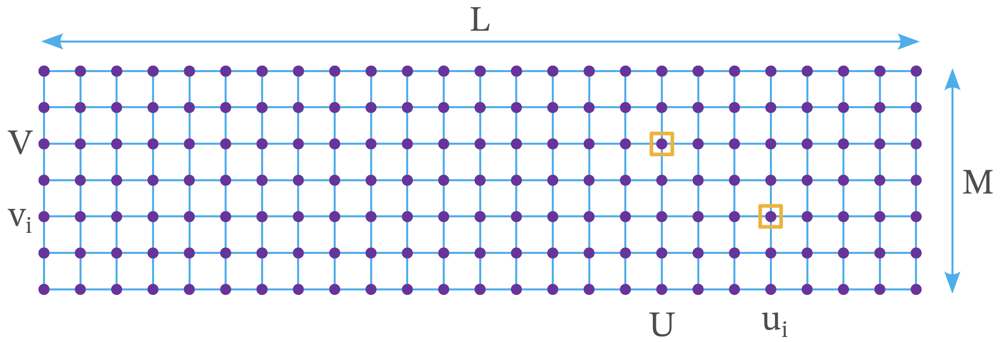

Figure 2.2: Schematic interpretation of the formula (2.12)

Due to the boundary conditions, the columns of A_α are in the subspace of negative Lyapunov exponents. We further choose them to be pairwise orthogonal.

Then we can write:

$$P_{(1,1,\ldots)}[\psi] = \left|\det\left(BP_{n\times n}\right)\det\left(P_{n\times n}T(U)P_{n\times n}\right)\det\left(P_{n\times n}A\right)\right|^2, \tag{2.13}$$

where the rows of B are given by the B_i and the A is the matrix formed by the A_α as columns. The projection $P_{n\times n}$ restricts to the space of the first n Lyapunov exponents.

Provided that the n points and energies are pairwise distinct, $\det\left(BP_{n\times n}\right)$ and $\det\left(P_{n\times n}A\right)$ are finite and do not scale with L. From Oseledecs theorem it then follows that

$$\ln\det\left(P_{n\times n}T(U)P_{n\times n}\right) \sim -\frac{1}{2}(\lambda_1 + \ldots + \lambda_n)U, \tag{2.14}$$

here, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n smallest Lyapunov exponents. From this, we can infer

$$\langle \ln P_{(1^n)}[\psi]\rangle_{Q1D} \sim -(\lambda_1 + \ldots + \lambda_n)U. \tag{2.15}$$

The logarithm appears, since the Lyapunov exponents describe typical localization lengths. It follows that in the Q1D strip, the operators scale as

$$\langle \mathcal{O}_{(q^n)}(w)\rangle_{Q1D} \sim e^{-q(\lambda_1+\ldots+\lambda_n)U+\mathcal{O}(q^2)} \tag{2.16}$$

in the limit of small q.

On the other hand, in the 2D system by definition

$$\langle \mathcal{O}_\lambda(z) \rangle_{2D} = R^{(2-x_\nu)|\lambda|} \langle P_\lambda[\psi] \rangle_{2D} \sim R^{-x_\lambda^{(s)}} \tag{2.17}$$

holds. Using the conformal relation Eq. (2.11), we now can relate the Q1D scaling to the 2D multifractal exponents:

$$\langle \mathcal{O}_\lambda(w) \rangle_{Q1D} \sim \left(\frac{\pi}{M}\right)^{x_\lambda^{(s)}} \exp\left[-\frac{\pi}{M} x_\lambda^{(s)} (L-u)\right]. \tag{2.18}$$

In particular for $\lambda = (q^n)$ with small q this becomes:

$$\langle \mathcal{O}_{(q^n)}(w) \rangle_{Q1D} \sim \left(\frac{\pi}{M}\right)^{x_{(q^n)}^{(s)}} \cdot \exp\left[-q \frac{\pi}{M} \left.\frac{dx_{(q^n)}^{(s)}}{dq}\right|_{q=0} (L-u)\right]. \tag{2.19}$$

Putting the equations (2.19) and (2.16) together and taking the limit $L \to \infty$, we obtain the connection between Lyapunov exponents and the multifractal spectrum:

$$\pi \left.\frac{dx_{(q^n)}^{(s)}}{dq}\right|_{q=0} = M \cdot \sum_{i=1}^{n} \lambda_i. \tag{2.20}$$

2.2 Numerical results

2.2.1 Subleading critical exponents

Pure scaling observables at Class C have form (2.6) because Class C is a class with (pseudo)spin degree of freedom. From the mapping to percolation, the exponents $\mu = 1/12$ or equivalently $x_{(1)}^{(s)} = 1/3$ are known [6]. In order to find numerically wave functions we use the $SU(2)$ generalization of the Chalker-Coddington random network model for Class C. In the framework of this model, we consider the propagation of wave function along the square network: wave function obtains random phase at every link, which corresponds to the potential disorder in the initial system, and scatters at every node, which corresponds to the tunneling between equipotential lines in the initial system. The scattering matrices have the following form (at the critical point: $\theta = \frac{\pi}{4}$):

$$\mathbf{S} = \mathbf{1}_2 \otimes \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.21}$$

$$\mathbf{S}' = \mathbf{1}_2 \otimes \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \tag{2.22}$$

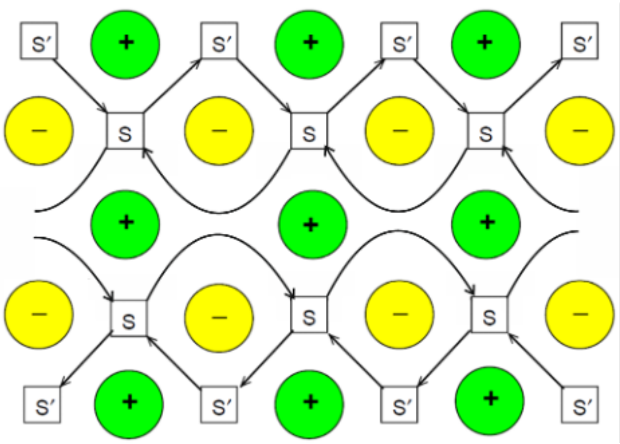

Figure 2.3: An illustration of square network for the Chalker-Coddington model.

We introduce a boundary to this system setting $\theta = 0$ for all nodes which lie on the straight line (see Fig.2.3). We treat $N = 10000$ realizations of disorder and the following sizes of the system: $L = 32, ...832$.

λ	τ^s_λ	$\tau^s_{\lambda(exact)}$	Δ^s_λ	x^s_λ
(1)	1.0815	13/12	-0.0018	0.3315 ± 0.0022
(2)	2.838	17/6	-0.329	0.338 ± 0.009
$(1,1)$	4.487	4.5	1.320	1.987 ± 0.007
(3)	4.36	4.25	-0.89	0.11 ± 0.05
$(2,1)$	6.27	6.25	1.02	2.02 ± 0.03
$(1,1,1)$	9.15	9.25	3.90	4.90 ± 0.04
(4)	5.74	-	-1.59	-0.26 ± 0.18
$(3,1)$	7.86	-	0.52	1.86 ± 0.09
$(2,2)$	8.92	-	1.58	2.92 ± 0.05
$(2,1,1)$	10.83	11	3.49	4.83 ± 0.12
$(1,1,1,1)$	14.41	$15\frac{1}{3}$	7.07	8.41 ± 0.06

Table 2.1: SQH network: numerically calculated subleading critical exponents $\tau^{(s)}_\lambda$, $\Delta^{(s)}_\lambda$ and $x^{(s)}_\lambda$ in comparison to exact values $\tau^{(s)}_{\lambda(exact)}$ from percolation mapping [6, 20, 21, 22] for pure-scaling observables with $q \equiv |\lambda| \leq 4$.

The data in the Table 2.1 is obtained from wavefunction observables with $r = 2$. Certain exponents can be calculated exactly ($\tau^s_{\lambda(exact)}$) using the mapping to percolation [6, 20, 21, 22]. We can see that numerical data is in good agreement with the analytical results from percolation. Numerical results for the dependence of observables $L^{d+\mu q}\langle P_\lambda^{\rm sp}[\psi]\rangle$ on r/L are presented in Fig.2.4, r lies in the region $[2, ..., 11]$. We expect the following power-law dependence:

$$L^{(d+\mu)q}\langle P_\lambda^{\rm sp}[\psi]\rangle \sim L^{-\Delta^{(s)}_{(q_1,q_2,...)}} \tag{2.23}$$

$$q \equiv |\lambda| \tag{2.24}$$

It is clear from the plots that P_λ has power-law dependence on L since the data shows straight lines on the log-log scale.

In addition, we can analyze dependence of critical exponents $x_{q_1,q_2,q_3,...}$ on q_1, q_2, q_3,... and compare this dependence with the prediction from generalized parabolicity (2.8), in order to check whether local conformal invariance is violated or not. The comparison of numerical result for critical exponents and prediction from generalized parabolicity is presented on the Fig.2.5. It is clear that generalized parabolicity is strongly violated in Class C, therefore, local conformal invariance is violated too.

2.2.2 Lyapunov exponents

We have derived the equality (2.20) between Lyapunov exponents for the quasi 1D strip and critical exponents for 2D system from the conformal mapping (2.9). Now we would like to test this equality numerically.

To calculate $\left.\frac{dx_{(q^n)}}{dq}\right|_{q=0}$ in the 2D systems, we numerically calculate critical surface exponents $x_{(q^n)}$ in the same way as it was done in the previous section (linear sizes lie in the range $[32, ..., 832]$ and the disorder averaging is made over $N = 10^4$ realizations). We do numerical calculations for several values of q in the small vicinity of 0 and then calculate the derivative $\left.\frac{dx_{(q^n)}}{dq}\right|_{q=0}$.

To find the Lyapunov exponents, we implement transfer matrix approach in the framework of Chalker-Coddington random network model [51]. Using transfer matrix approach for the strip we find Lyapunov exponents λ_i in the following way:

$$\lambda_i = \frac{\ln t_i^* t_i}{L} \tag{2.25}$$

here, t_i is the eigenvalue of transfer matrix T and L is the length of the strip. For calculation of Lyapunov exponents λ_i we implement the transfer-matrix approach for quasi-1D systems: the strip length is $L = 10^5$ and width M lies in the range $[32, ..., 160]$ in order to be in the Q1D limit $L/M \gg 1$. To estimate the statistical errors of this approach, an average over 10 disorder configurations is computed.

Numerical results for Lypunov exponents for several values of width M are presented in Fig.2.6 and in Table 2.2.

	$\pi \frac{dx_{(q^n)}}{dq}\Big\vert_{q=0}$	$M \cdot \sum_{i=1}^{n} \lambda_i$
$n = 1$	1.805 ± 0.007	1.821 ± 0.016
$n = 2$	8.60 ± 0.02	8.46 ± 0.05
$n = 3$	19.51 ± 0.08	19.81 ± 0.19
$n = 4$	32.84 ± 0.09	35.54 ± 0.43

Table 2.2: Numerical results for the derivative of subleading exponent x_{q^n} and n smallest Lyapunov exponents λ_i. We observe slight discrepancies for big n, which happens because higher moments with components $\lambda \gg 1$ are prone to suffer from insufficient statistics.

From this data we can see that the equality (2.20) holds with good precision.

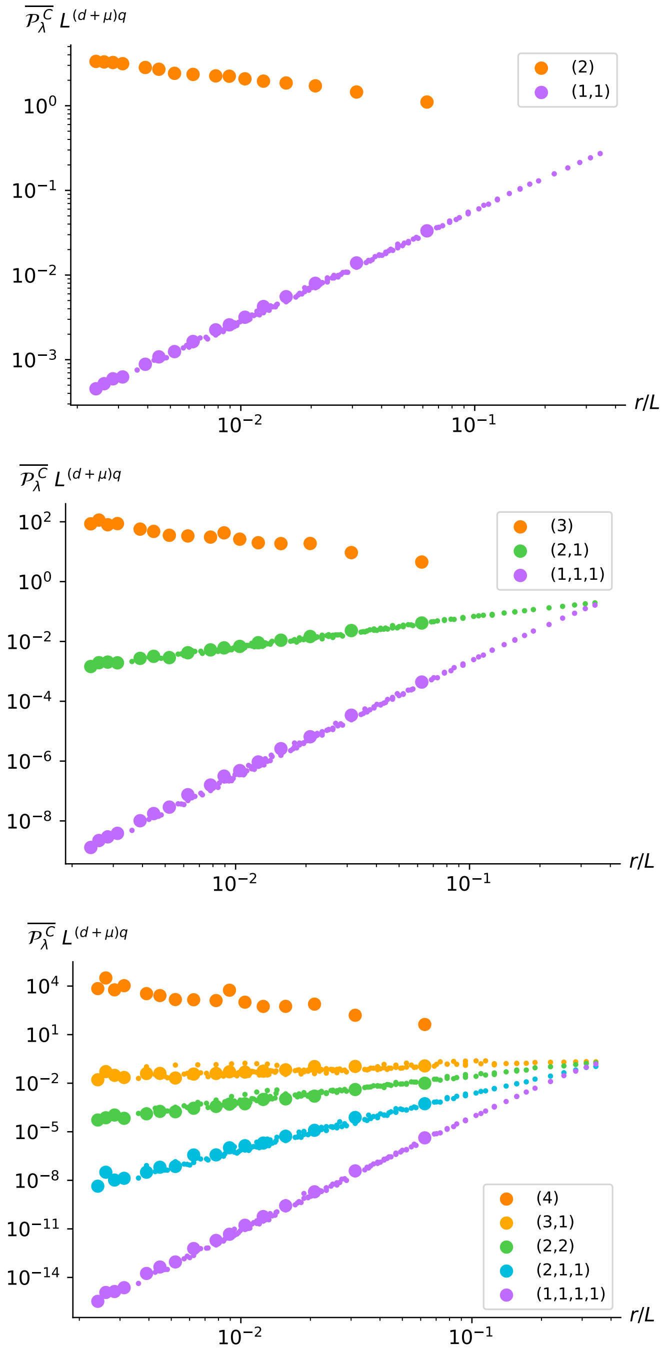

Figure 2.4: Generalized surface multifractality in metal phase at Class C for $q = 2$ (top), $q = 3$ (middle) and $q = 4$ (bottom). The pure-scaling observables are averaged over $N = 10000$ realizations of disorder and over the boundary. The data is scaled with $r^{\Delta_{q_1}+\Delta_{q_2}+\ldots+\Delta_{q_n}}$, due to this we observe the collapse as a function of r/L. Data which correspond to the smallest r ($r = 2$) is highlighted as large dots.

