Around turbulence

Waves in turbulent media

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Presentation for

students of the department

problems in theoretical physics

Turbulence – chaotic state of a fluid appearing at high Reynolds numbers

$$\mathrm{Re} = VL/\nu.$$

Water: $\nu = 10^{-2}\, cm^2/s$. Air: $\nu = 0.15\, cm^2/s$. Flow in a pipe: transition to turbulence at $\mathrm{Re} \sim 10^3$. In turbulent regime drag is independent of ν: $F \sim \rho V^2 L^2$.

Energy is pumped into a fluid at the integral scale L. Power per unit mass $\epsilon \sim V^3/L$. What further?? Strong non-linear interaction produces eddies of smaller and smaller sizes and velocities v_r. Direct cascade or energy cascade. The cascade is stopped by viscosity at a scale r_d: $r_d v_d \sim \nu$. Large value of Re leads to an inequality $r_d \ll L$.

Scales between L and r_d – inertial interval. Characterized by chaotic behavior of eddies. It is convenient to characterize statistical properties of the flow in the inertial interval by the structure functions

$$S_n(r) = \langle |\boldsymbol{v}(\boldsymbol{r}_1) - \boldsymbol{v}(\boldsymbol{r}_2)|^n \rangle.$$

Angular brackets mean averaging over time. The observation time should be larger than the formation time $\sim L/V$.

Though just scales of the order L are relevant from the engineering viewpoint, the properties of the turbulence are sensitive to geometry there. At smaller scales the situation is more universal. There the model of statistically homogeneous and isotropic turbulence is applied. Kolmogorov theorem, $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$

$$\langle [(\boldsymbol{v}_2 - \boldsymbol{v}_1)\boldsymbol{r}/r]^3 \rangle = -(4/5)\epsilon r,$$

Therefore $S_3 \sim \epsilon r$. Hypothesis: $S_n \sim (\epsilon r)^{n/3}$ (normal scaling). Then, particularly,

$$r_d \sim (\nu^3/\epsilon)^{1/4} \sim \mathrm{Re}^{-3/4} L.$$

Thus, inertial interval exists at $\mathrm{Re} \gg 1$.

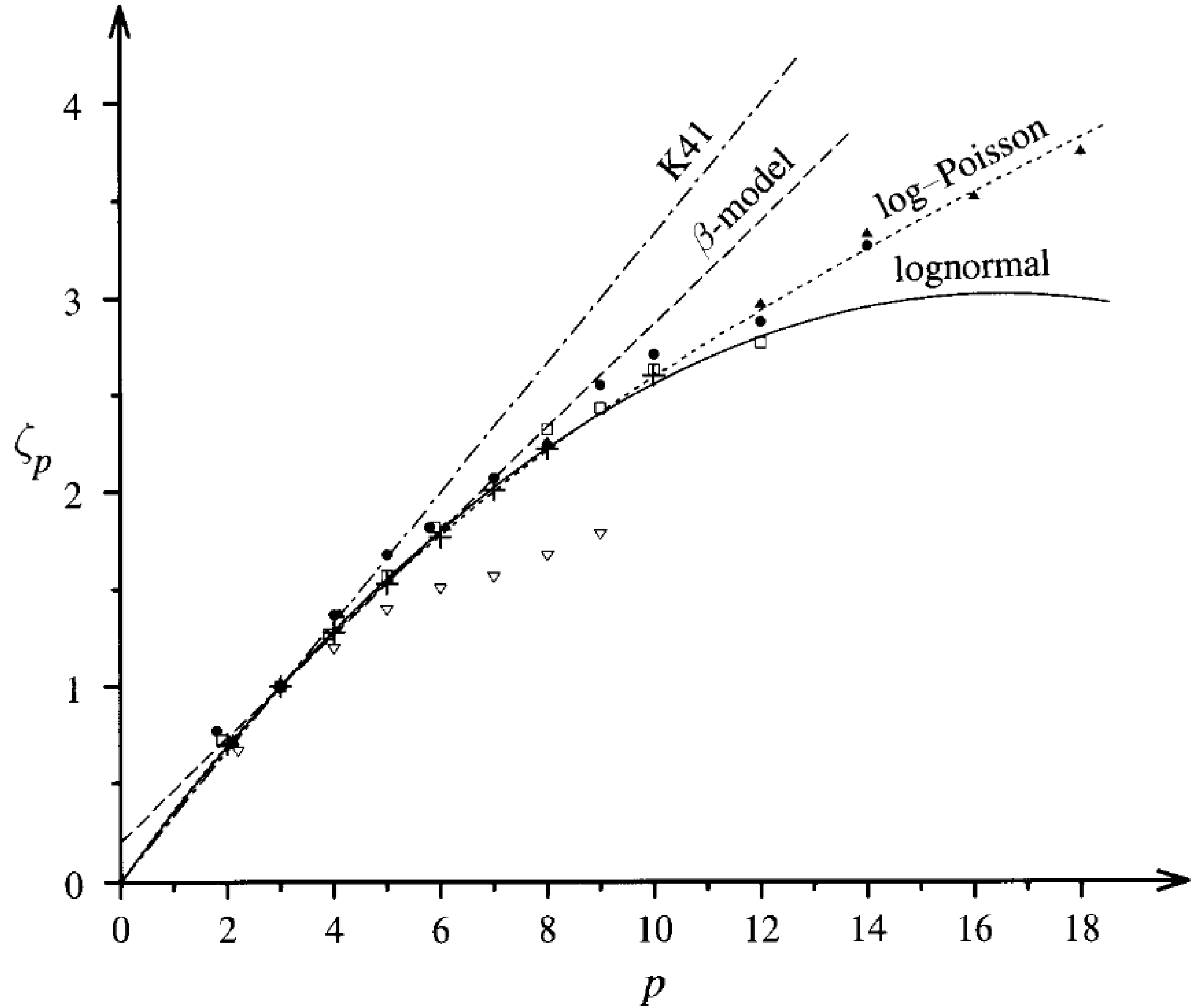
Kolmogorov spectrum:

$$\langle \boldsymbol{v}(\boldsymbol{r}_1)\boldsymbol{v}(\boldsymbol{r}_2)\rangle \sim \int dk\ \exp(ikr)\epsilon^{2/3}k^{-5/3}.$$

In reality the structure functions deviate from the normal scaling

$$S_n \sim (\epsilon r)^{n/3}(L/r)^{\xi_n} \propto r^{\zeta_n},$$

where $\xi_n > 0$ for $n > 3$. The structure functions are much larger than in accordance with the normal scaling. Intermittency!


K41
β-model
log–Poisson
lognormal
ζ_p
p
0
1
2
3
4
0
2
4
6
8
10
12
14
16
18

We pass to two-dimensional turbulence. Thin fluid layers or films (say, soap film). An interesting object – atmosphere at scales larger than its width (near $10\ km$). It is convenient to describe a two-dimensional flow in terms of its vorticity $\omega = \partial_x v_y - \partial_y v_x$, that is a scalar. It characterizes completely a two-dimensional flow due to incompressibility $\partial_x v_x + \partial_y v_y = 0$.

