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Non-Born scattering effects in resistivity of quasi-one-dimensional systems

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Abstract

Quasi-one-dimensional systems demonstrate Van Hove singularities in the density of states ν_F and the resistivity ρ, occurring when the Fermi level E crosses a bottom E_N of some subband of transverse quantization. We demonstrate that the character of smearing of the singularities crucially depends on the concentration of impurities. There is a crossover concentration $n_c \propto |\lambda|$, $\lambda \ll 1$ being the dimensionless amplitude of scattering. For $n \gg n_c$ the singularities are simply rounded at $\varepsilon \equiv E - E_N \sim \tau^{-1}$ – the Born scattering rate. For $n \ll n_c$ the single-impurity non-Born effects in scattering become essential despite $\lambda \ll 1$. The peak of the resistivity is asymmetrically split in a Fano-resonance manner (however with a more complex structure). Namely, for $\varepsilon > 0$ there is a broad maximum at $\varepsilon \propto \lambda^2$ while for $\varepsilon < 0$ there is a deep minimum at $|\varepsilon| \propto n^2 \ll \lambda^2$. The behaviour of ρ below the minimum depends on the sign of λ. In case of repulsion ρ monotonically grows with $|\varepsilon|$ and saturates for $|\varepsilon| \gg \lambda^2$. In case of attraction ρ has sharp maximum at $|\varepsilon| \propto \lambda^2$. The latter feature is due to resonant scattering at quasistationary bound states that inevitably arise just below the bottom of each subband for any attracting impurity.

Contents

1 Introduction

In this study we consider clean multichannel quasi-one-dimensional metallic systems: wires, tubes, strips etc. We revisit a seemingly well-understood problem of semiclassical (i.e. without localization effects) resistivity for such systems in the presence of weak short-range impurities at low concentration. It is well known, that this resistivity (as well as the density of states at the Fermi level) has a square-root Van Hove singularities as a function of the Fermi level position E, occurring when E crosses a bottom E_N of certain subband of transverse quantization [1]. These singularities are expected to be smeared due to scattering of electrons by impurities and (at least in the Born approximation) the width of the peak $\Gamma_{\rm B} \sim \tau_{\rm min}^{-1}$ can be estimated as an electronic scattering rate at maximum of resistivity. This smearing was theoretically studied within the self-consistent Born approximation by different groups of authors [2, 3, 4, 5, 6, 7, 8].

We demonstrate that the above picture is valid only if the concentration of impurities is relatively high while for low concentration due to specifics of the quasi-one-dimensional systems the non-Born effects become essential despite the nominal weakness of scattering. These effects lead to dramatic restructuring of the Van Hove singularities.

Complex asymmetric features were experimentally observed in many quasi-one-dimensional systems, such as nanotubes (both single-wall [9, 10] and multi-wall [11, 12] ones). These features were attributed to Fano resonance [13], arising due to interference of the scattering at some narrow resonant state with the scattering at background continuum. The E-dependence of resistivity ρ at the Fano resonance is usually described by the formula

$$\rho(E) \propto \frac{(E - E_N + q\Gamma/2)^2}{(E - E_N)^2 + (\Gamma/2)^2} \tag{1.1}$$

with phenomenological parameters q and Γ (see, e.g., [14]). There were attempts [9, 10, 11] to fit the experimental data on the Van Hove singularities in nanotubes with the formula (1.1) with an appropriate choice of Γ and q. We will show, however, that this phenomenological expression is not sufficient to describe the entire zoo of possible $\rho(E)$ shapes. In this diploma work we will give the microscopic derivation of the actual $\rho(E)$ dependence. The main ingredient of our theory is the non-Born effects in scattering.

1.1 Statement of the problem

In this study we restrict our consideration to 2 simple realizations of quasi-one-dimensional system: a single-wall metallic tube in a strong longitudinal magnetic field and a conducting strip. The zero (or weak) field case for a tube is more difficult for theoretical study because of the chiral degeneracy of the electronic states that is only lifted due to an interaction with an impurity.

Besides the simplicity of the theoretical interpretation the case of strong magnetic field is convenient practically since the changing of magnetic field is an effective instrument for tuning the energy distance $E - E_N$, so it is easy to sweep the Van Hove singularity in a controllable way.