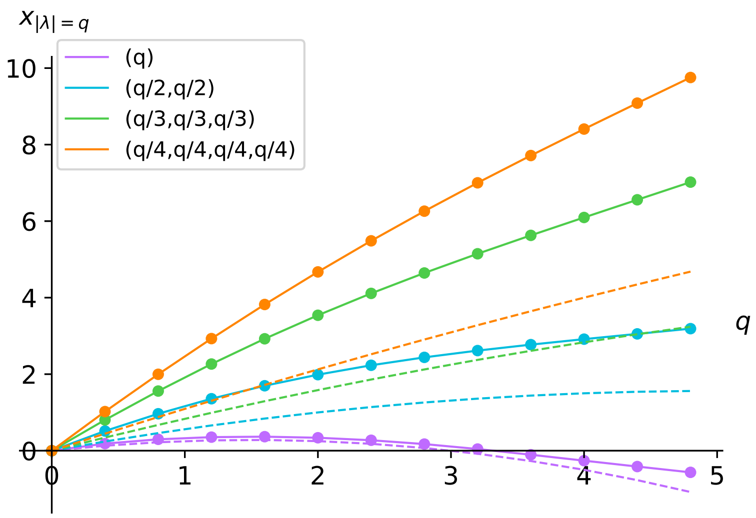

Figure 2.5: Dependence of critical surface exponents $x_{(\frac{q}{n})^n}$ on q for n=1,2,3,4 at Class C. Solid line corresponds to numerical data, dashed line - to generalized parabolicity, ($b = 1/8$,$c_j = 1 - 4j$). It is clear that generalized parabolicity is strongly violated.

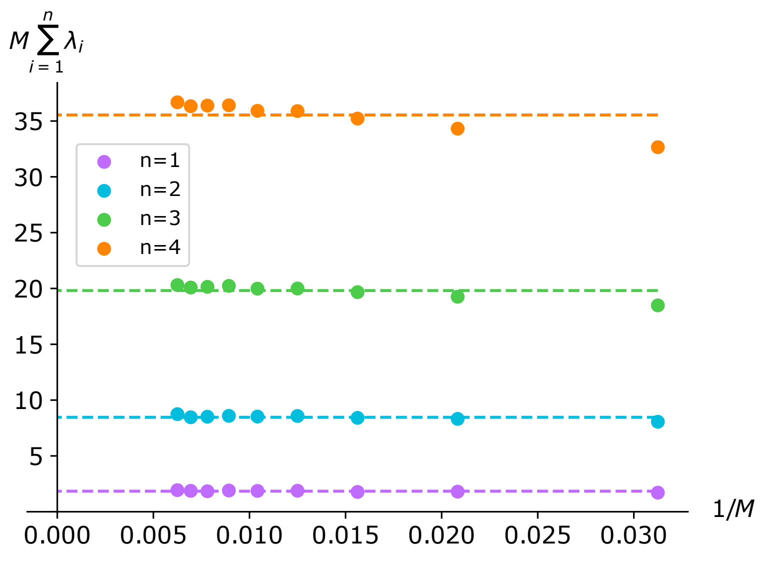

Figure 2.6: Dependence of sum of Lyapunov exponents $\sum_{i=1}^{n} \lambda_i$ for the class C on the inversed width of the strip $1/M$ for $n = 1, 2, 3, 4$. Dashed lines correspond to the values, averaged over the system sizes.

Chapter 3

Generalized bulk multifractality with interaction

3.1 Formalism of Finkel'stein NLσM

3.1.1 NLσM action

We start with the description of formalism of the Finkel'stein NLσM applied to class C. We follow approach of Refs. [52, 53]. As usual, the NLσM action is given as a sum of the noninteracting part, S_0, and the term S_{int}, describing interaction:

$$Z = \int D[Q] \exp S, \qquad S = S_0 + S_{\text{int}}, \tag{3.1}$$

where

$$S_0 = -\frac{g}{16}\int_{\boldsymbol{x}} \operatorname{Tr}(\nabla Q)^2 + Z_\omega \int_{\boldsymbol{x}} \operatorname{Tr}\hat{\varepsilon} Q, \tag{3.2a}$$

$$S_{\text{int}} = -\frac{\pi T \Gamma_t}{4}\sum_{\alpha,n}\int_{\boldsymbol{x}} \operatorname{Tr}(I_n^\alpha \mathsf{s} Q)\operatorname{Tr}\big(I_{-n}^\alpha \mathsf{s} Q\big). \tag{3.2b}$$

Here and in what follows, we use the shorthand notation $\int_{\boldsymbol{x}} \equiv \int d^d\boldsymbol{x}$. The field Q is Hermitian matrix, $Q^\dagger = Q$, which satisfies a standard nonlinear local constraint

$$Q^2(\boldsymbol{x}) = 1. \tag{3.3}$$

The matrix field Q acts in the $N_r \times N_r$ replica space, in the 2×2 spin space and in the $2N_m \times 2N_m$ space of the Matsubara fermionic energies, $\varepsilon_n = \pi T(2n+1)$. The action (3.1) involves the following matrices

$$(I_k^\gamma)_{nm}^{\alpha\beta} = \delta_{n-m,k}\delta^{\alpha\beta}\delta^{\alpha\gamma}\,\mathsf{s}_0, \quad \hat{\varepsilon}_{nm}^{\alpha\beta} = \varepsilon_n\,\delta_{nm}\delta^{\alpha\beta}\,\mathsf{s}_0\,. \tag{3.4}$$

Here s_0 denotes the 2×2 identity matrix in the spin space. We note that Greek indices represent replica space whereas Latin indices corresponds to Matsubara energies. The vector $\mathsf{s} = \{\mathsf{s}_1, \mathsf{s}_2, \mathsf{s}_3\}$ is the vector of three nontrivial Pauli matrices

$$\mathsf{s}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{s}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathsf{s}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.5}$$

Since spin quantum Hall symmetry class (class C) belongs to the Bogolubov – de Gennes symmetry classes there is an additional symmetry that relates positive and negative Matsubara energies,

$$\begin{gathered} Q = -\bar{Q}, \qquad \bar{Q} = \mathsf{s}_2 \, L_0 Q^{\mathsf{T}} L_0 \, \mathsf{s}_2, \\ (L_0)^{\alpha\beta}_{nm} = \delta_{\varepsilon_n, -\varepsilon_m} \delta^{\alpha\beta} \, \mathsf{s}_0 \, . \end{gathered} \tag{3.6}$$

Here superscript $^{\mathsf{T}}$ denotes the matrix transposition operation.

Nonlinear constraint (3.3) can be resolved by representing the matrix Q as rotation around the fixed matrix Λ:

$$Q = \mathtt{T}^{-1} \Lambda \mathtt{T}, \qquad \Lambda^{\alpha\beta}_{nm} = \operatorname{sgn} \varepsilon_n \, \delta_{nm} \delta^{\alpha\beta} \, \mathsf{s}_0 \tag{3.7}$$

Here the rotation $\mathtt{T}$ is a unitary matrix satisfying

$$\mathtt{T}^{-1} = \mathtt{T}^{\dagger}, \quad (\mathtt{T}^{-1})^{\mathsf{T}} L_0 \, \mathsf{s}_2 = \mathsf{s}_2 \, L_0 \mathtt{T}. \tag{3.8}$$

The NLσM action (3.1) involves three parameters. Bare dimensionless spin conductance is denoted as g. Bare strength of exchange interaction (interaction in the triplet particle-hole channel) is Γ_t. The parameter Z_ω describes the renormalization of the frequency term. Generically, g, Γ_t, and Z_ω are subjected to renormalization. Finally, temperature is denoted by T.

We note that the symmetry (3.6) forbids the interaction in singlet particle-hole channel since $\operatorname{Tr} I^\alpha_n \, \mathsf{s}_0 \, Q \equiv 0$. The Cooper channel interaction is suppressed by the absence of time-reversal symmetry.

The action (3.1) of the Finkel'stein NLσM for the class C is similar to the one for a standard Wigner-Dyson class A in the presence of spin rotation symmetry (see Refs. [40, 41, 54] for review). We emphasize two distinctions. At first, there is no interaction in the singlet particle-hole channel. Secondly, there is the additional symmetry relation (3.6). These two features make class C in the presence of interaction to be different from interacting class A.

Similarly to the class A, the action (3.1) can be supplemented by the Pruisken's theta-term. Being topological this term does not affect perturbative analysis presented in this paper however it is responsible for the existence of the spin quantum Hall transition in $d = 2$ dimensions.

3.1.2 Perturbation theory

In order to construct perturbation theory, we prefer to use the square-root parametrization of the Q matrix:

$$Q = W + \Lambda\sqrt{1-W^2}, \qquad W = \begin{pmatrix} 0 & w \\ w^\dagger & 0 \end{pmatrix}. \tag{3.9}$$

Here the block structure of matrix W is with respect to positive and negative Matsubara frequencies. In particular, we adopt the following notations: $W_{n_1 n_2} = w_{n_1 n_2}$ and $W_{n_2 n_1} = w^\dagger_{n_2 n_1}$ with $\varepsilon_{n_1} > 0$ and $\varepsilon_{n_2} < 0$. It is convenient to use the following expansion $w = \sum_{\mathsf{j}=0}^{3} w_{\mathsf{j}}\, \mathsf{s}_{\mathsf{j}}$. As a consequence of the constraint (3.6), the elements of w_{j} satisfy the symmetry relations

$$\begin{aligned} (w_{\mathsf{j}})^{\alpha\beta}_{n_1 n_2} &= \mathsf{v}_{\mathsf{j}} (w_{\mathsf{j}})^{\beta\alpha}_{-n_2,-n_1}, \\ \mathsf{v}_{\mathsf{j}} &= -\operatorname{tr}\big(\mathsf{s}_{\mathsf{j}}\, \mathsf{s}_2\, \mathsf{s}_{\mathsf{j}}^T\, \mathsf{s}_2\big)/2 = \{-1, 1, 1, 1\}. \end{aligned} \tag{3.10}$$

In particular, Eq. (3.10) implies $(w_0)^{\alpha\alpha}_{n_1,-n_1} \equiv 0$.

Expanding the action (3.1) to the second order in W, we find the propagators of Gaussian theory:

$$\Big\langle (w_{\mathsf{j}})^{\alpha\beta}_{n_1 n_2}(\boldsymbol{q}) (w^\dagger_{\mathsf{j}})^{\mu\nu}_{n_4 n_3}(-\boldsymbol{q}) \Big\rangle = \frac{2}{g}\Big[\delta^{\alpha\nu}\delta^{\beta\mu}\delta_{n_1 n_3}\delta_{n_2 n_4} + \mathsf{v}_{\mathsf{j}}\delta^{\alpha\mu}\delta^{\beta\nu}\delta_{n_1,-n_4}\delta_{n_2,-n_3} \\ - \frac{4\pi T\gamma}{D}(1-\delta_{\mathsf{j}0})\delta^{\alpha\nu}\delta^{\beta\mu}\delta^{\alpha\beta}\delta_{n_{12},n_{34}}\mathcal{D}^t_q(i\omega_{n_{12}})\Big]\mathcal{D}_q(i\omega_{n_{12}}). \tag{3.11}$$

Here we use the following shorthand notations $n_{12} = n_1 - n_2$ and $\omega_{n_{12}} = \varepsilon_{n_1} - \varepsilon_{n_2}$. Also we introduced the bare diffusion coefficient $D = g/(4Z_\omega)$ and dimensionless interaction strength $\gamma = \Gamma_t/Z_\omega$. Next,

$$\mathcal{D}_q(i\omega_n) = \Big[q^2 + \omega_n/D\Big]^{-1}, \tag{3.12a}$$

$$\mathcal{D}^t_q(i\omega_n) = \Big[q^2 + (1+\gamma)\omega_n/D\Big]^{-1} \tag{3.12b}$$

stand for diffuson and diffuson dressed by interaction via ladder resummation, respectively. We mention that the product $\gamma\mathcal{D}^{-1}_q(i\omega_n)\mathcal{D}^t_q(i\omega_n)$ gives the dynamically screened exchange interaction in the random phase approximation.