Two-dimensional hydrodynamics is described by the equation for the vorticity ω

$$\partial_t \omega + \boldsymbol{v} \nabla \omega = \nabla \times \boldsymbol{f} + \nu \nabla^2 \omega - \alpha \omega,$$

where $\boldsymbol{v}$ is velocity, $\boldsymbol{f}$ is pumping force per unit mass, ν is viscosity and α is bottom friction coefficient. We assume that the pumping force is correlated at a scale l and is random in time.

There are two quadratic dissipationless integrals of motion, energy and enstrophy:

$$\int dx\ dy\ v^2, \qquad \int dx\ dy\ \omega^2.$$

Pumped turbulence – two cascades: enstrophy flows to small scales whereas energy flows to large scales, being dissipated by viscosity and friction, respectively (Kraichnan 1967, Leith 1968, Batchelor 1969).

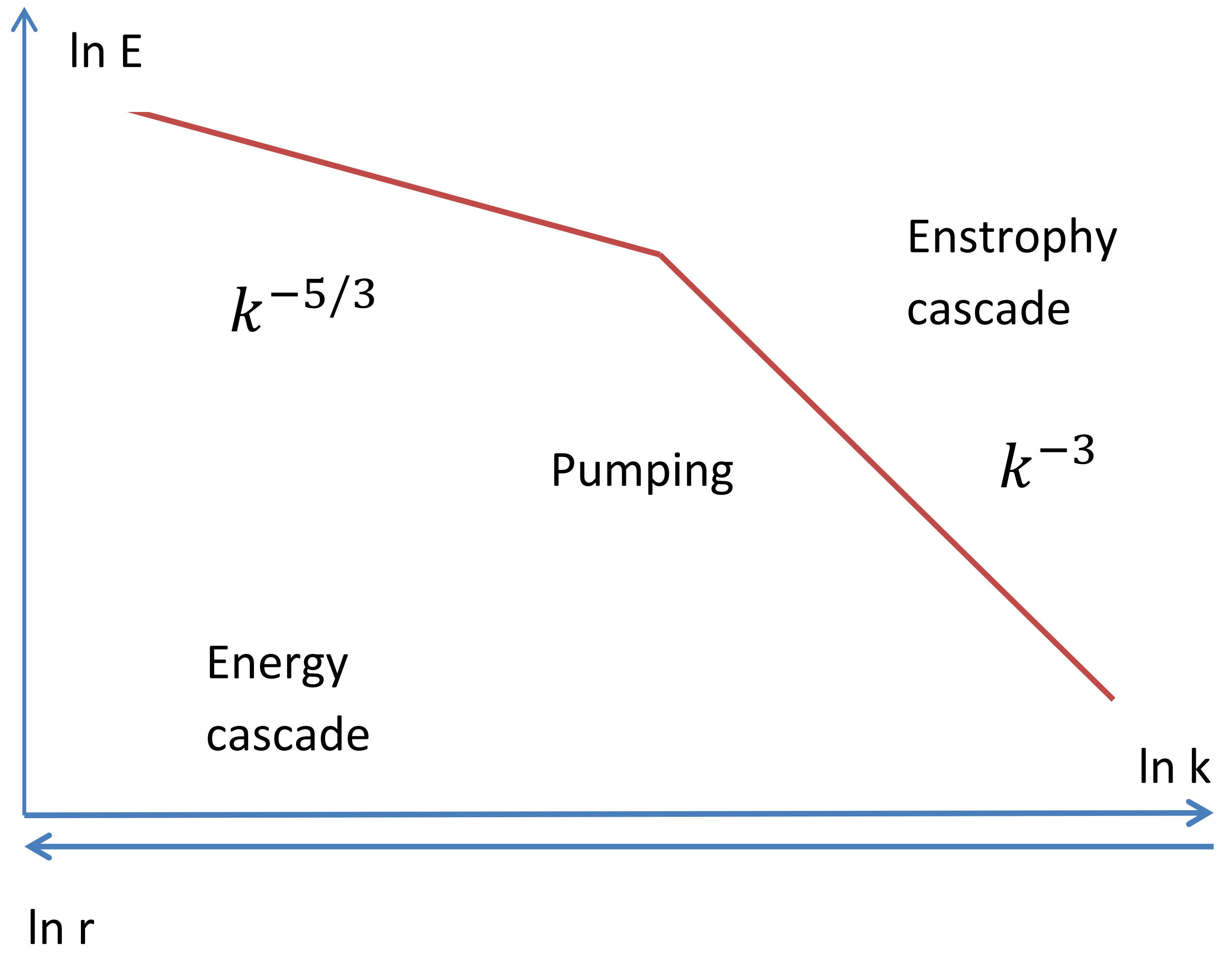
Constancy of the energy and enstrophy fluxes imply the proportionality laws

$$\langle (v_1 - v_2)\omega_1\omega_2 \rangle \propto r, \quad r \ll l;$$
$$\langle |v_1 - v_2|^3 \rangle \propto r, \quad r \gg l.$$

Suggest the normal scaling $v_1 - v_2 \propto r$ in the direct cascade and $v_1 - v_2 \propto r^{1/3}$ in the inverse cascade. The spectrum