Oscillations of the longitudinal resistivity with the magnetic flux Φ threading the tube is a well known effect that was experimentally observed in various tubes and wires (especially semimetallic) [15, 16, 17]. These oscillations are the direct manifestation of the Aharonov-Bohm effect [18] – the interference of electronic waves with opposite chiralities. From the semiclassical point of view it is instructive to write the resistivity ρ in the form of Fourier series:

$$\rho = \rho_0 \left(1 + \sum_{n=1}^{\infty} A_n \cos(\pi n \Phi / \Phi_0) \right) \tag{1.2}$$

where $\Phi_0 = \pi c\hbar/e = ch/2e$ is the flux quantum. The oscillations can be observed in both dirty and clean systems. As it was shown in a seminal paper by Altshuler, Aronov and Spivak [19] in dirty (diffusive) systems the odd-n harmonics of the Aharonov-Bohm oscillations (1.2) are washed out due to strong variations in the length of different diffusive trajectories that lead to randomisation of the non-magnetic part of the phases of electronic wave-functions. The even-n harmonics – the oscillations associated with a special sort of trajectories (the ones containing closed topologically non-trivial loops on the cylinder) survive the randomisation. This effect was observed in experiments (see [20, 21]). The odd-n harmonics are in general very fragile: they may be suppressed also in nominally clean systems [17] due to the fluctuations of the tube's parameters: e.g., the radius [22]. The even-n harmonics are less fragile but still, in the presence of any kind of disorder the amplitudes A_n rapidly decrease with n so that the oscillations in the imperfect systems usually look roughly harmonic.

It is not the case for the geometrically perfect clean systems where A_n decrease with n only as $n^{-1/2}$ so that the series diverges at $\Phi \to 2M\Phi_0$ with integer M. This divergency is nothing else but the Van Hove singularity (see, e.g., [22]). So, the shape of oscillations is very different for perfect and for imperfect systems (see Fig.1).

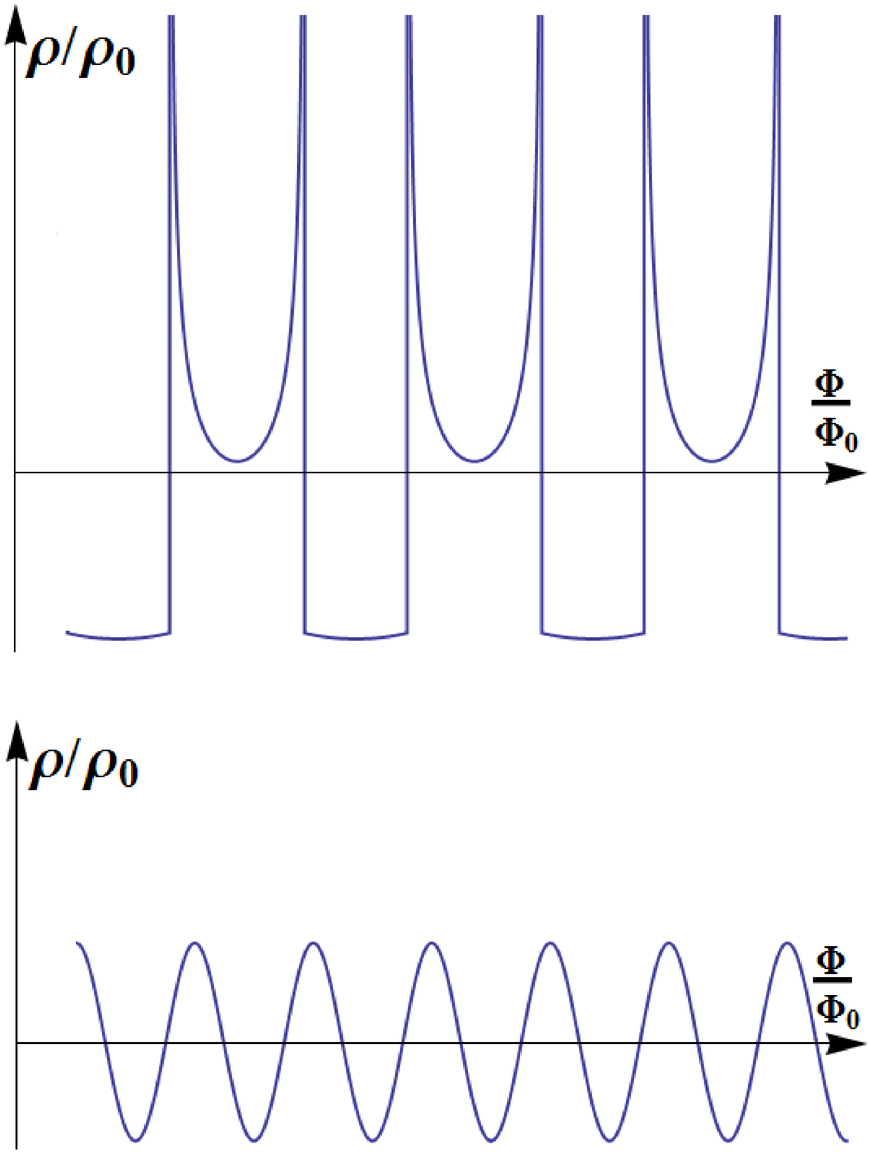

Figure 1. $\rho(\Phi)$ dependence for clean and dirty cases. Top: clean case, $\rho(\Phi)$ is periodic with a period $2\Phi_0$ and Van Hove square root singularities are present for $\Phi = 2n\Phi_0$. Bottom: dirty case, $\rho(\Phi)$ is periodic with a period Φ_0 - odd harmonics are suppressed.