As we shall see below, in the process of renormalization of the NLσM action it is convenient not to keep track on Matsubara frequencies of slow fields in the propagators. Then, in order to regularize the infrared, it is convenient to add the following regulator into the action (3.1):

$$S_{\mathrm{h}} = \frac{gh^2}{8}\int_{\boldsymbol{x}} \operatorname{Tr}\Lambda Q. \tag{3.13}$$

On the level of the Gaussian theory S_{h} results in the change $q^2 \to q^2 + h^2$ in the diffusive propagators (3.12a) and (3.12b).

3.2 Background field renormalization of the action

In this section we present details of the one-loop renormalization of the NLσM action within the background field method. Although one-loop results for parameters g, Z_ω, and γ has been reported before in Refs. [55, 56, 53, 57], this section serves place to set notations.

Let us split the matrix field: $Q \to \mathtt{T}^{-1} Q\, \mathtt{T}$ where Q now plays the role of "fast" mode and $\underline{Q} = \mathtt{T}^{-1} \Lambda\, \mathtt{T}$ is a "slow" field. We assume that the matrix field $\mathtt{T}$ deviates from the unit matrix only at small frequencies such that $\mathtt{T}_{nm}^{\alpha\beta} = \delta_{nm} \delta^{\alpha\beta}\, \mathsf{s}_0$ for $\max\{|\varepsilon_n|, |\varepsilon_m|\} \gg \Omega$. Here Ω plays the role of the ultra-violet cutoff for "slow" modes. Next we write

$$S[\mathtt{T}^{-1} Q\, \mathtt{T}] = S[\underline{Q}] + S[Q] + \delta S_0 + \delta S_{\mathrm{int}}, \tag{3.14}$$

where

$$\delta S_0 = \delta S_0^{(1)} + \delta S_0^{(2),1} + \delta S_0^{(2),2} + \delta S_0^{(\varepsilon)} + \delta S_0^{(\mathrm{h})}, \tag{3.15a}$$

$$\delta S_{\mathrm{int}} = \delta S_{\mathrm{int}}^{(1),1} + \delta S_{\mathrm{int}}^{(1),2} + \delta S_{\mathrm{int}}^{(2),1} + \delta S_{\mathrm{int}}^{(2),2}. \tag{3.15b}$$

Here following Ref. [58], we introduce the following notations

$$\delta S_0^{(1)} = -\frac{g}{4} \int_{\boldsymbol{x}} \operatorname{Tr} \mathtt{A} \overline{\delta Q} \nabla \overline{\delta Q}, \tag{3.16a}$$

$$\delta S_0^{(2),1} = -\frac{g}{4} \int_{\boldsymbol{x}} \operatorname{Tr} \mathtt{A} \overline{\delta Q} \mathtt{A} \Lambda, \tag{3.16b}$$

$$\delta S_0^{(2),2} = -\frac{g}{8} \int_{\boldsymbol{x}} \operatorname{Tr} \mathtt{A} \overline{\delta Q} \mathtt{A} \overline{\delta Q}, \tag{3.16c}$$

$$\delta S_0^{(\varepsilon)} = Z_\omega \int_{\boldsymbol{x}} \operatorname{Tr} \big[\mathtt{T}\, \hat{\varepsilon}, \mathtt{T}^{-1}\big] \overline{\delta Q}, \tag{3.16d}$$

$$\delta S_0^{(h)} = \frac{g h^2}{8} \int_{\boldsymbol{x}} \operatorname{Tr} \big[\mathtt{T}\, \Lambda, \mathtt{T}^{-1}\big] \overline{\delta Q}, \tag{3.16e}$$

$$\delta S_{\rm int}^{(1),1} = -\frac{1}{2}\pi T\Gamma_t \sum_{\alpha,n}\int_{\boldsymbol{x}} \mathrm{tr}\, I_n^\alpha \mathtt{s}\underline{Q}\, \mathrm{tr}\, I_{-n}^\alpha \mathtt{s}\overline{\delta Q}, \tag{3.16f}$$

$$\delta S_{\rm int}^{(2),1} = -\frac{1}{2}\pi T\Gamma_t \sum_{\alpha,n}\int_{\boldsymbol{x}} \mathrm{tr}\, I_n^\alpha \mathtt{s}\underline{Q}\, \mathrm{tr}\, \mathtt{A}_{-n}^\alpha \overline{\delta Q}, \tag{3.16g}$$

$$\delta S_{\rm int}^{(1),2} = -\frac{1}{2}\pi T\Gamma_t \sum_{\alpha,n}\int_{\boldsymbol{x}} \mathrm{tr}\, I_n^\alpha \mathtt{s}\overline{\delta Q}\, \mathrm{tr}\, \mathtt{A}_{-n}^\alpha \overline{\delta Q}, \tag{3.16h}$$

$$\delta S_{\rm int}^{(2),2} = -\frac{1}{4}\pi T\Gamma_t \sum_{\alpha,n}\int_{\boldsymbol{x}} \mathrm{tr}\, \mathtt{A}_n^\alpha \overline{\delta Q}\, \mathrm{tr}\, \mathtt{A}_{-n}^\alpha \overline{\delta Q}, \tag{3.16i}$$

where $\overline{\delta Q} = Q - \Lambda$, $\mathtt{A} = \mathtt{T}\,\nabla \mathtt{T}^{-1}$, and $\mathtt{A}_n^\alpha = [\mathtt{T}\, I_n^\alpha \mathtt{s}, \mathtt{T}^{-1}]$.

Within the one-loop approximation, the effective action for the "slow" field $\underline{Q}$ can be obtain as

$$S_{\rm eff}[\underline{Q}] = \ln \int D[Q] e^{S[\mathtt{T}^{-1} Q\, \mathtt{T}]} \simeq S[\underline{Q}] + \langle \delta S_0 + \delta S_{\rm int}\rangle$$
$$+\frac{1}{2}\langle\!\langle (\delta S_0 + \delta S_{\rm int})^2 \rangle\!\rangle\,. \tag{3.17}$$

Here $\langle\!\langle A^2 \rangle\!\rangle$ denotes the irreducible average. We note that it is enough to expand $\overline{\delta Q}$ to the second order in W for computation of the averages in the above expressions.

3.2.1 Background field renormalization of Γ_t

We start computation from renormalization of the exchange interaction Γ_t. There are several contributions. At first, we find

$$\langle \delta S_{\rm int}^{(2),1}\rangle \simeq \frac{1}{4}\pi T\Gamma_t \sum_{\alpha,n}\int_{\boldsymbol{x}} \mathrm{tr}\, I_n^\alpha \mathtt{s}\underline{Q}\, \mathrm{tr}\, \mathtt{A}_{-n}^\alpha \Lambda \langle W^2\rangle \to \frac{\delta_{(2),1}\Gamma_t}{\Gamma_t} S_{\rm int}[\underline{Q}], \tag{3.18}$$

where

$$\frac{\delta_{(2),1}\Gamma_t}{\Gamma_t} = -\frac{2\mathsf{v}}{g}\int_p \mathcal{D}_p(0) + \frac{24\pi T\gamma}{gD}\sum_{m>0}\int_p \mathcal{D}\!\mathcal{D}_p^t(i\omega_m). \tag{3.19}$$

Here, in order to keep track for anomalous contribution related with the additional symmetry (3.6), we introduce the parameter $\mathsf{v} = \sum_{\mathsf{j}=0}^{3} \mathsf{v}_{\mathsf{j}} = 2$. Also we introduce the shorthand notations which will be intensively used below: $\int_p \equiv \int d^d\boldsymbol{p}/(2\pi)^d$ and $\mathcal{D}\!\mathcal{D}_p^t(i\omega_m) \equiv \mathcal{D}_p(i\omega_m)\mathcal{D}_p^t(i\omega_m)$.

Next contribution to Γ_t comes from

$$\frac{1}{2}\,\langle\!\langle(\delta S_{\rm int}^{(1),1})^2\rangle\!\rangle \simeq \frac{1}{2}\left\langle\!\!\left\langle\left(\frac{\pi T\Gamma_t}{4}\sum_{\alpha;n}\int_x \operatorname{tr} I_n^\alpha \mathsf{s}\underline{Q}\,\operatorname{tr} I_{-n}^\alpha \mathsf{s}\Lambda W^2\right)^2\right\rangle\!\!\right\rangle$$

$$\to \frac{\delta_{(1),1;(1),1}\Gamma_t}{\Gamma_t} S_{\rm int}[\underline{Q}], \qquad (3.20)$$

where

$$\frac{\delta_{(1),1;(1),1}\Gamma_t}{\Gamma_t} = -\frac{8\pi T\gamma}{gD}\sum_{n>0}\int_p \mathcal{D}_p^2(2i\varepsilon_n) + \frac{16\pi T\gamma}{gD}\sum_{m>0}\int_p \Big[\mathcal{D}_p^2(i\omega_m) - \mathcal{D}_p^{t2}(i\omega_m)\Big]. \qquad (3.21)$$

One more contribution is as follows

$$\langle\!\langle\delta S_{\rm int}^{(1),1}\delta S_{\rm int}^{(1),2}\rangle\!\rangle = -\frac{(\pi T\Gamma_t)^2}{8}\left\langle\!\!\left\langle\sum_{\alpha,n}\int_{\boldsymbol{x}} \operatorname{tr} I_n^\alpha \mathsf{s}\underline{Q}\,\operatorname{tr} I_{-n}^\alpha \mathsf{s}\Lambda W^2\right.\right.$$

$$\left.\left.\times\sum_{\beta,m}\int_{\boldsymbol{x}'} \operatorname{tr} I_m^\beta \mathsf{s}W\,\operatorname{tr} \mathtt{A}_{-m}^\beta W\right\rangle\!\!\right\rangle \to \frac{\delta_{(1),1;(1),2}\Gamma_t}{\Gamma_t} S_{\rm int}[\underline{Q}], \qquad (3.22)$$

where

$$\frac{\delta_{(1),1;(1),2}\Gamma_t}{\Gamma_t} = -\frac{24\pi T\gamma}{gD}\sum_{m>0}\int_p \mathcal{D}\mathcal{D}_p^t(i\omega_m) + \frac{32\pi T\gamma}{gD}\sum_{m>0}\int_p \mathcal{D}_p^{t2}(i\omega_m). \qquad (3.23)$$

Last contribution to Γ_t is provided by the following combination

$$\left\langle\!\!\left\langle\,\delta S_{\rm int}^{(2),2} + \frac{1}{2}\big(\delta S_{\rm int}^{(1),2}\big)^2\right\rangle\!\!\right\rangle = -\frac{\pi T\Gamma_t}{4}\sum_{\alpha,\beta;nm}\sum_{\mathsf{j},\mathsf{k}=1}^{3}\int_{\boldsymbol{x},\boldsymbol{x}'}\left\langle\!\!\left\langle\Big(\delta_{nm}\right.\right.$$

$$\times\delta^{\alpha\beta}\delta_{\mathsf{jk}}\delta(\boldsymbol{x}-\boldsymbol{x}') - 2\pi T\Gamma_t\,\operatorname{tr} I_n^\alpha \mathsf{s}_{\mathsf{j}} W\,\operatorname{tr} I_{-m}^\beta \mathsf{s}_{\mathsf{k}} W\Big)$$

$$\left.\left.\times\operatorname{tr} \mathtt{A}_{-n,\mathsf{j}}^\alpha W\,\operatorname{tr} \mathtt{A}_{m,\mathsf{k}}^\beta W\right\rangle\!\!\right\rangle \to \frac{\delta_{(2),2}\Gamma_t}{\Gamma_t} S_{\rm int}[\underline{Q}], \qquad (3.24)$$

where

$$\frac{\delta_{(2),2}\Gamma_t}{\Gamma_t} = \frac{2}{g}\int_p \mathcal{D}_p(0) - \frac{16\pi T\gamma}{gD}\sum_{m>0}\int_p \mathcal{D}_p^{t2}(i\omega_m). \qquad (3.25)$$

Combining all the above contributions (cf. Eqs. (3.19), (3.21), (3.23), and (3.25)) together, we find

$$\frac{\delta\Gamma_t}{\Gamma_t} = -\frac{2(\mathsf{v}-1)}{g}\int_p \mathcal{D}_p(0) - \frac{8\pi T\gamma}{gD}\sum_{n>0}\int_p \mathcal{D}_p^2(2i\varepsilon_n)$$
$$+\frac{16\pi T\gamma}{gD}\sum_{m>0}\int_p \mathcal{D}_p^2(i\omega_m) \to (1-3\gamma)\frac{th^{\epsilon}}{\epsilon}. \tag{3.26}$$

Here the final expression of the last line is obtained by setting $T=0$ and using h as an infrared regulator together with dimensional regularization in $d=2+\epsilon$ dimension. Parameter t controlling disorder is defined as

$$t = \frac{2^{2-d}\Gamma(2-d/2)}{g\pi^{d/2}} \quad \xrightarrow{d=2} \quad t = \frac{1}{\pi g}\ . \tag{3.27}$$