$$\langle v_1 v_2 \rangle = \int \frac{dk}{2\pi} e^{ikr} E(k),$$

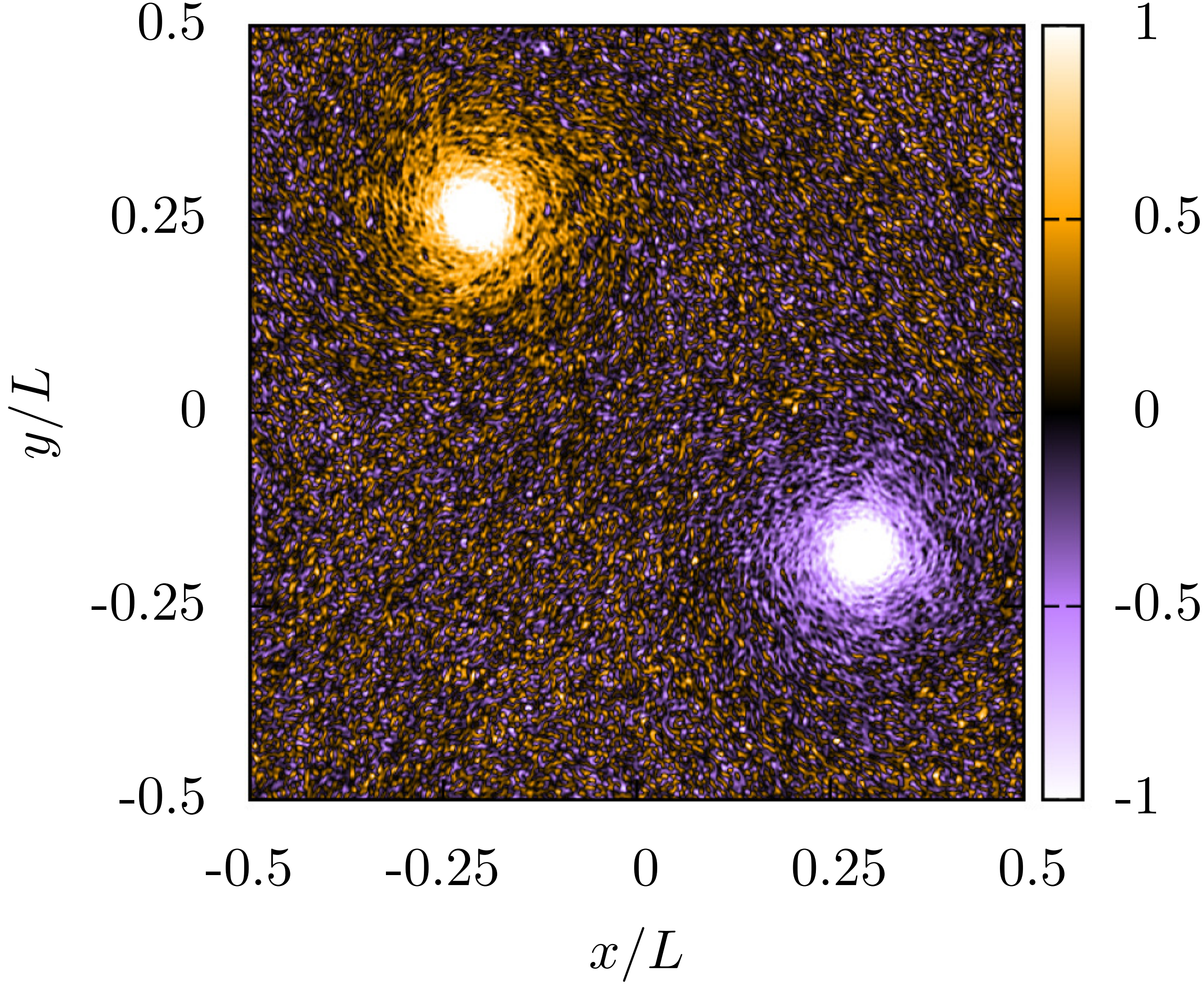
Then $E(k) \propto k^{-3}$ for the direct (enstrophy) cascade $E(k) \propto k^{-5/3}$ for the inverse (energy) cascade. Direct cascade – logarithmic correlation functions of vorticity (Falkovich, Lebedev 1994). Inverse cascade – an absence of anomalous scaling (Paret and Tabeling 1998, Boffetta, Celani and Vergassola 2000).


ln E
$k^{-5/3}$
Enstrophy
cascade
Pumping
k^{-3}
Energy
cascade
ln k
ln r

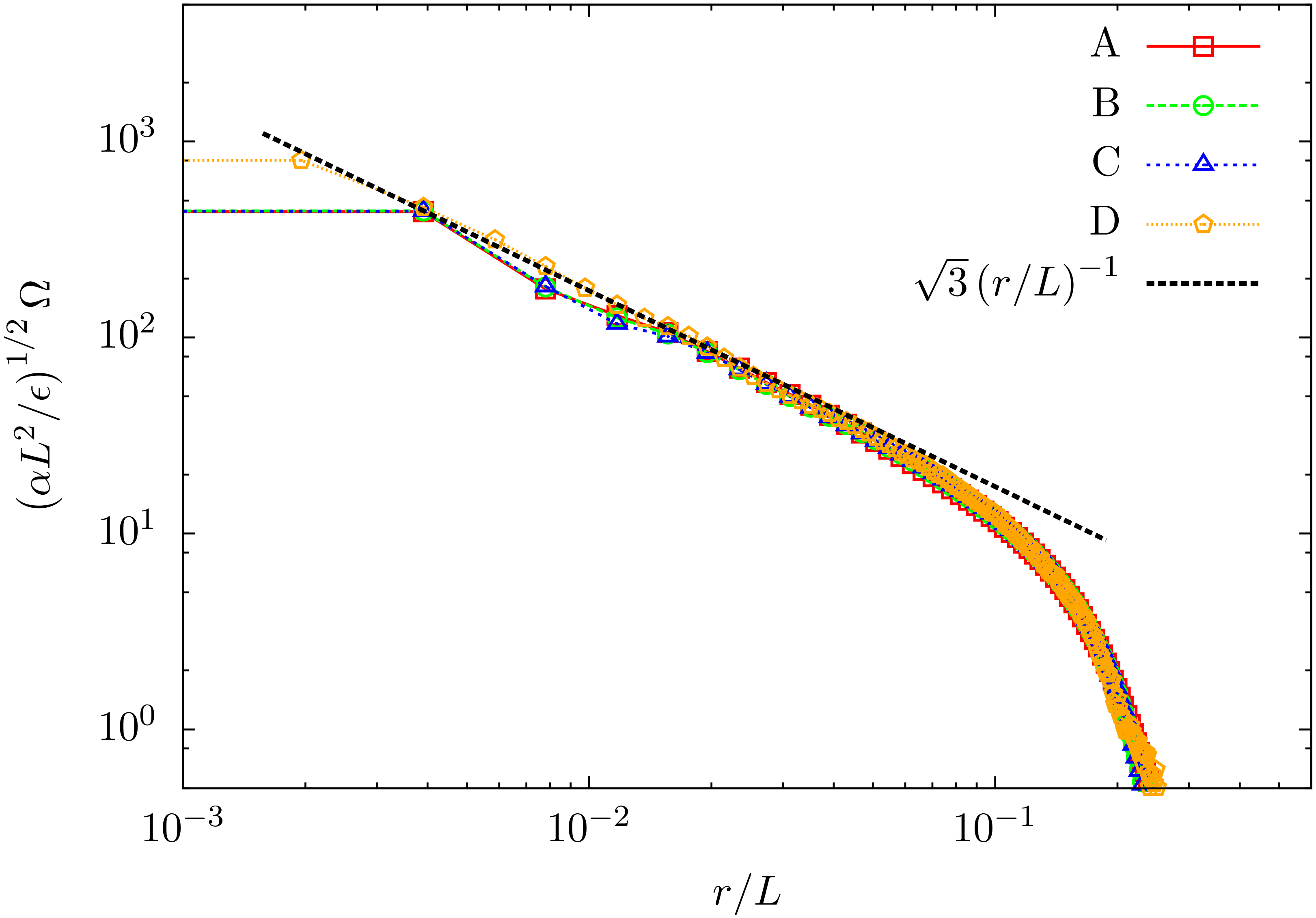
In an unbounded system the inverse cascade is terminated by the friction at the scale $L_\alpha \sim \epsilon^{1/2}\alpha^{-3/2}$ where ϵ is the energy production rate per unit mass. If the size box $L < L_\alpha$ then the energy accumulates at L: experiment (Shats, Xia, Punzmann and Falkovich 2007) and numerics (Chertkov, Connaughton, Kolokolov and Lebedev 2007). Coherent structures are formed!