3.2.2 Background field renormalization of Z_ω

Now we compute the one-loop renormalization of Z_ω. We start from the following contribution

$$\langle \delta S_0^{(\varepsilon)}\rangle = -\frac{Z_\omega}{2}\int_{\boldsymbol{x}} \mathrm{Tr}\big[\mathtt{T}\,\hat{\varepsilon}, \mathtt{T}^{-1}\big]\Lambda\langle W^2\rangle \to \delta_\varepsilon Z_\omega\,\mathrm{Tr}\,\hat{\varepsilon}\underline{Q}, \tag{3.28}$$

where (cf. Eq. (3.19))

$$\frac{\delta_\varepsilon Z_\omega}{Z_\omega} = -\frac{\mathsf{v}}{g}\int_p \mathcal{D}_p(0) + \frac{12\pi T\gamma}{gD}\sum_{n>0}\int_p \mathcal{D}\mathcal{D}_p^t(i\omega_n). \tag{3.29}$$

The second contribution comes from

$$\frac{1}{2}\left\langle\!\!\left\langle \big(\delta S_{\mathrm{int}}^{(1);2}\big)^2 \right\rangle\!\!\right\rangle = \frac{(\pi T\Gamma_t)^2}{8}\left\langle\!\!\left\langle \Big(\sum_{\alpha,n}\int_{\boldsymbol{x}} \mathrm{tr}\, I_n^\alpha \mathsf{s} W\,\mathrm{tr}\,\mathtt{A}_{-n}^\alpha W\Big)^2 \right\rangle\!\!\right\rangle$$
$$\to \frac{2\pi T\Gamma_t^2}{g}\int_{\boldsymbol{x}\boldsymbol{x}'}\sum_{\alpha,\omega_n>\Omega}\omega_n \mathcal{D}_{\boldsymbol{x}-\boldsymbol{x}'}^t(i\omega_n)\langle \mathrm{tr}\,\mathtt{A}_n^\alpha(\boldsymbol{x})W(\boldsymbol{x})\,\mathrm{tr}\,\mathtt{A}_{-n}^\alpha(\boldsymbol{x}')W(\boldsymbol{x}')\rangle. \tag{3.30}$$

Here we singled out the contribution from large values of n. The part with $\omega_n < \Omega$ does not contribute to the renormalization of Z_ω. We note that due "largeness" of ω_n, either $\mathtt{T}$ or $\mathtt{T}^{-1}$ in the expression for $\mathtt{A}_n^\alpha$ is the identity matrix. Then we find

$$\frac{1}{2}\left\langle\!\!\left\langle \big(\delta S_{\mathrm{int}}^{(1);2}\big)^2 \right\rangle\!\!\right\rangle \to \frac{12\pi T\gamma\Gamma_t}{gD}\int_{\boldsymbol{x}\boldsymbol{x}'}\sum_{\omega_n>0}\omega_n\mathcal{D}_{\boldsymbol{x}-\boldsymbol{x}'}^t(i\omega_n)\sum_{\alpha,\beta;k,m}$$
$$\times\mathcal{D}_{\boldsymbol{x}-\boldsymbol{x}'}(i\omega_{n+|m|} - i\omega_k\,\mathrm{sgn}\,\omega_m)\,\mathrm{tr}\,\mathtt{T}_{mk}^{\beta\alpha}(\boldsymbol{x})[\mathtt{T}^{-1}(\boldsymbol{x}')]_{km}^{\alpha\beta}. \tag{3.31}$$

Expanding the propagator to the first power in "small" frequencies ω_k and ω_m, we obtain

$$\frac{1}{2}\left\langle\!\!\left\langle\left(\delta S_{\text{int}}^{(1);2}\right)^2\right\rangle\!\!\right\rangle \to \delta_{(1);2;(1);2)} Z_\omega \operatorname{Tr} \hat{\varepsilon} \underline{Q}, \tag{3.32}$$

where

$$\frac{\delta_{(1);2;(1);2)} Z_\omega}{Z_\omega} = \frac{12\pi T\gamma}{gD} \sum_{n>0} \int_p \Big[\mathcal{D}_p^2(i\omega_n) - \mathcal{D}\!\!\!\!\mathcal{D}_p^t(i\omega_n)\Big]. \tag{3.33}$$

Combining together both contributions, cf. Eqs. (3.29) and (3.33), we find

$$\begin{aligned}\frac{\delta Z_\omega}{Z_\omega} &= -\frac{\mathsf{v}}{g}\int_p \mathcal{D}_p(0) + \frac{12\pi T\gamma}{gD}\sum_{n>0}\int_p \mathcal{D}_p^2(i\omega_n) \\ &\to (1-3\gamma)\frac{th^\epsilon}{\epsilon}.\end{aligned} \tag{3.34}$$

We note that inspection of Eqs. (3.26) and (3.34) demonstrates that $\delta Z_\omega/Z_\omega \equiv \delta\Gamma_t/\Gamma_t$ within the one-loop approximation (the lowest order in t). It implies that the dimensionless interaction parameter is not renormalized,

$$\delta\gamma \equiv 0. \tag{3.35}$$

3.2.3 Background field renormalization of g

We start from the following comment. The matrix vector field $\mathtt{A}$ is related with the matrix $\underline{Q}$ as: $\operatorname{Tr}[\mathtt{A},\Lambda]^2 = \operatorname{Tr}\big(\nabla \underline{Q}\big)^2$. Therefore, components of $\mathtt{A}$ anticommuting with Λ can only contribute to renormalized effective action. Thus, for a sake of simplicity we can assume that $\mathtt{A}\Lambda = -\Lambda\mathtt{A}$.

There are two contributions to renormalization of g. At first, we have

$$\langle \delta S_0^{(2),1}\rangle = \frac{g}{8}\int_{\boldsymbol{x}} \operatorname{Tr} \mathtt{A}\Lambda\mathtt{A}\Lambda\langle W^2\rangle \to -\frac{\delta_{(2),1} g}{16}\int_{\boldsymbol{x}} \operatorname{Tr}\big(\nabla\underline{Q}\big)^2, \tag{3.36}$$

where (cf. Eq. (3.19))

$$\frac{\delta_{(2),1} g}{g} = -\frac{\mathsf{v}}{g}\int_p \mathcal{D}_p(0) + \frac{12\pi T\gamma}{gD}\sum_{n>0}\int_p \mathcal{D}\!\!\!\!\mathcal{D}_p^t(i\omega_n). \tag{3.37}$$

The other contribution comes from $\frac{1}{2}\langle\!\langle\big(\delta S_{\text{int}}^{(1);2}\big)^2\rangle\!\rangle$. Using Eq. (3.31), we write

$$\begin{aligned}\frac{1}{2}\left\langle\!\!\left\langle\left(\delta S_{\text{int}}^{(1);2}\right)^2\right\rangle\!\!\right\rangle &\to \frac{12\pi T\gamma\Gamma_t}{gD}\int_{\boldsymbol{x}\boldsymbol{x}'}\sum_{\omega_n>0}\omega_n\, \mathcal{D}\!\!\!\!\mathcal{D}^t_{x-x'}(i\omega_n) \\ &\times \operatorname{tr}\mathtt{T}(\boldsymbol{x})\mathtt{T}^{-1}(\boldsymbol{x}').\end{aligned} \tag{3.38}$$

Here we neglect "small" frequencies in comparison with "large" frequency ω_n. Next expanding $\mathtt{T}(\boldsymbol{x})$ and $\mathtt{T}^{-1}(\boldsymbol{x}')$ near the point $(\boldsymbol{x}+\boldsymbol{x}')/2$ to the second order $\boldsymbol{x}-\boldsymbol{x}'$ and using the identity

$$\int\limits_{\boldsymbol{x}} \boldsymbol{x}^2\, \mathcal{D}\!\!\!\mathcal{D}^t_{\boldsymbol{x}}(i\omega_n) = 4\int\limits_p \mathcal{D}^t_p(i\omega_n)\mathcal{D}^2_p(i\omega_n)\Bigl[1-\frac{4p^2}{d}\mathcal{D}_p(i\omega_n)\Bigr], \tag{3.39}$$

we find

$$\frac{1}{2}\Bigl\langle\!\Bigl\langle\bigl(\delta S^{(1);2}_{\rm int}\bigr)^2\Bigr\rangle\!\Bigr\rangle \to -\frac{\delta_{(1),2;(1),2}g}{16}\int_{\boldsymbol{x}} \mathrm{Tr}\bigl(\nabla \underline{Q}\bigr)^2, \tag{3.40}$$

where

$$\begin{aligned}\frac{\delta_{(1),2;(1),2}g}{g} = \frac{12\pi T\gamma}{gD}\int_p \sum_{n>0}\Bigl[\mathcal{D}^2_p(i\omega_n) - \mathcal{D}\!\!\!\mathcal{D}^t_p(i\omega_n)\Bigr] \\ \times\Bigl[1-\frac{4p^2}{d}\mathcal{D}_p(i\omega_n)\Bigr].\end{aligned} \tag{3.41}$$

We note that the above contribution contains the full derivative

$$\int_p \mathcal{D}^2_p(i\omega_n)\Bigl[1-\frac{4p^2}{d}\mathcal{D}_p(i\omega_n)\Bigr] = -\frac{1}{4d}\int_p \partial_{p_\mu}\partial_{p_\mu}\mathcal{D}^2_p(i\omega_n). \tag{3.42}$$

This term being full derivative does not contribute to renormalization and can be safely neglected.

Combing both contributions to g, cf. Eqs. (3.37) and (3.41), together, we obtain

$$\begin{aligned}\frac{\delta g}{g} &= -\frac{\mathsf{v}}{g}\int\limits_p \mathcal{D}_p(0) + \frac{48\pi T\gamma}{dgD}\sum_{m>0}\int\limits_p p^2\mathcal{D}^2_p(i\omega_m)\mathcal{D}^t_p(i\omega_m) \\ &\to \Bigl[1+6f(\gamma)\Bigr]\frac{th^\epsilon}{\epsilon}, \quad f(\gamma) = 1-\frac{1+\gamma}{\gamma}\ln(1+\gamma).\end{aligned} \tag{3.43}$$

3.2.4 Background field renormalization of h^2

Finally, we discuss the background field renormalization of the $S_{\rm h}$ regulator. This is intimately related with the renormalization of the Q matrix itself, so-called Z-factor. As we shall discuss below, the latter is also related with the renormalization of the LDoS. There is a single contribution to renormalization of h^2:

$$\langle \delta S_0^{(h)}\rangle = -\frac{gh^2}{16}\int\limits_{\boldsymbol{x}} \mathrm{Tr}\bigl[\mathtt{T}\,\Lambda, \mathtt{T}^{-1}\bigr]\Lambda\langle W^2\rangle \to \frac{gh^2\delta Z}{8}\int\limits_{\boldsymbol{x}} \mathrm{Tr}\,\Lambda \underline{Q}, \tag{3.44}$$

where (cf. Eq. (3.19))

$$\delta Z = -\frac{\mathsf{v}}{g}\int_p \mathcal{D}_p(0) + \frac{12\pi T\gamma}{gD}\sum_{n>0}\int_p \mathcal{D}\!\!\!\!\mathcal{D}^t_p(i\omega_n)$$
$$\to \Bigl[1 - 3\ln(1+\gamma)\Bigr]\frac{th^\epsilon}{\epsilon}. \qquad (3.45)$$

Introducing the renormalization of h^2 according to $\delta(gh^2) = gh^2\delta Z$ we find

$$\frac{\delta h^2}{h^2} = \delta Z - \frac{\delta g}{g} = -3\Bigl[\ln(1+\gamma) + 2f(\gamma)\Bigr]\frac{th^\epsilon}{\epsilon}. \qquad (3.46)$$

We note that there is no renormalization of h^2 in the absence of interaction, $\gamma = 0$.

3.2.5 One-loop renormalization

We introduce the renormalized infrared scale h' and renormalized conductance g' as

$$h'^2 = \frac{gh^2 Z}{g'} = h^2\Bigl[1 - \frac{bth^\epsilon}{\epsilon}\Bigr], \qquad g' = g\Bigl[1 + \frac{a_1 th^\epsilon}{\epsilon}\Bigr],$$
$$a_1 = 1 + 6f(\gamma), \qquad b = 3\ln(1+\gamma) + 6f(\gamma). \qquad (3.47)$$

Also we introduce renormalized Z_ω and Γ_t:

$$\frac{Z'_\omega}{Z_\omega} = \frac{\Gamma'_t}{\Gamma_t} = 1 + (1 - 3\gamma)\frac{th^\epsilon}{\epsilon}. \qquad (3.48)$$

Then applying the minimal subtraction scheme (see e.g. the book [59] for details), we can formulate the one-loop renormalization group equations as

$$\frac{dt}{d\ell} = -\epsilon t + \bigl[\mathsf{v}/2 + 6f(\gamma)\bigr]t^2 + O(t^3), \qquad (3.49a)$$
$$\frac{d\gamma}{d\ell} = 0 + O(t^2), \qquad (3.49b)$$
$$\frac{d\ln Z_\omega}{d\ell} = -(\mathsf{v}/2 - 3\gamma)t + O(t^2), \qquad (3.49c)$$
$$\frac{d\ln Z}{d\ell} = -\bigl[\mathsf{v}/2 - 3\ln(1+\gamma)\bigr]t + O(t^2). \qquad (3.49d)$$

where $\ell = \ln 1/h'$ plays the role of the logarithm of the infrared lengthscale. At $T = 0$ the later is just a system size. At finite temperature the infrared scale is set by the temperature length $\sim \sqrt{D/T}$. We remind that $\mathsf{v} = 2$.

We mention that Eqs. (3.49) coincide with renormalization group equations obtained in Refs. [55, 56, 53, 57].

3.3 Local derivativeless operators

3.3.1 General construction

In this section we construct the local pure scaling operators without spatial derivatives. These operators are eigenoperators with respect to the renormalization group, i.e. the renormalization group flow preserves their form. We shall follow the approach of Ref. [60].

The simplest local derivativeless operator is related with the LDoS. It can be written as

$$\mathcal{K}_1(E) = \frac{1}{4}\sum_{p=\pm} \mathcal{P}_1^{\alpha;p}(E). \tag{3.50}$$

Here the retarded/advanced correlation function $\mathcal{P}_1^{\alpha;\pm}(E)$ is defined from its Matsubara counterpart

$$P_1^{\alpha}(i\varepsilon_n) = \operatorname{tr}\langle Q_{nn}^{\alpha\alpha}\rangle \tag{3.51}$$

as a result of a standard analytic continuation, $i\varepsilon_n \to E+ip0^+$. We emphasize that a replica index α and Matsubara energy index n are fixed. Since the operator $\mathcal{K}_1(E)$ corresponds to the disorder-average LDoS, it stays invariant under the action of the renormalization group. We shall demonstrate this statement explicitly below.

Next we turn to the local operators without derivatives with two Q matrices. Let us introduce

$$\mathcal{K}_2(E_1,E_2) = \frac{1}{16}\sum_{p_1,p_2=\pm} p_1p_2\mathcal{P}_2^{\alpha_1\alpha_2;p_1p_2}(E_1,E_2), \tag{3.52}$$

where the correlation function $\mathcal{P}_2^{\alpha_1\alpha_2;p_1p_2}(E_1,E_2)$ is related with its Matsubara counterpart

$$\begin{aligned} P_2^{\alpha_1\alpha_2}(i\varepsilon_n,i\varepsilon_m) = {}& \bigl\langle \operatorname{tr} Q_{nn}^{\alpha_1\alpha_1}(\boldsymbol{r}) \operatorname{tr} Q_{mm}^{\alpha_2\alpha_2}(\boldsymbol{r})\bigr\rangle \\ & + \mu_2\bigl\langle \operatorname{tr}\bigl[Q_{nm}^{\alpha_1\alpha_2}(\boldsymbol{r})Q_{mn}^{\alpha_2\alpha_1}(\boldsymbol{r})\bigr]\bigr\rangle \end{aligned} \tag{3.53}$$

by standard analytic continuation to the real frequencies: $\varepsilon_n \to E_1 + ip_1 0^+$ and $\varepsilon_m \to E_2 + ip_2 0^+$. We note that no summation over α_1, α_2, n, and m is assumed and $\alpha_1 \neq \alpha_2$. The latter inequality reflects the fact that we are interested in mesoscopic fluctuations in the presence of interaction. We mention that if one interested in the renormalization of operator (3.52) alone then one can use the following simplified definition

$$K_2 = \frac{1}{16}\lim_{\varepsilon_n,\varepsilon_m\to 0}\sum_{p_1,p_2=\pm} p_1p_2 P_2^{\alpha_1\alpha_2}(ip_1|\varepsilon_n|, ip_2|\varepsilon_m|), \tag{3.54}$$

The operator $\mathcal{K}_2(E_1, E_2)$ depends on a parameter μ_2. There are particular (integer) values of μ_2 for which $\mathcal{K}_2(E_1, E_2)$ becomes the eigenoperator under the action of renormalization group.

An operator which involves the number q of matrix fields Q can be constructed in a similar way as above. We introduce

$$\mathcal{K}_q(E_1, \dots, E_q) = \frac{1}{4^q} \sum_{p_1,\dots p_q=\pm} \left(\prod_{j=1}^{q} p_j\right) \mathcal{P}_q^{\alpha_1,\dots,\alpha_q;p_1,\dots,p_q}(E_1, \dots, E_q), \quad (3.55)$$

where $\mathcal{P}_q^{\alpha_1,\dots,\alpha_q;p_1,\dots,p_q}(E_1, \dots, E_q)$ is related with the Matsubara correlation function $P_q^{\alpha_1,\dots,\alpha_q}(i\varepsilon_{n_1}, \dots, i\varepsilon_{n_q})$ by the analytic continuation to the real frequencies: $\varepsilon_{n_j} \to E_j + ip_j 0^+$. The later is given as

$$P_q^{\alpha_1,\dots,\alpha_q}(i\varepsilon_{n_1}, \dots, i\varepsilon_{n_q}) = \sum_{\{k_1,\dots,k_q\}} \mu_{k_1,\dots,k_s} \langle R_{k_1,\dots,k_s} \rangle,$$
$$R_{k_1,\dots,k_s} = \prod_{r=k_1}^{k_s} \operatorname{tr} Q_{n_{j_1} n_{j_2}}^{\alpha_{j_1}\alpha_{j_2}} Q_{n_{j_2} n_{j_3}}^{\alpha_{j_2}\alpha_{j_3}} \dots Q_{n_{j_r} n_{j_1}}^{\alpha_{j_r}\alpha_{j_1}}. \quad (3.56)$$

The summation in the right hand side of Eq. (3.56) is performed over all partitions of the integer number q, i.e. over all sets of positive integer numbers $\{k_1, \dots, k_s\}$ which satisfy the following conditions: $k_1 + k_2 + \dots k_s = q$ and $k_1 \geqslant k_2 \geqslant \dots \geqslant k_s > 0$. As above all replica indices are different: $\alpha_j \neq \alpha_k$ if $j \neq k$ for $j, k = 1, \dots, q$. One coefficient among the set $\{\mu_{k_1,\dots,k_s}\}$ can be chosen arbitrary. We stick to the normalization:

$$\mu_{1,1,\dots,1} = 1. \quad (3.57)$$

As we shall see below, for a given q the number of eigenoperators coincide with the number of partitions $(k_1, \dots, k_s)$. Therefore, it will be convenient to denote the eigenoperators by the partitions $(k_1, \dots, k_s)$ of the integer number q (see details in Ref. [8]).

In the absence of interaction, $\Gamma_t = 0$, the NLσM action reduces to Eq. (3.2b). Since the Matsubara indices of the Q matrix are not mixed without interaction (the energy of diffusive modes conserves), one can project Q matrix to the 2×2 subspace of a given positive and a given negative Matsubara frequencies. Then the group G reduces to $\tilde{G} = \mathrm{Sp}(4N_r)$ and the effective action becomes K-invariant, i.e. invariant under rotations $Q \to \mathtt{T}^{-1} Q \mathtt{T}$ with $\mathtt{T} \in \tilde{K} = \mathrm{U}(2N_r)$. Then operators $\mathcal{K}_q$ can be averaged over K rotations and resulting K-invariant operators can be classified with respect to the irreducible representations of $\tilde{G}$. Each irreducible representation contains single

K-invariant pure scaling operator [61, 12, 8]. We note that in the noninteracting case one can use also the highest weight vectors approach or the Iwasawa decomposition in order to construct eigenoperators with respect to the renormalization group transformation [12, 8].

The presence of interaction in the NLσM action introduces several complications for the approach of construction of pure scaling operators developed in Refs. [12, 8]. At first, we have to deal with fermionic representation of the NLσM. We note that works [12, 8] deal with bosonic realisation of the NLσM. As discussed in Refs. [12, 8], to extend their analysis to fermionic NLσM is far from being obvious. Secondly, it is not clear how to extend the classification of $\mathcal{K}_q$ operators with respect to the irreducible representations of the group G. For a given N_m this group is finite but we need to take the limit $N_m \to \infty$. In third, the NLσM action is no more K-invariant, that is why we have to work with non-K-invariant operators. Because of these circumstances we employ two complementary approaches. In order to determine the pure scaling operators (to fix the set of the coefficients $\{\mu_{k_1,\dots,k_s}\}$) we shall employ the background field renormalization. In order to check that the structure of the eigenoperators is not affected by the interaction we shall perform the two-loop renormalization procedure. The latter allows us to see the effect of interaction on anomalous dimensions of pure scaling operators.

3.3.2 Operator with single Q matrix

We start analysis from the operator with single Q matrix. At first, we shall perform the background field renormalization of this operator in order to demonstrate that it is the eigenoperator indeed. As we shall see, the presence of interaction affects the renormalization of this operator already at the one-loop approximation.

Employing the background field method, we find

$$\begin{aligned} P_1^{\alpha}(i\varepsilon_n) \to \operatorname{tr}\langle[\mathsf{T}^{-1}Q\mathsf{T}]^{\alpha\alpha}_{nn}\rangle &= \operatorname{Tr}\mathsf{T}\mathsf{P}^{\alpha}_{n}\mathsf{T}^{-1}\langle Q\rangle \\ &= Z(i\varepsilon_n)\operatorname{tr}\underline{Q}^{\alpha\alpha}_{nn}, \end{aligned} \tag{3.58}$$

where we introduced the projection operator to the fix replica and Matsubara energy,

$$(\mathsf{P}^{\alpha}_{n})^{\beta\gamma}_{mk} = \delta^{\alpha\beta}\delta^{\alpha\gamma}\delta_{nk}\delta_{nm}\,\mathsf{s}_0\,. \tag{3.59}$$

and (cf. Eq. (3.45))

$$Z(i\varepsilon_n) = 1 - \frac{\mathsf{v}}{g}\int_p \mathcal{D}_p(2i|\varepsilon_n|) \qquad + \frac{12\pi T\gamma}{gD}\int_p \sum_{\omega_m>|\varepsilon_n|} \mathcal{D}\mathcal{D}^t_p(i\omega_m). \tag{3.60}$$

As we see from Eq. (3.58), the operator K_1 is the eigenoperator under the action of the renormalization group. Using Eqs. (3.58) and (3.60), we obtain

$$K_1 = Z K_1[\Lambda], \quad Z = 1 + \left(\frac{\mathsf{v}}{2} - 3\ln(1+\gamma)\right)\frac{t h^{\epsilon}}{\epsilon}$$
$$= 1 + \left(\frac{\mathsf{v}}{2} - 3\ln(1+\gamma)\right)\frac{t h'^{\epsilon}}{\epsilon}. \tag{3.61}$$

Applying the minimal subtraction scheme, we deduce the anomalous dimension of the operator K_1,

$$\eta_{(1)} = -\frac{d\ln Z}{d\ell} = [1 - 3\ln(1+\gamma)]\, t + O(t^2). \tag{3.62}$$

The interaction affects the anomalous dimension of Z in the one-loop approximation. Therefore, we shall not compute its two-loop contribution here. As we shall see below, in order to perform two-loop renormalization of operators involving $q \geqslant 2$ Q matrices, one-loop result (3.62) is enough.

3.3.3 Operators with two Q matrices

Now we move on to the eigenoperators with two Q matrices. At first, in order to find them, we shall perform the background field renormalization. However, as we shall see, the background field renormalization is insensitive to the electron-electron interaction. Therefore, we shall also employ two-loop renormalization of the corresponding eigenoperators.

Background field renormalization

We start from the operator with two traces in Eq. (3.53),

$$\begin{aligned}
\operatorname{tr} Q^{\alpha\alpha}_{nn} \operatorname{tr} Q^{\beta\beta}_{mm} &\to \langle \operatorname{tr}\big[\mathtt{T}^{-1} Q \mathtt{T}\, \mathtt{P}^{\alpha}_{n}\big] \operatorname{tr}\big[\mathtt{T}^{-1} Q \mathtt{T}\, \mathtt{P}^{\beta}_{m}\big]\rangle \\
&\to \operatorname{tr} \underline{Q}^{\alpha\alpha}_{nn} \operatorname{tr} \underline{Q}^{\beta\beta}_{mm} - \frac{1}{2}\operatorname{tr} \underline{Q}^{\alpha\alpha}_{nn} \operatorname{tr}\big[\mathtt{T}^{-1}\Lambda\langle W^2\rangle \mathtt{T}\, \mathtt{P}^{\beta}_{m}\big] \\
+\langle \operatorname{tr}\big[W \mathtt{T}\, \mathtt{P}^{\alpha}_{n}\, \mathtt{T}^{-1}\big] &\operatorname{tr}\big[W \mathtt{T}\, \mathtt{P}^{\beta}_{m}\, \mathtt{T}^{-1}\big]\rangle - \frac{1}{2}\operatorname{tr}\big[\mathtt{T}^{-1}\Lambda\langle W^2\rangle \mathtt{T}\, \mathtt{P}^{\alpha}_{n}\big] \operatorname{tr} \underline{Q}^{\beta\beta}_{mm} \\
&\simeq Z^2 \operatorname{tr} \underline{Q}^{\alpha\alpha}_{nn} \operatorname{tr} \underline{Q}^{\beta\beta}_{mm} + \langle \operatorname{tr}\big[W \mathtt{T}\, \mathtt{P}^{\alpha}_{n}\, \mathtt{T}^{-1}\big] \operatorname{tr}\big[W \mathtt{T}\, \mathtt{P}^{\beta}_{m}\, \mathtt{T}^{-1}\big]\rangle.
\end{aligned} \tag{3.63}$$