The coherent velocity profile arises at a time $t \sim t_L = L^{2/3}\epsilon^{-1/3}$. After that the major part of the pumped energy is accumulated at scales $\sim L$. Therefore typical large-scale velocity $\sim \sqrt{\epsilon t}$ increases as time grows. The stage is terminated at time $t \sim \alpha^{-1}$. After that some steady (statistically homogeneous in time) state is realized.

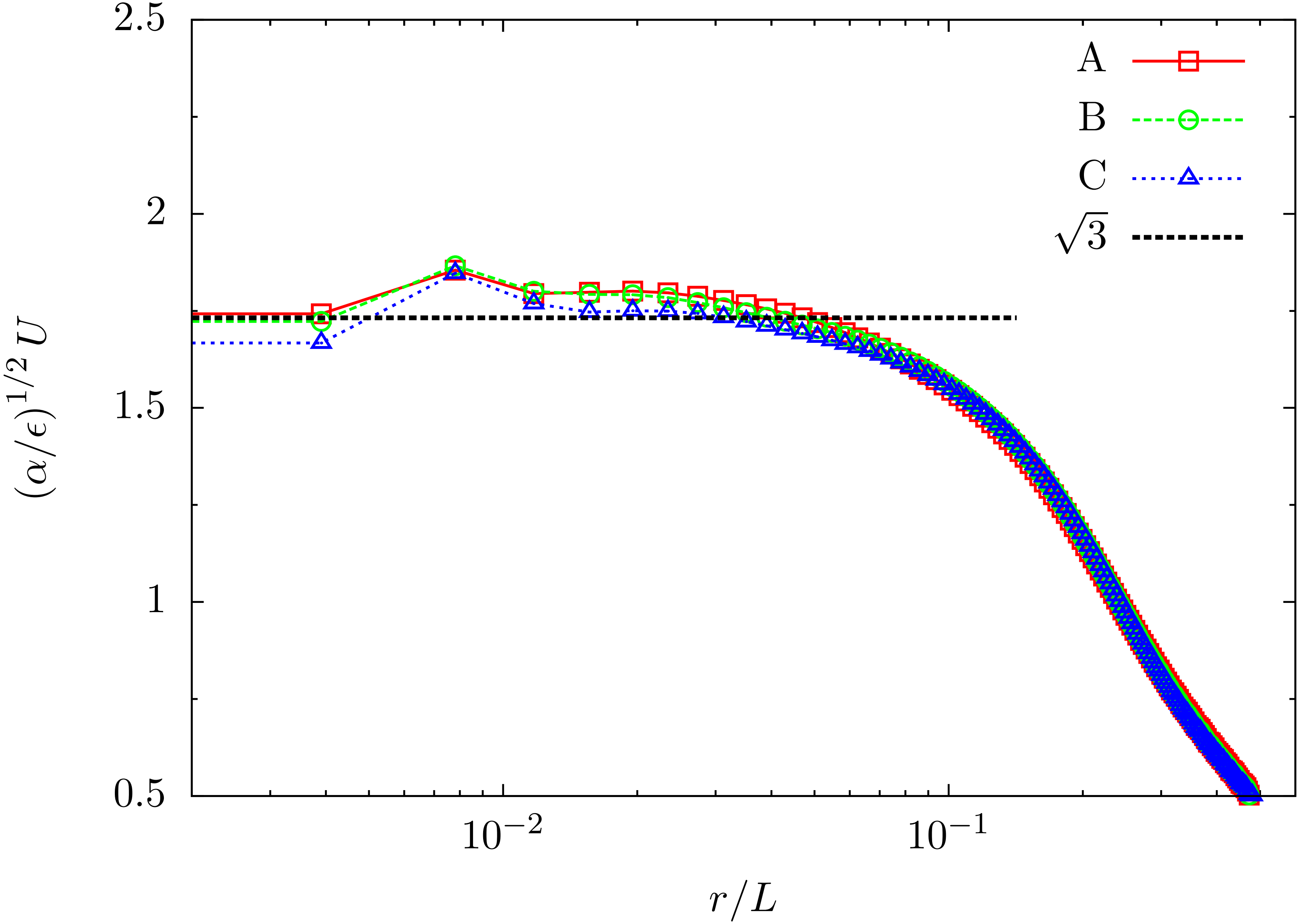
In the steady state one can find an average velocity profile. Both, experiment and numerics, show that the vortices are isotropic in average: the mean polar velocity U and the mean vorticity Ω are functions of the separation from the vortex center r. There is the hyperbolic region where the average velocity is estimated as $\sqrt{\epsilon/\alpha}$ and the average vorticity Ω is estimated as $\sqrt{\epsilon/\alpha}/L$.


0.5
0.25
0
-0.25
-0.5
y/L
-0.5
-0.25
0
0.25
0.5
x/L
1
0.5
0
-0.5
-1

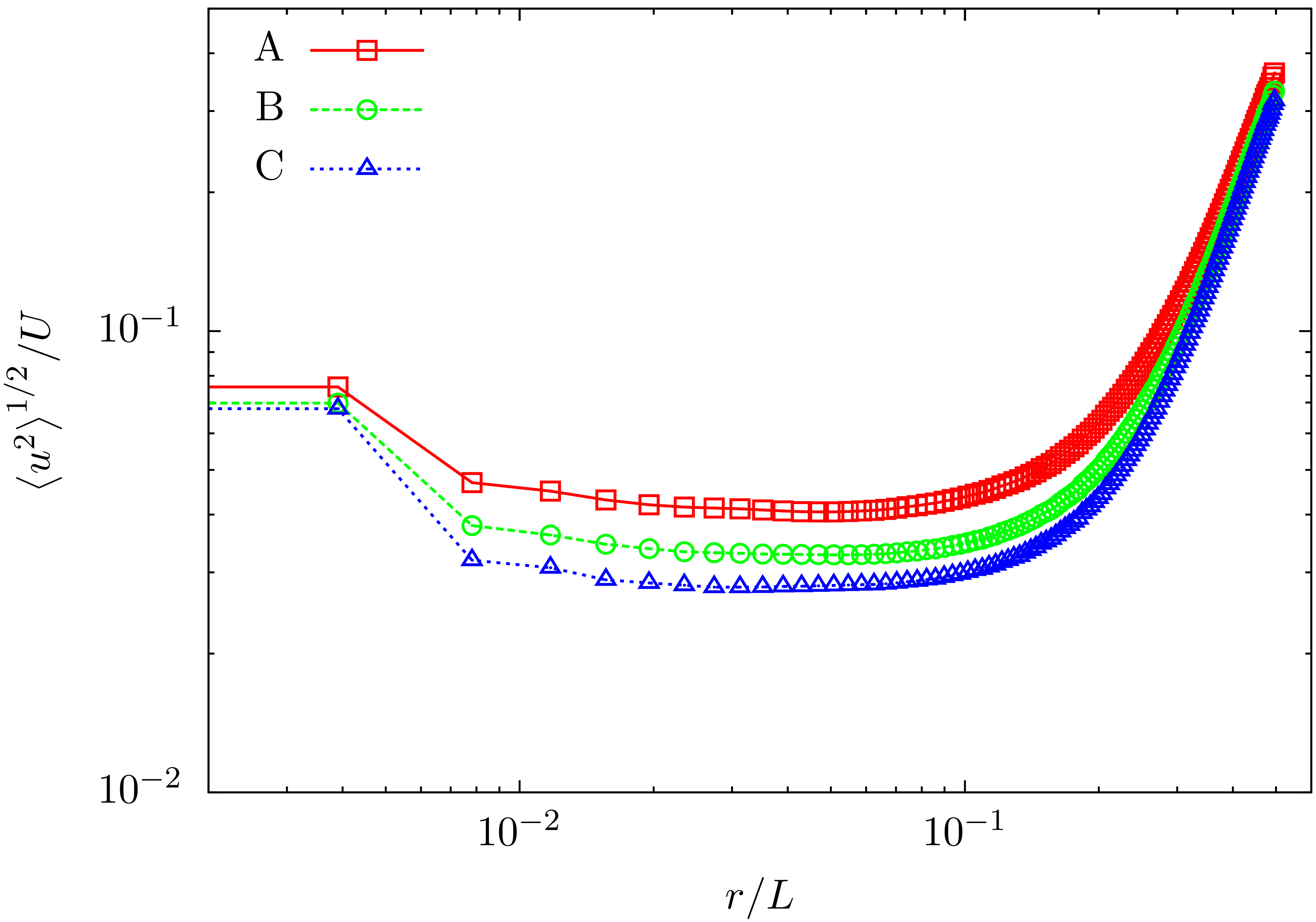
In numerics (Laurie at al. 2014) the vortex diameters are approximately $1/10$ of the box size. Our numerics gives the average flow profile inside the vortex. There exists the region of scales where the universal behavior $U = \mathrm{const}$ and $\Omega \propto r^{-1}$ is observed. In the vortex core (determined by viscosity) the average vorticity Ω is saturated and the average velocity U tends to zero.