Important property of the average over W in the above equation is that both Matsubara indices of matrix $\mathtt{T}\,\mathtt{P}^{\beta}_{m}\,\mathtt{T}^{-1}$ should be small since the rotation $\mathtt{T}$ represents slow mode. Therefore, the interaction part of the propagator (3.11) can be omitted since it does not lead to infrared divergent terms. Then it is straightforward to derive the following identity (for slow matrices A and B):

$$\langle \operatorname{tr} Aw \operatorname{tr} Bw^\dagger \rangle \simeq 2Y \operatorname{tr}\Big[\Lambda_- A \Lambda_+ (B - \bar{B})\Big], \quad Y = -\frac{th^\epsilon}{\epsilon}. \tag{3.64}$$

where $\Lambda_\pm = (1 \pm \Lambda)/2$ are projectors onto the subspace of positive and negative Matsubara energies. Using Eq. (3.64), we obtain

$$\begin{aligned}\langle \operatorname{tr} Q^{\alpha\alpha}_{nn} \operatorname{tr} Q^{\beta\beta}_{mm} \rangle \to Z^2 \operatorname{tr} \underline{Q}^{\alpha\alpha}_{nn} \operatorname{tr} \underline{Q}^{\beta\beta}_{mm} - Y \operatorname{tr} \underline{Q}^{\alpha\beta}_{nm} \underline{Q}^{\beta\alpha}_{mn} \\ + Y \operatorname{tr} \underline{Q}^{\alpha\beta}_{n,-m} \underline{Q}^{\beta\alpha}_{-mn}.\end{aligned} \tag{3.65}$$

The background field renormalization of the operator with single trace reads

$$\begin{aligned}\operatorname{tr} Q^{\alpha\beta}_{nm} Q^{\beta\alpha}_{mn} \to \langle \operatorname{tr} Q \, \mathsf{T} \, \mathsf{P}^\alpha_n \, \mathsf{T}^{-1} Q \, \mathsf{T} \, \mathsf{P}^\beta_m \, \mathsf{T}^{-1} \rangle \to \operatorname{tr} \underline{Q}^{\alpha\beta}_{n,m} \underline{Q}^{\beta\alpha}_{mn} \\ - \frac{1}{2} \operatorname{tr} \mathsf{T}^{-1} \Lambda \langle W^2 \rangle \, \mathsf{T} \Big[\mathsf{P}^\alpha_n \, \mathsf{T}^{-1} \Lambda \, \mathsf{T} \, \mathsf{P}^\beta_m + \mathsf{P}^\beta_m \, \mathsf{T}^{-1} \Lambda \, \mathsf{T} \, \mathsf{P}^\alpha_n\Big] \\ + \langle \operatorname{tr} W \, \mathsf{T} \, \mathsf{P}^\alpha_n \, \mathsf{T}^{-1} W \, \mathsf{T} \, \mathsf{P}^\beta_m \, \mathsf{T}^{-1} \rangle \simeq Z^2 \operatorname{tr} \underline{Q}^{\alpha\beta}_{n,m} \underline{Q}^{\beta\alpha}_{mn} \\ + \langle \operatorname{tr} W \, \mathsf{T} \, \mathsf{P}^\alpha_n \, \mathsf{T}^{-1} W \, \mathsf{T} \, \mathsf{P}^\beta_m \, \mathsf{T}^{-1} \rangle.\end{aligned} \tag{3.66}$$

Now we shall use the following identity (we assume A and B being slow matrices as above)

$$\langle \operatorname{tr} AwBw^\dagger \rangle \simeq 2Y \Big[\operatorname{tr} \Lambda_+ A \operatorname{tr} \Lambda_- B + \operatorname{tr} \Lambda_+ A \Lambda_+ \bar{B}\Big]. \tag{3.67}$$

We note that in derivation of Eq. (3.67) the following relations were used:

$$\sum_{\mathsf{j}} \operatorname{tr} \mathsf{s}_\mathsf{k} \, \mathsf{s}_\mathsf{j} \, \mathsf{s}_\mathsf{k} \, \mathsf{s}_\mathsf{j} = 8\delta_{\mathsf{k}0}, \qquad \sum_{\mathsf{j}} \mathsf{v}_\mathsf{j} \operatorname{tr} \mathsf{s}_\mathsf{k} \, \mathsf{s}_\mathsf{j} \, \mathsf{s}_\mathsf{k} \, \mathsf{s}_\mathsf{j} = -4\mathsf{v}_\mathsf{k}. \tag{3.68}$$

Using Eq. (3.67), we obtain

$$\begin{aligned}\operatorname{tr} Q^{\alpha\beta}_{nm} Q^{\beta\alpha}_{mn} \to Z^2 \operatorname{tr} \underline{Q}^{\alpha\beta}_{n,m} \underline{Q}^{\beta\alpha}_{mn} - Y \operatorname{tr} \underline{Q}^{\alpha\alpha}_{nn} \operatorname{tr} \underline{Q}^{\beta\beta}_{mm} \\ + Y \operatorname{tr} \underline{Q}^{\alpha\beta}_{n,-m} \underline{Q}^{\beta\alpha}_{-mn}.\end{aligned} \tag{3.69}$$

We emphasize that as follows from Eqs. (3.65) and (3.69), the background field renormalization mixes not only operators with one and two traces but also the ones with positive and negative Matsubara frequencies. The later is the consequence of the additional symmetry (3.6). Finally, using Eqs. (3.65) and (3.69), we find

$$\begin{aligned}K_2[Q] &\to Z^2 K_2[\underline{Q}] - \mu_2 Y O_{1,1}[\underline{Q}] - (2 + \mu_2) Y O_2[\underline{Q}] \\ &\simeq Z^2 (1 - \mu_2 Y) \Big[O_{1,1} + (\mu_2 + (\mu_2^2 - \mu_2 - 2) Y) O_2\Big].\end{aligned} \tag{3.70}$$

Here we introduce generalization of the operator $R_{k_1,\dots,k_q}$ in a way similar to transformation from $P_2^{\alpha\beta}(i\varepsilon_n, i\varepsilon_m)$ to K_2, cf. Eq. (3.54):

$$O_{k_1,\dots,k_q} = \frac{1}{4^q}\prod_{j=1}^{q}\left(\lim_{\varepsilon_{n_j}\to 0}\sum_{p_j=\operatorname{sgn}\varepsilon_{n_j}} p_j\right) R_{k_1,\dots,k_q}. \tag{3.71}$$

As follows from Eq. (3.70), in order combination $O_{1,1}+\mu_2 O_2$ serves as the eigenoperator under renormalization group transformation, μ_2 should solve the equation

$$\mu_2^2 - \mu_2 = 2 \quad\Longrightarrow\quad \mu_2 = 2, -1. \tag{3.72}$$

Therefore, we find two eigenoperators corresponding to $\mu_2 = 2$ and $\mu_2 = -1$. As it follows from Eq. (3.70), the renormalization of these eigenoperators are immune to interaction within one-loop approximation.

We note that the eigenoperators constructed in Ref. [8] differ from ones in this work by the sign of μ_2. This difference in sign is explained by usage of bosonic replica in Ref. [8] whereas we are employing fermionic replica. Translation from one approach to the other can be done by the sign change of all trace ("tr") operations.

One-loop renormalization

After construction of the eigenoperators with two Q matrices with the help of the background field renormalization method we study how the interaction affects their anomalous dimensions. It will be convenient to consider the irreducible part of the correlation function (3.53),

$$P_2^{\alpha\beta;(\mathrm{irr})}(i\varepsilon_n, i\varepsilon_m) = \langle\!\langle \operatorname{tr} Q_{nn}^{\alpha\alpha} \operatorname{tr} Q_{mm}^{\beta\beta} \rangle\!\rangle + \mu_2 \langle \operatorname{tr} Q_{nm}^{\alpha\beta} Q_{mn}^{\beta\alpha}\rangle. \tag{3.73}$$

With the help of the irreducible part the full correlation function can be restored as follows

$$P_2^{\alpha\beta}(i\varepsilon_n, i\varepsilon_m) = 4Z^2 \operatorname{sgn}\varepsilon_n \operatorname{sgn}\varepsilon_m + P_2^{\alpha\beta;(\mathrm{irr})}(i\varepsilon_n, i\varepsilon_m). \tag{3.74}$$

Expanding Q to the first order in W, we obtain the one-loop contribution,

$$P_{2,1}^{\alpha\beta;(\mathrm{irr})}(i\varepsilon_n, i\varepsilon_m) = \mu_2 \langle \operatorname{tr} W_{nm}^{\alpha\beta} W_{mn}^{\beta\alpha}\rangle = \frac{16\mu_2}{g} \times \frac{1-\operatorname{sgn}\varepsilon_n \operatorname{sgn}\varepsilon_m}{2}\int_q \mathcal{D}_q(i|\varepsilon_n| + i|\varepsilon_m|). \tag{3.75}$$

Neglecting the energy dependence in the diffusive propagator (for the reasons explained above Eq. (3.54)), we find the following one-loop result for the irreducible part of the operator K_2,

$$K_{2,1}^{(\mathrm{irr})} = \frac{\mu_2 t h^\epsilon}{\epsilon}. \tag{3.76}$$

Two-loop renormalization

Expanding Q to the second order in W, we obtain the two-loop contribution as

$$\begin{aligned} P_{2,2}^{\alpha\beta;(\mathrm{irr})} = \frac{1}{4} \operatorname{sgn}\varepsilon_n \operatorname{sgn}\varepsilon_m \left\langle\!\!\left\langle \operatorname{tr}\bigl(W^2\bigr)_{nn}^{\alpha\alpha} \operatorname{tr}\bigl(W^2\bigr)_{mm}^{\beta\beta} \right\rangle\!\!\right\rangle \\ +\mu_2 \frac{1+\operatorname{sgn}\varepsilon_n \operatorname{sgn}\varepsilon_m}{8} \left\langle \operatorname{tr}\bigl(W^2\bigr)_{nm}^{\alpha\beta} (W^2)_{mn}^{\beta\alpha} \right\rangle \\ +\mu_2 \frac{1-\operatorname{sgn}\varepsilon_n \operatorname{sgn}\varepsilon_m}{2} \left\langle\!\!\left\langle \operatorname{tr} W_{nm}^{\alpha\beta} W_{mn}^{\beta\alpha} \begin{bmatrix} S_0^{(4)} + S_{\mathrm{int}}^{(4)} \\ +\frac{1}{2}(S_{\mathrm{int}}^{(3)})^2 \end{bmatrix} \right\rangle\!\!\right\rangle . \end{aligned} \tag{3.77}$$

Then, using Eq. (3.11), we obtain

$$\begin{aligned} \left\langle\!\!\left\langle \operatorname{tr}\bigl(W^2\bigr)_{nn}^{\alpha\alpha} \operatorname{tr}\bigl(W^2\bigr)_{mm}^{\beta\beta} \right\rangle\!\!\right\rangle = \frac{64}{g^2} \left(\int_q \mathcal{D}_q(i|\varepsilon_n| + i|\varepsilon_m|) \right)^2 \\ \to 16 \frac{t^2 h^{2\epsilon}}{\epsilon^2}. \end{aligned} \tag{3.78}$$

In the last line we neglect the energy dependence in the propagators. We emphasize that this contribution is immune to the interaction. Next, we find

$$\begin{aligned} \left\langle \operatorname{tr}\bigl(W^2\bigr)_{nm}^{\alpha\beta} (W^2)_{mn}^{\beta\alpha} \right\rangle = \frac{32\mathsf{v}}{g^2} \int_{qp} \mathcal{D}_q(2i|\varepsilon_n|) \mathcal{D}_p(i|\varepsilon_n| + i|\varepsilon_m|) \\ -3 \frac{128\pi T \gamma}{g^2 D} \sum_{\varepsilon_k > 0} \int_{qp} \mathcal{D}_p(i|\varepsilon_m| + i\varepsilon_k)\, \mathcal{D}\mathcal{D}_q^t(i|\varepsilon_n| + i\varepsilon_k) \\ +(n \leftrightarrow m) \to 16\mathsf{v} \frac{t^2 h^{2\epsilon}}{\epsilon^2} - 3 \frac{128\gamma}{g^2} J_{101}^0 (1+\gamma) \\ \simeq 16\mathsf{v} \frac{t^2 h^{2\epsilon}}{\epsilon^2} - 48 \frac{t^2 h^{2\epsilon}}{\epsilon^2} \Bigl[\ln(1+\gamma) - \frac{\epsilon}{4} \ln^2(1+\gamma) \Bigr]. \end{aligned} \tag{3.79}$$

Here we introduce the following notation for integral over momenta and frequency,

$$J^{\delta}_{\nu\mu\eta}(a) = \int_{qp} \int\limits_0^\infty ds\, s^{\delta} \frac{1}{(p^2+h^2+s)^{\nu}} \frac{1}{p^2+h^2+as} \times \frac{1}{(q^2+h^2)^{\mu}} \frac{1}{(\boldsymbol{p}+\boldsymbol{q})^2+h^2+s)^{\eta}}. \tag{3.80}$$

The integrals $J^{\delta}_{\nu\mu\eta}$ were computed in Ref. [62].

For the term with $S_0^{(4)}$, we obtain

$$\begin{aligned}\Big\langle\!\Big\langle \operatorname{tr}\big[W^{\alpha\beta}_{nm} W^{\beta\alpha}_{mn}\big] S_0^{(4)} \Big\rangle\!\Big\rangle = & -\frac{8\mathsf{v}}{g^2}\int\limits_{qp} \mathcal{D}_q(i|\omega_{nm}|)\Big[2\mathcal{D}_q(i|\omega_{nm}|) \\ & + \mathcal{D}_p(i2|\varepsilon_n|) + \mathcal{D}_p(i2|\varepsilon_m|)\Big] + 3\frac{32\pi T\gamma}{g^2 D}\int\limits_{qp}\Big[\sum_{\omega_k>|\varepsilon_n|} \\ & + \sum_{\omega_k>|\varepsilon_m|}\Big]\Big[\mathcal{D}_q(i|\omega_{nm}|)+\mathcal{D}_p(i\omega_k)\Big]\mathcal{D}^t_p(i\omega_k)\mathcal{D}_q(i|\omega_{nm}|) \\ & \to -4\mathsf{v}\frac{t^2h^{2\epsilon}}{\epsilon^2} + 3\frac{32\gamma}{g^2}\Big[J^0_{020}(1+\gamma) + J^0_{110}(1+\gamma)\Big] \\ & \simeq -4\mathsf{v}\frac{t^2h^{2\epsilon}}{\epsilon^2} + 24\frac{t^2h^{2\epsilon}}{\epsilon^2}\Big[\ln(1+\gamma) + \frac{\epsilon\gamma}{2(1+\gamma)}\Big].\end{aligned} \tag{3.81}$$

In order to compute the last contribution in Eq. (3.77), we use the following simplification (we note that it is possible due to different replica indices, $\alpha \neq \beta$):

$$\begin{aligned}S^{(4)}_{\rm int} + \frac{1}{2}(S^{(3)}_{\rm int})^2 \to & -\sum_{\alpha n}\int\limits_{\boldsymbol{x},\boldsymbol{x'}}\Big[1-\frac{\gamma|\omega_n|}{D}\mathcal{D}^t_{\boldsymbol{x}-\boldsymbol{x'}}(i|\omega_n|)\Big] \\ & \times\frac{\pi T\Gamma_t}{4}\sum_{\mathsf{j}=1}^{3}\operatorname{Tr} I^{\alpha}_n\,\mathsf{s}_{\mathsf{j}}\,\Lambda W^2(\boldsymbol{x})\operatorname{Tr} I^{\alpha}_{-n}\,\mathsf{s}_{\mathsf{j}}\,\Lambda W^2(\boldsymbol{x'}).\end{aligned} \tag{3.82}$$

After tedious but straightforward calculations, we find

$$\left\langle\!\!\left\langle \operatorname{tr} W^{\alpha\beta}_{nm} W^{\beta\alpha}_{mn} \Big[S^{(4)}_{\rm int} + \frac{1}{2} (S^{(3)}_{\rm int})^2 \Big] \right\rangle\!\!\right\rangle = -3 \frac{32\pi T\gamma}{g^2 D} \int\limits_{pq}$$

$$\times \Big(\sum_{|\varepsilon_n|>\omega_k} + \sum_{|\varepsilon_m|>\omega_k} \Big) \Big[1 - \frac{\gamma|\omega_k|}{D} \mathcal{D}^t_{\boldsymbol{p}+\boldsymbol{q}}(i|\omega_k|) \Big]$$

$$\times \mathcal{D}^2_p(i|\varepsilon_n| + i|\varepsilon_m|) \mathcal{D}_q(i|\varepsilon_n| + i|\varepsilon_m| - i\omega_k)$$

$$\to -\frac{96\gamma}{g^2} \Big[J^0_{020}(1) - \gamma J^1_{021}(1+\gamma) \Big] \simeq -12\gamma \frac{t^2 h^{2\epsilon}}{\epsilon^2}$$

$$\times \Big[\frac{2\gamma - (2+\gamma)\ln(1+\gamma)}{\gamma^2} + \epsilon \frac{(2+\gamma)\ln(1+\gamma)}{\gamma^2}$$

$$+ \frac{\epsilon}{1+\gamma} + \epsilon \frac{2+\gamma}{\gamma^2} \Big(\operatorname{li}_2(-\gamma) + \frac{1}{4}\ln^2(1+\gamma) \Big) \Big]. \tag{3.83}$$

Here $\operatorname{li}_2(z) = \sum_{k=1}^{\infty} z^k/k^2$ denotes the polylogarithm. We note that the factor 32 in the first line of the above equation appears as the result of the following identity for $\mathsf{j} = 1, 2, 3$:

$$\sum_{\mathsf{j}_{1,2}=0}^{3} \operatorname{tr}\big(\mathsf{s}_\mathsf{j}\, \mathsf{s}_{\mathsf{j}_1} \mathsf{s}_{\mathsf{j}_2}\big) \Big[\operatorname{tr}\big(\mathsf{s}_\mathsf{j}\, \mathsf{s}_{\mathsf{j}_2} \mathsf{s}_{\mathsf{j}_1}\big) - \mathsf{v}_{\mathsf{j}_1} \mathsf{v}_{\mathsf{j}_2} \operatorname{tr}\big(\mathsf{s}_\mathsf{j}\, \mathsf{s}_{\mathsf{j}_1} \mathsf{s}_{\mathsf{j}_2}\big) \Big] = 32. \tag{3.84}$$

Combing the above results, Eqs. (3.78)–(3.81) and (3.83), we find

$$K^{(\rm irr)}_{2,2} = \Big[\mu_2(\mathsf{v} - 6\ln(1+\gamma)) + (b^{(2)}_2 + \epsilon\mu_2 b_3) \Big] \frac{t^2 h^{2\epsilon}}{\epsilon^2}, \tag{3.85}$$

where

$$b^{(2)}_2 = 1 - 3\mu_2 f(\gamma), \quad b_3 = \frac{3}{2} \Big\{ \frac{1+\gamma}{2\gamma} \ln^2(1+\gamma)$$

$$+ \frac{2+\gamma}{\gamma} \Big[\operatorname{li}_2(-\gamma) + \ln(1+\gamma) \Big] \Big\}. \tag{3.86}$$

Anomalous dimension

Employing the one-loop (see Eq. (3.76)) and two-loop (see Eq. (3.85)) results, we write the operator K_2 in the following form

$$K_2 = Z^2 M_2 K_2[\Lambda]. \tag{3.87}$$

Here $K_2[\Lambda] = 1$ is the classical value of K_2 and

$$M_2 = 1 + Z^{-2}(K_{2,1}^{(\mathrm{irr})} + K_{2,2}^{(\mathrm{irr})}) = 1 + \mu_2 \frac{th^\epsilon}{\epsilon} + (b_2^{(2)} + \epsilon\mu_2 b_3)\frac{t^2 h^{2\epsilon}}{\epsilon^2} = 1 + \mu_2 \frac{th'^\epsilon}{\epsilon} + (b_2^{(2)} + \epsilon\mu_2 \tilde{b}_3)\frac{t^2 h'^{2\epsilon}}{\epsilon^2}. \qquad (3.88)$$

where $\tilde{b}_3 = b_3 + b/2$ with b given by Eq. (3.47). We note that in order to determine M_2 within the two-loop approximation it is enough to know the factor Z in the one-loop approximation only. That is why we considered irreducible part of K_2.

Applying the minimal subtraction scheme to Eq. (3.88), we obtain the anomalous dimension of M_2 upto the second order in t:

$$\eta^{(\mu_2)} = -\frac{d \ln M_2}{d\ell} = \mu_2 \Big[t + 3c(\gamma)t^2\Big] + O(t^3). \qquad (3.89)$$

Here we introduce the function (cf. Refs. [63, 62, 60])

$$c(\gamma) = 2 + \frac{1+\gamma}{2\gamma}\ln^2(1+\gamma) + \frac{2+\gamma}{\gamma}\,\mathrm{li}_2(-\gamma). \qquad (3.90)$$

The finiteness of η_{μ_2} in the limit $\epsilon \to 0$ is guaranteed if the following condition holds:

$$\mu_2\,(\mu_2 - a_1) = 2b_2^{(2)} \quad \Leftrightarrow \quad \mu_2(\mu_2 - 1) = 2. \qquad (3.91)$$

We emphasize that the self-consistent condition (3.91) is (i) nonlinear in μ_2 and (ii) independent of the interaction strength γ. The former implies that it cannot be satisfied by a linear combination of two or more eigenoperators. The later guarantees that the eigenoperators in the absence of interaction remain eigenoperators in the presence of interaction.

Solving Eq. (3.91) we find two solutions: $\mu_2 = 2, -1$ in full agreement with the background field renormalization scheme above. We emphasize that Eq. (3.91) uniquely determines the value of μ_2. Denoting the corresponding eigenoperator as in the noninteracting case, we find

$$\begin{aligned} \mu_2 &= -1, \qquad && \eta_{(2)} = -t(1 + 3c(\gamma)t) + O(t^3), \\ \mu_2 &= 2, && \eta_{(1,1)} = 2t(1 + 3c(\gamma)t) + O(t^3). \end{aligned} \qquad (3.92)$$

3.4 Implications for Weyl symmetry

At a fixed point of the renormalization group flow corresponding to the spin quantum Hall transition, the scaling with the system size L of an eigenoperator characterized by the Young tableau $\lambda = (\mathsf{k}_1, \dots, \mathsf{k}_\mathsf{s})$ (with $\sum_{\mathsf{j}=1}^{\mathsf{s}} \mathsf{k}_\mathsf{j} = q$) is given as

$$K_\lambda \sim L^{-x_\lambda}, \quad x_\lambda = q x_{(1)} + \Delta_\lambda. \qquad (3.93)$$

Here the exponent $x_{(1)}$ describes the scaling of the disorder averaged LDoS and is equal to the magnitude of $\eta_{(1)}$ at the fixed point, $x_{(1)} = \eta^*_{(1)}$. Similarly, the exponent Δ_λ describes the scaling of the operator M_λ and coincides with its anomalous dimension at the fixed point, $\Delta_\lambda = \eta^*_\lambda$. We note in passing that Eq. (3.93) states explicitly that the eigenoperators are just the pure scaling operators.

In the absence of interaction, the generalized multifractal dimensions x_λ are known to obey symmetry relations as consequence of Weyl-group invariance [12]. The exponents x_λ are the same for the eigenoperators related by the following symmetry operations: reflection, $\mathsf{k_j} \to -\mathsf{c_j} - \mathsf{k_j}$, and permutation of some pair, $\mathsf{k_{j/i}} \to \mathsf{k_{i/j}} + (\mathsf{c_{i/j}} - \mathsf{c_{j/i}})/2$. For example, reflection symmetry implies that $x_{(\mathsf{q})} = x_{(\mathsf{3\text{-}q})}$, i.e., in particular, $x_{(1)} = x_{(2)}$ and $x_{(3)} = 0$. Of course, the one-loop results for the anomalous dimensions (in the absence of interaction) is consistent with the symmetry relations.

We emphasize that the presence of interaction seems to break the symmetry relations between exponents. It can be seen already at one-loop order. The interaction affects $\eta_{(1)}$ but leaves anomalous exponents η_λ for the eigenoperators with $q \geqslant 2$ intact (to the order ϵ).

Chapter 4

Summary and Conclusions

In the first chapter of this work, we numerically investigated generalized multifractality on the surface of two-dimensional disordered systems belonging to symmetry class C. Using the Chalker-Codington network model, we calculated observables (2.6), (2.7) on the surface of the systems and analyzed their dependence on the size of the system. As a result, we verified that these observables have pure-scaling dependence on the size of the system and calculated the corresponding subleading critical exponents. Besides, we showed that the numerically calculated subleading surface exponents are in good agreement with known analytical results. We also observed a strong violation of generalized parabolicity, which means a violation of local conformal invariance at the critical point. This means that the theory is not invariant under some conformal transformations (at least one). Then, we numerically analyzed the conformal transformation (2.9). Provided that the disorder averaged theory is invariant under this transformation, the relation (2.20) for subleading surface critical exponents and linear combinations of Lyapunov exponents holds. Our numerical results indicate that the relation holds with small deviations. Thus, we have demonstrated an example of a conformal transformation under the action of which the theory is invariant. It may also be of interest to find the explicit conformal transformation under the action of which the theory is not invariant. This is particularly relevant since we have already demonstrated the violation of local conformal invariance, indicating the existence of such transformations.

In the second chapter of this work, we developed the theory of generalized multifractality in class C in the presence of interaction. Using the background field method we constructed the pure scaling derivativeless operators in the Finkel'stein NLσM in class C. As in the standard Wigner-Dyson classes these operators in the interacting theory are straightforward generalization of corresponding operators for the noninteracting case. Employing second order perturbation theory in inverse spin conductance, we computed the anomalous dimensions of the pure scaling operators. These anomalous dimensions are affected by the interaction. Additionally, we checked that the constructed operators are indeed eigenoperators with respect to the renormalization group. Application of our results to the transition in $d = 2 + \epsilon$ dimensions demonstrates that interaction breaks the exact symmetry relations between generalized multifractal exponents x_λ known in the absence of interaction.