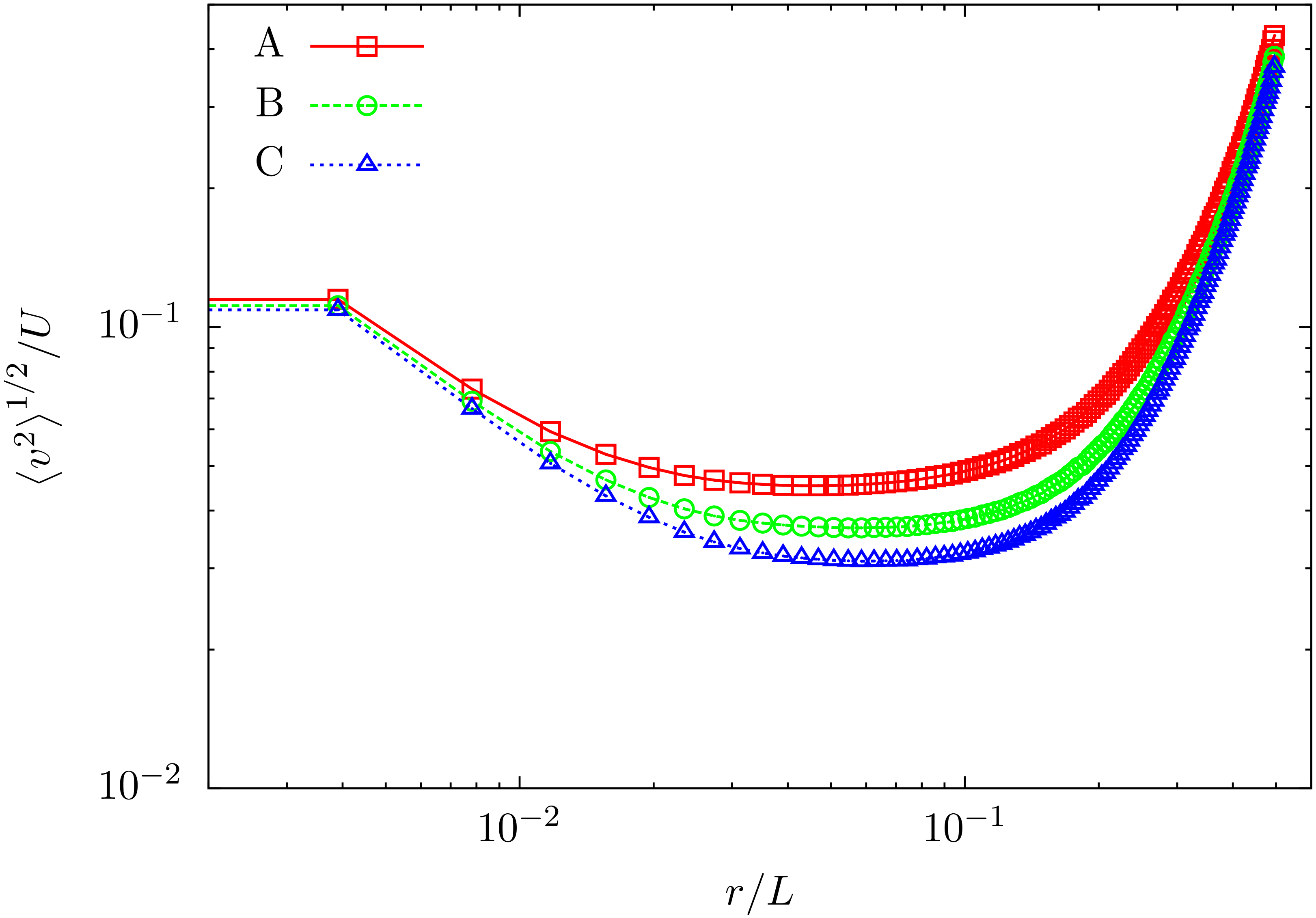
A
B
C
D
$\sqrt{3}\,(r/L)^{-1}$
$(\alpha L^2/\epsilon)^{1/2}\,\Omega$
10^3
10^2
10^1
10^0
10^{-3}
10^{-2}
10^{-1}
r/L


2.5
2
1.5
1
0.5
$(\alpha/\epsilon)^{1/2}\,U$
10^{-2}
10^{-1}
r/L
A
B
C
$\sqrt{3}$

In the universal region of the vortex $u, v \ll U$. It is a consequence of the large value of the mean velocity gradient $\sim U/r$, growing toward the center of the vortex. The relative strength of fluctuations increases as r grows and on the periphery where $r \sim L$, fluctuations become of the order of the average flow.



A
B
C
10^{-1}
10^{-2}
$\langle u^2 \rangle^{1/2}/U$
10^{-2}
10^{-1}
r/L


A
B
C
10^{-1}
10^{-2}
$\langle v^2 \rangle^{1/2}/U$
10^{-2}
10^{-1}
r/L

Smallness of fluctuations enables one to construct a consistent theory of the coherent vortex (Kolokolov and Lebedev 2016). It leads to the profile

$$U = \sqrt{3\epsilon/\alpha},$$

for pumping, short correlated in time. It is in accordance with the numerics. Now we examine other types of pumping.