To conclude, we studied both surface and bulk multifractality. For the analytical treatment of bulk multifractality, we simply excluded the boundaries from the system. For the numerical study of surface multifractality, we explicitly introduced the boundary to the Chalker-Coddington model and applied periodic boundary conditions, therefore, we investigated a torus with the boundary wrapped around it. To understand the connection of these models to a real sample, which has a finite size and boundaries, we can consider the following approach: there exists a characteristic spatial scale at a quantum Hall transition known as the dephasing length (L_ϕ) [64]. Surface exponents are observed when the distance to the boundary is much smaller than the dephasing length (L_ϕ), while bulk exponents are observed when the distance to the boundary is much larger than L_ϕ (assuming that the sample size is much larger than the dephasing length).

In addition, it may be intriguing to examine the surface's generalized multifractality in the presence of interactions. We can expect that interactions might have a nontrivial effect on the critical exponents, similar to their impact in bulk studies. It would also be of interest to investigate whether the presence of interactions breaks the Weyl symmetry relations between the surface critical exponents or not.

The current master's thesis is based on our article published in Physical Review B [47], where we study bulk generalized multifractality in class C with interactions. Additionally, it incorporates the results of our ongoing research on surface generalized multifractality in class C, which will be published soon.

Bibliography

[1] Elihu Abrahams, editor. 50 Years of Anderson Localization. World Scientific, 2010.

[2] Philip W Anderson. Absence of diffusion in certain random lattices. Phys. Rev., 109(5):1492, 1958.

[3] P. W. Anderson. Absence of diffusion in certain random lattices. Physical Review, 109(5):1492–1505, March 1958.

[4] Ferdinand Evers and Alexander D. Mirlin. Anderson transitions. Reviews of Modern Physics, 80(4):1355–1417, October 2008.

[5] A. R. Subramaniam, I. A. Gruzberg, A. W. W. Ludwig, F. Evers, A. Mildenberger, and A. D. Mirlin. Surface criticality and multifractality at localization transitions. Physical Review Letters, 96(12), March 2006.

[6] Arvind R. Subramaniam, Ilya A. Gruzberg, and Andreas W. W. Ludwig. Boundary criticality and multifractality at the two-dimensional spin quantum Hall transition. , 78:245105, Dec 2008.

[7] F. Evers, A. Mildenberger, and A. D. Mirlin. Multifractality at the quantum hall transition: Beyond the parabolic paradigm. Physical Review Letters, 101(11), September 2008.

[8] Jonas F. Karcher, Noah Charles, Ilya A. Gruzberg, and Alexander D. Mirlin. Generalized multifractality at spin quantum hall transition. Annals of Physics, 435:168584, December 2021.

[9] F. Wegner. The mobility edge problem: continuous symmetry and a conjecture. Zeitschrift für Physik B, 35:207–210, 1979.

[10] Franz Wegner. Anomalous dimesions for the nonlinear sigma-model in $2 + \epsilon$ dimensions (I). Nucl. Phys. B, 280:193–209, 1987.

[11] Franz Wegner. Anomalous dimensions for the nonlinear sigma-model, in $2 + \epsilon$ dimensions (II). Nucl. Phys. B, 280:210–224, 1987.

[12] I. A. Gruzberg, A. D. Mirlin, and M. R. Zirnbauer. Classification and symmetry properties of scaling dimensions at anderson transitions. Physical Review B, 87(12), March 2013.

[13] A. D. Mirlin, Y. V. Fyodorov, A. Mildenberger, and F. Evers. Exact relations between multifractal exponents at the anderson transition. Physical Review Letters, 97(4), July 2006.

[14] I. A. Gruzberg, A. W. W. Ludwig, A. D. Mirlin, and M. R. Zirnbauer. Symmetries of multifractal spectra and field theories of anderson localization. Physical Review Letters, 107(8), August 2011.

[15] Jonas F. Karcher, Ilya A. Gruzberg, and Alexander D. Mirlin. Metal-insulator transition in a two-dimensional system of chiral unitary class. Phys. Rev. B, 107:L020201, Jan 2023.

[16] Jonas F. Karcher, Ilya A. Gruzberg, and Alexander D. Mirlin. Generalized multifractality in 2d disordered systems of chiral symmetry classes. arxiv, 2301:10851, Jan 2023.

[17] Alexander Altland and Martin R. Zirnbauer. Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures. Physical Review B, 55(2):1142–1161, January 1997.

[18] Martin R. Zirnbauer. Riemannian symmetric superspaces and their origin in random-matrix theory. Journal of Mathematical Physics, 37(10):4986–5018, October 1996.

[19] P. Heinzner, A. Huckleberry, and M.R. Zirnbauer. Symmetry classes of disordered fermions. Communications in Mathematical Physics, 257(3):725–771, May 2005.

[20] Ilya A. Gruzberg, Andreas W. W. Ludwig, and N. Read. Exact exponents for the spin quantum hall transition. Physical Review Letters, 82(22):4524–4527, May 1999.

[21] E. J. Beamond, John Cardy, and J. T. Chalker. Quantum and classical localization, the spin quantum hall effect, and generalizations. Physical Review B, 65(21), May 2002.

[22] Jonas F. Karcher, Ilya A. Gruzberg, and Alexander D. Mirlin. Generalized multifractality at the spin quantum hall transition: Percolation mapping and pure-scaling observables. Physical Review B, 105(18), May 2022.

[23] M. V. Feigel'man, L. B. Ioffe, V. E. Kravtsov, and E. A. Yuzbashyan. Eigenfunction fractality and pseudogap state near the superconductor-insulator transition. Phys. Rev. Lett., 98:027001, 2007.

[24] M.V. Feigel'man, L.B. Ioffe, V.E. Kravtsov, and E. Cuevas. Fractal superconductivity near localization threshold. Ann. Phys. (N.Y.), 325:1390, 2010.

[25] I. S. Burmistrov, I. V. Gornyi, and A. D. Mirlin. Enhancement of the critical temperature of superconductors by Anderson localization. Physical Review Letters, 108:017002, 2012.

[26] I. S. Burmistrov, I. V. Gornyi, and A. D. Mirlin. Superconductor-insulator transitions: phase diagram and magnetoresistance. Phys. Rev. B, 92:014506, 2015.

[27] L. Dell'Anna. Enhancement of critical temperatures in disordered bipartite lattices. Phys. Rev. B, 88:195139, 2013.

[28] I. S. Burmistrov, I. V. Gornyi, and A. D. Mirlin. Multifractally-enhanced superconductivity in thin films. Ann. Phys. (N.Y.), 435:168499, 2021. Special Issue on Localisation 2020.

[29] Bo Fan and A. M. García-García. Enhanced phase-coherent multifractal two-dimensional superconductivity. Phys. Rev. B, 101:104509, 2020.

[30] M. Stosiek, B. Lang, and F. Evers. Self-consistent-field ensembles of disordered Hamiltonians: Efficient solver and application to superconducting films. Phys. Rev. B, 101:144503, 2020.

[31] M. Stosiek, F. Evers, and I. S. Burmistrov. Multifractal correlations of the local density of states in dirty superconducting films. Phys. Rev. Research, 3:L042016, 2021.

[32] M. S. Foster and E. A. Yuzbashyan. Interaction-mediated surface-state instability in disordered three-dimensional topological superconductors with spin SU(2) symmetry. Phys. Rev. Lett., 109:246801, 2012.

[33] M. S. Foster, H.-Y. Xie, and Y.-Z. Chou. Topological protection, disorder, and interactions: Survival at the surface of three-dimensional topological superconductors. Phys. Rev. B, 89:155140, 2014.

[34] S. Kettemann and E. R. Mucciolo. Free magnetic moments in disordered systems. JETP Lett., 83:284, 2006.

[35] T. Micklitz, A. Altland, T. A. Costi, and A. Rosch. Universal dephasing rate due to diluted Kondo impurities. Phys. Rev. Lett., 96:226601, 2006.

[36] S. Kettemann and E. R. Mucciolo. Disorder-quenched Kondo effect in mesoscopic electronic systems. Phys. Rev. B, 75:184407, 2007.

[37] I. S. Burmistrov and M. A. Skvortsov. Magnetic disorder in superconductors: Enhancement by mesoscopic fluctuations. Phys. Rev. B, 97:014515, 2018.

[38] M. V. Feigel'man and V. E. Kravtsov. Electron-phonon cooling power in Anderson insulators. Phys. Rev. B, 99:125415, 2019.

[39] S. Kettemann. Exponential orthogonality catastrophe at the Anderson metal-insulator transition. Phys. Rev. Lett., 117:146602, 2016.

[40] A. M. Finkel'stein. Electron liquid in disordered conductors. In I. M. Khalatnikov, editor, Soviet Scientific Reviews, volume 14. Harwood Academic Publishers, London, 1990.

[41] D. Belitz and T. R. Kirkpatrick. The Anderson-Mott transition. Rev. Mod. Phys., 66:261, 1994.

[42] Y. Harashima and K. Slevin. Effect of electron-electron interaction near the metal-insulator transition in doped semiconductors studied within the local density approximation. Int. J. Mod. Phys. Conf. Ser., 11:90, 2012.

[43] Keith Slevin and Tomi Ohtsuki. Critical exponent for the anderson transition in the three-dimensional orthogonal universality class. New Journal of Physics, 16(1):015012, January 2014.

[44] M. Amini, V. E. Kravtsov, and M. Müller. Multifractality and quantum-to-classical crossover in the Coulomb anomaly at the Mott–Anderson metal–insulator transition. New J. Phys., 16:015022, 2014.

[45] H.-J. Lee and K.-S. Kim. Hartree-Fock study of the Anderson metal-insulator transition in the presence of Coulomb interaction: Two types of mobility edges and their multifractal scaling exponents. Phys. Rev. B, 97:155105, 2018.

[46] H. Obuse, A. R. Subramaniam, A. Furusaki, I. A. Gruzberg, and A. W. W. Ludwig. Conformal invariance, multifractality, and finite-size scaling at anderson localization transitions in two dimensions. Physical Review B, 82(3), July 2010.

[47] S. S. Babkin and I. S. Burmistrov. Generalized multifractality in the spin quantum hall symmetry class with interaction. Physical Review B, 106(12), September 2022.

[48] I. A. Gruzberg, A. D. Mirlin, and M. R. Zirnbauer. Classification and symmetry properties of scaling dimensions at Anderson transitions. , 87:125144, Mar 2013.

[49] Jonas F. Karcher, Ilya A. Gruzberg, and Alexander D. Mirlin. Generalized multifractality at metal-insulator transitions and in metallic phases of two-dimensional disordered systems. Physical Review B, 106(10), September 2022.

[50] Jonas F. Karcher, Noah Charles, Ilya A. Gruzberg, and Alexander D. Mirlin. Generalized multifractality at spin quantum Hall transition. Annals of Physics, 435:168584, 2021.

[51] B. Kramer, T. Ohtsuki, and S. Kettemann. Random network models and quantum phase transitions in two dimensions. Physics Reports, 417(5):211–342, 2005.

[52] M. Bruno, A. Toschi, L. Dell'Anna, and C. Castellani. Quasiparticle dephasing time in disordered d-wave superconductors. Phys. Rev. B, 72:104512, 2005.

[53] Luca Dell'Anna. Disordered d-wave superconductors with interactions. Nucl. Phys. B, 758:255, 2006.

[54] I. S. Burmistrov. Finkel'stein nonlinear sigma model: interplay of disorder and interaction in 2D electron systems. JETP, 129:669, 2019.

[55] M. Jeng, A. W. W. Ludwig, T. Senthil, and C. Chamon. Interaction effects on quasiparticle localization in dirty superconductors. Bull. Am. Phys. Soc., 46:231, 2001.

[56] M. Jeng, A. W. W. Ludwig, T. Senthil, and C. Chamon. Interaction effects on quasiparticle localization in dirty superconductors. arXiv:cond-mat/0112044, 2001.

[57] Yunxiang Liao, Alex Levchenko, and Matthew S. Foster. Response theory of the ergodic many-body delocalized phase: Keldysh Finkel'stein sigma models and the 10-fold way. Ann. Phys. (N.Y.), 386:97, 2017.

[58] M. A. Baranov, A. M. M. Pruisken, and B. Škorić. (Mis-)handling gauge invariance in the theory of the quantum Hall effect. II. Perturbative results. Phys. Rev. B, 60:16821, 1999.

[59] D. J. Amit. Field Theory, the Renormalization Group, and Critical Phenomena. World Scientific, Singapore, 1993.

[60] E. V. Repin and I. S. Burmistrov. Mesoscopic fluctuations of the single-particle Green's function at Anderson transitions with Coulomb interaction. Phys. Rev. B, 94:245442, 2016.

[61] D. Höf and F. Wegner. Calculation of anomalous dimensions for the nonlinear sigma model. Nucl. Phys. B, 275:561, 1986.

[62] I. S. Burmistrov, I. V. Gornyi, and A. D. Mirlin. Multifractality and electron-electron interaction at Anderson transitions. Phys. Rev. B, 91:085427, 2015.

[63] I. S. Burmistrov, I. V. Gornyi, and A. D. Mirlin. Multifractality at Anderson transitions with Coulomb interaction. Phys. Rev. Lett., 111:066601, 2013.

[64] I. S. Burmistrov, S. Bera, F. Evers, I. V. Gornyi, and A. D. Mirlin. Wave function multifractality and dephasing at metal–insulator and quantum Hall transitions. Annals of Physics, 326(6):1457–1478, 2011.