There is a tendency in $2d$ turbulence of forming coherent structures: creation of order from chaos. Main features of the process are understood. However, there is a lot of questions: influence of the box geometry, of the pumping types, of inhomogeneity. If we are thinking about atmosphere, we should take into account Koriolis forces and landscape.

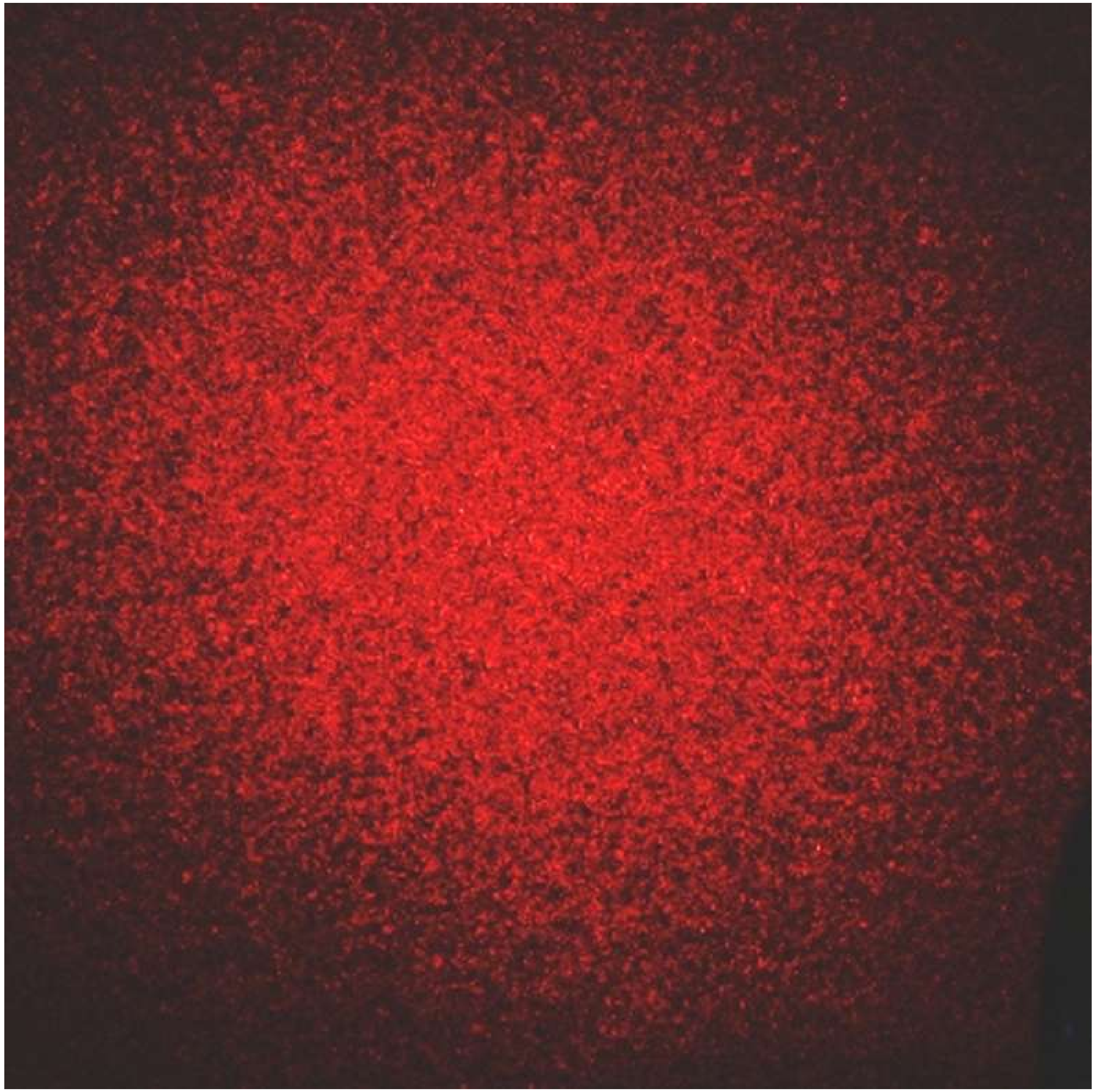
Rotating fluid. In the rotating $3d$ fluid the inertial waves propagate with the frequency $2\Omega\cos\theta$, where Ω – angular velocity and θ is the angle between the direction of the wave propagation and the rotation axis. Thus there is a $2d$ subsystem with zero frequency that behaves as a $2d$ fluid. Therefore one expects coherent structures.

One examines a laser beam propagating in turbulent atmosphere. There is the diffraction of the beam on fluctuations of the refractive index, induced by pressure fluctuations. They are a random field whose properties are described statistically. Therefore, theoretical predictions of the behavior of the laser beam concern mean values, which are obtained by averaging.

Due to the large value of the speed of light the propagation time of the laser beam is extremely small. Say for the distance 3 km it is near 10^{-5} s. It is smaller than all characteristic turbulent times. Therefore one can treat the turbulent state as static during the beam propagation. Besides, the refractive index varies rapidly along the beam trajectory.

The typical viscous length in the turbulent atmosphere is 10^{-4} m. We assume that the laser beam width is larger than the length. Then the refractive index varies essentially in the lateral plane inside the beam as well. Thus the refractive index fluctuations are functions of the lateral coordinates r and of the coordinate z in the direction of the beam propagation.

Diffraction of the laser beam on the refractive index fluctuations leads to its distortions. At large enough distances the beam is separated on speckles that are bright spots on the background of weaker intensity. The typical intensity of the electromagnetic field in a cross-section of the strongly distorted laser beam is presented in the figure.



The turbulent state of the atmosphere varies on times larger than the beam propagation time. The structure of the beam varies as well. We are interested in quantities averaged over times larger than the characteristic times of the atmosphere fluctuations. Such averaging reveals statistical properties of the propagation process, we are interested in.

The starting point of the theoretical analysis is the equation for the envelop Ψ of the electromagnetic field varying on distances much larger than the wavelength. The intensity of the electromagnetic field is $I = |\Psi|^2$. The complex field Ψ depends on $\boldsymbol{r}$ and z and tends to zero if we go away from the beam in the lateral direction, that is at large $\boldsymbol{r}$.

We assume that the beam intensity is small and neglect the non-linear effects. Then the equation for the envelop is the linear "Schrödinger" equation

$$i\partial_z\Psi + (\partial_x^2 + \partial_y^2)\Psi + \xi\Psi = 0,$$

for an appropriate choice of the units for the coordinates. Here the factor ξ represents the refractive index fluctuations.

The equation has to be supplemented by an initial condition Ψ_{in} (determined by the laser output) that we put to the point $z = 0$. Typically, one deals with a Gaussian profile of the envelop

$$\Psi_{in}(\boldsymbol{r}) \propto \exp\left(-r^2/l^2\right),$$

where l is the initial beam width. More complicated initial conditions are possible.

At solving the equation for the envelop Ψ the refractive index ξ is assumed to be dependent solely on r, z since the propagation time is small. However, the refractive index ξ chaotically varies as time goes on time scales much larger than the propagation time. As a consequence, the envelop Ψ becomes dependent on time as well.

We examine statistic of the intensity $I = |\Psi|^2$ at some distance from the source. The complete statistic of the intensity is determined by its moments $\langle I^n \rangle$, where angular brackets mean averaging over time. To get the moments one should find Ψ for a given realization of $\xi(\boldsymbol{r}, z)$ and then average I^n over realizations of $\xi(\boldsymbol{r}, z)$.

Solving the equation for $\Psi(\boldsymbol{r})$ by iterations on an interval $z_1 < z < z_2$, we get

$$\Delta\Psi(\boldsymbol{r}) = i(z_2 - z_1)(\partial_x^2 + \partial_y^2)\Psi(\boldsymbol{r}, z_1)$$
$$+ i\int_{z_1}^{z_2} dz\, \xi(\boldsymbol{r}, z)\Psi(\boldsymbol{r}, z_1)$$
$$- \int_{z_1}^{z_2} dz \int_{z_1}^{z} d\zeta\, \xi(\boldsymbol{r}, z)\xi(\boldsymbol{r}, \zeta)\Psi(\boldsymbol{r}, z_1).$$

Here we kept first contributions to the increment $\Delta\Psi$, assuming that it is small.

The main contribution to the refractive index fluctuations $\xi(\boldsymbol{r}, z)$ goes from the integral scale of turbulence. The fluctuations of such scales produce homogeneous phase shifts of $\boldsymbol{\Psi}$ and are, consequently, irrelevant, since the homogeneous phase shift does not contribute to the observable quantities like the intensity $I = |\Psi|^2$.

The fluctuations $\xi(\boldsymbol{r}, z)$ with the scale of the order of the beam width are relevant. The quantity enters the increment $\Delta\Psi$ via the integrals $\int dz\,\xi$. If the propagation distance $z_2 - z_1$ is much larger than the beam width, then the integral $\int dz\,\xi$ (a sum of big number of random variables) possesses a Gaussian statistic due to the Central limit theorem.

Since $\langle \xi \rangle = 0$, the statistic of $\int dz\, \xi$ is completely characterized by its pair correlation function. At an appropriate choice of units

$$\left\langle \left[\int dz (\xi(r_1, z) - \xi(r_2, z)) \right]^2 \right\rangle = 2(z_2 - z_1) r_{12}^c.$$

Here we have taken the difference of ξ in close points to exclude the irrelevant big homogeneous contribution to ξ and $r_{12} = |r_1 - r_2|$.

The average value is proportional to the first power of $z_2 - z_1$ since the characteristic length of ξ along the propagation direction is much smaller than $z_2 - z_1$. In the lateral direction the correlation function is power-like, it is explained by the scaling properties of fluctuations in the inertial interval of turbulence. For the Kolmogorov spectrum the exponent c is $c = 5/3$.

It is impossible to obtain a closed equation for the intensity $I = |\Psi|^2$. That is why one should first formulate and solve the equations for the correlation functions of the envelop Ψ. Then it is possible to extract the moments $\langle I^n \rangle$ by merging points in the correlation functions. Of course it is extremely hard program that cannot be realized up to the end.

Due to the strong fluctuations of the phase of Ψ the averages like $\langle \Psi \rangle$ are zero. Thus the simplest object is the pair correlation function

$$F(\boldsymbol{r}_1, \boldsymbol{r}_2, z) = \langle \Psi(\boldsymbol{r}_1, z) \Psi^{\star}(\boldsymbol{r}_2, z) \rangle,$$

where $\Psi^{\star}$ is complex conjugated to Ψ. The first moment of the intensity $\langle I \rangle$ is expressed as $\langle I \rangle = F(z, \boldsymbol{r}, \boldsymbol{r})$.

Taking the increment of $\Psi_1\Psi_2^{\star}$, averaging it and passing from the increment to the differential equation, one obtains

$$\partial_z F = i(\nabla_1^2 - \nabla_2^2)F - r_{12}^c F,$$

where $\nabla^2 = \partial_x^2 + \partial_y^2$. The first term describes the homogeneous diffraction whereas the second term describes the diffraction on the fluctuations of the refractive index.

The solution of the equation can be written as the integral over the initial profile

$$F(\boldsymbol{r}_1, \boldsymbol{r}_2, z) = \int d^2q_1\, d^2q_2 \mathcal{G} \Psi_{in}(\boldsymbol{q}_1) \Psi^{\star}_{in}(\boldsymbol{q}_2).$$

Here $\Psi_{in}(\boldsymbol{q}_1)\Psi^{\star}_{in}(\boldsymbol{q}_2)$ is the initial value of the pair correlation function F and $\mathcal{G}$ is the Green function. The representation is a consequence of the linearity of the equation for F.

It is possible to find the explicit expression for the Green function

$$\mathcal{G} = \frac{1}{16\pi^2 z^2} \exp\left[\frac{i}{2z}(\boldsymbol{r}-\boldsymbol{q})(\boldsymbol{R}-\boldsymbol{Q}) - z\int_0^1 d\chi \, |\chi\boldsymbol{q} + (1-\chi)\boldsymbol{r}|^c\right],$$

where $\boldsymbol{q} = \boldsymbol{q}_1 - \boldsymbol{q}_2$, $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, $\boldsymbol{Q} = (\boldsymbol{q}_1 + \boldsymbol{q}_2)/2$, $\boldsymbol{R} = (\boldsymbol{R}_1 + \boldsymbol{R}_2)/2$. Again, it reflects the interplay of two types of the diffraction.

The expression enables one to draw some conclusions concerning the beam structure at $z \gg 1$. If the initial size of the beam $l \lesssim 1$ then there are two characteristic length. The separation r is estimated as $z^{-1/c}$, it is the size of the speckles. The value of R is estimated as $z^{1+1/c}$, it is the beam width. Note that $r \ll R$ at $z \gg 1$.

Analogously to the pair correlation function, one can derive the equation for the forth-order correlation function

$$F_4(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4, z) = \langle \Psi_1 \Psi_2 \Psi_3^\star \Psi_4^\star \rangle.$$

To obtain the second moment of the intensity $\langle I^2 \rangle$, one should merge all the points in the forth-order correlation function, $\langle I^2 \rangle = F_4(\boldsymbol{r}, \boldsymbol{r}, \boldsymbol{r}, \boldsymbol{r}, z)$.

The equation is

$$\partial_z F_4 = i(\nabla_1^2 + \nabla_2^2 - \nabla_3^2 - \nabla_4^2)F_4$$
$$-[-r_{12}^c + r_{13}^c + r_{14}^c - r_{34}^c + r_{23}^c + r_{24}^c]F_4.$$

If $\boldsymbol{r}_1$ is close to $\boldsymbol{r}_3$ and $\boldsymbol{r}_2$ is close to $\boldsymbol{r}_4$, then, due to the cancellations the expression in square brackets is $r_{13}^c + r_{24}^c$. Thus $F_4 = F(\boldsymbol{r}_1, \boldsymbol{r}_3, z)F(\boldsymbol{r}_2, \boldsymbol{r}_4, z)$. Analogously for close points $1, 4$ and $2, 3$.

We conclude that in the main approximation the forth-order correlation function is

$$F_4 = F(\boldsymbol{r}_1, \boldsymbol{r}_3, z)F(\boldsymbol{r}_2, \boldsymbol{r}_4, z) + F(\boldsymbol{r}_1, \boldsymbol{r}_4, z)F(\boldsymbol{r}_2, \boldsymbol{r}_3, z).$$

The expression reflects an absence of correlations between the speckles. Merging the points $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, $\boldsymbol{r}_4$,, we find the relation $\langle I^2 \rangle = 2\langle I \rangle^2$.

Analogously the higher order correlation functions of the envelop $\langle \Psi\Psi \ldots \Psi^\star\Psi^\star \ldots \rangle$ can be analyzed. The $2n$-order correlation functions can be presented in the main approximation as the sum of $n!$ products of n pair correlation functions. Here $n!$ is the number of possible couplings. Merging the points, we conclude that $\langle I^n \rangle = n! \langle I \rangle^n$.

The moments correspond to the exponential probability density function (PDF)

$$P(I) = \langle I \rangle^{-1} \, \mathrm{exp} \, (-I/\langle I \rangle) \, .$$

One can say that Ψ possesses a Gaussian statistic at large z, converted to the exponential one for I. Thus we confirm the general expectation that the complex field with random phase possesses a Gaussian statistic.

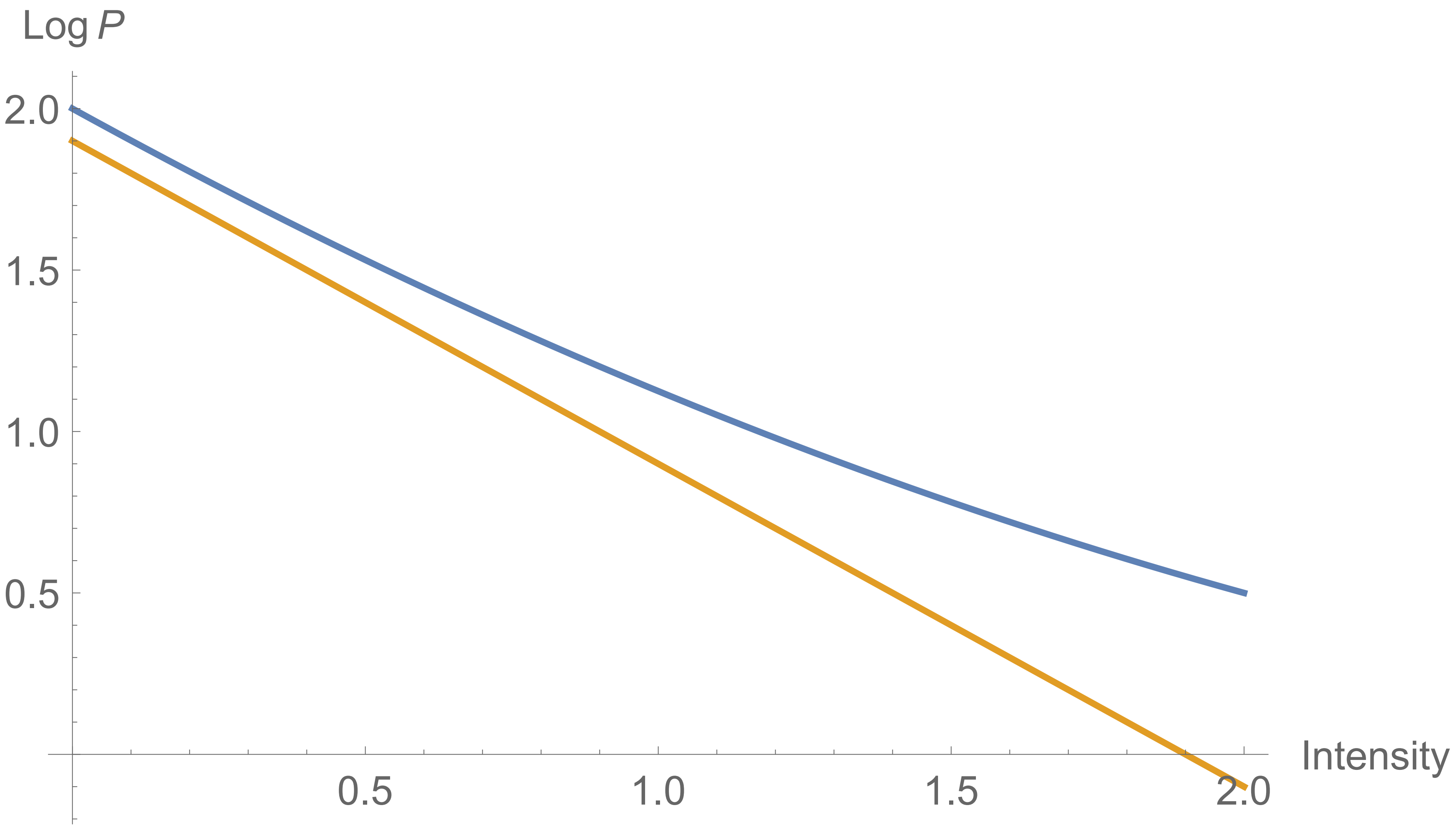
Due to the multiplicative character of the random field ξ one expects an anomalously large probability of rare events, determining large values of the intensity I. To find the probability one should go outside the established approximation. Formally, one should take into account the neglected terms in the equations for the high-order correlation functions.

Relative corrections to the exponential probability density function are controlled by the parameter $z^{-4/c+c}$ (V.U.Zavorotnyi, V.I.Klyatskin, V.I.Tatarskii, 1977), that is small at large z. The corrections are estimated as $z^{-4/c+c}I/\langle I\rangle$. When at increasing the intensity I the parameter becomes of order unity then the exponential PDF is broken.

The case $z^{-4/c+c}I \gg \langle I \rangle$ needs a special analysis (I.V.Kolokolov, V.V.Lebedev, 2023). The principal contribution to PDF is given by special configurations of the field ξ, leading to the stretched exponent

$$\ln P(I) \propto -I^{(4-c)(6-c)}.$$

The exponent here is equal to $7/13$ for the Kolmogorov spectrum.



Log P
2.0
1.5
1.0
0.5
Intensity
0.5
1.0
1.5
2.0

The result is illustrated by the figure where a deviation from the exponential PDF is presented. It means an essential increase of probability comparing to the exponential PDF for $I > z^{4/c-c}\langle I \rangle$. If to extrapolate the observation to $z \sim 1$ then one could expect anomalously high probabilities for I larger than typical. It is a subject of future investigations.