It this work we concentrate on geometrically perfect tubes with low concentration of weak short-range impurities, where one can expect strongly unharmonic oscillations dominated by the Van Hove singularities as in the upper panel of Fig.1.

Thus, we consider a single-wall tube of radius R threaded by magnetic flux Φ and a strip of width D. Both the tube and the strip are supposed to be cut from a sheet of two-dimensional metal with simple quadratic spectrum [23] of electrons $E = \hbar^2\mathbf{k}^2/2m^*$. Impurities are embedded in this sheet with two-dimensional concentration n_2. They are supposed to be short-range and weakly scattering ones.

1.2 Principal approximations

It is convenient to measure all the energies in the units of $E_0^{(t)} = \hbar^2/2m^* R^2$ for a tube and $E_0^{(s)} = 2\pi^2\hbar^2/m^* D^2$ for a strip. We will assume the semiclassical condition throughout this paper:

$$\varepsilon_0 \gg 1, \quad \varepsilon_0 \equiv E/E_0, \quad N \sim \varepsilon_0^{1/2} \gg 1. \tag{1.3}$$

where N is the label of a subband whose bottom is the closest to the Fermi level and has the meaning of the number of open channels in the system.

The magnetic field (tube case) is assumed to be strong enough so that the splitting between E_N and E_{-N} is larger than the width of the peaks Γ. Besides that, the parameter $2\sqrt{\varepsilon_0}$ should not be close to any integer K to avoid resonance between the subbands with $m = N$ and $m' = N \pm 2\sqrt{\varepsilon_0}$.

All the interesting effects associated with the Van Hove singularities occur in the range where

$$|\varepsilon| \ll 1, \qquad \varepsilon \equiv (E - E_N)/E_0. \tag{1.4}$$

Besides the semiclassical parameter $N \gg 1$ there are two additional dimensionless small parameters in this problem:

(i) The dimensionless concentration of impurities

$$n \equiv \begin{cases} n_2(2\pi R)^2, & \text{tube} \\ n_2 D^2 & \text{strip} \end{cases} \qquad n \ll 1. \tag{1.5}$$

It is assumed to be small which in particular means that the average distance between impurities is larger than the transverse size of the system.

(ii) Dimensionless scattering amplitude

$$\Lambda_{2d} = \lambda - i\lambda^2 \tag{1.6}$$

of the background two-dimensional problem ($\lambda > 0$ corresponds to repulsion, $\lambda < 0$ – to attraction). It is also supposed to be small:

$$|\lambda| \ll 1. \tag{1.7}$$

The imaginary part of complex Λ_{2d} in (1.6) is necessary to fulfil the unitarity requirement [24] (the optical theorem):

$$\mathrm{Im}\,\Lambda_{2d} = -|\Lambda_{2d}|^2. \tag{1.8}$$

Actually we will need this imaginary part only for proper treatment of quasistationary states arising in the case of attracting $\lambda < 0$. In all other cases we can simply put $\Lambda_{2d} \to \lambda$.

There is also a condition imposed on the length L of the system: it should satisfy the inequality

$$l(\varepsilon) \ll L \ll \mathcal{L}_{\mathrm{loc}}(\varepsilon), \tag{1.9}$$

where $l(\varepsilon)$ is the mean free path and $\mathcal{L}_{\mathrm{loc}} \sim Nl(\varepsilon)$ is the localization length. The large parameter $N \gg 1$ assures at least the possibility for inequality (1.9) to be fulfilled.

Indeed, it is very well known that weak localization effects in quasi-one-dimensional systems lead (in the absence of inelastic processes) to an ultimate localization on all electronic states. However, for the tubes or strips with lengths L in the range (1.9) the localization corrections are still small so that the results obtained throughout this paper are well justified and should give a valid expressions for the resistivity $\rho(\varepsilon)$. Moreover, these results provide a possibility to estimate the dependence of the localization length on the parameter ε:

$$\mathcal{L}_{\mathrm{loc}}(\varepsilon) = \left[e^2\rho(\varepsilon)\right]^{-1}. \tag{1.10}$$

1.3 The structure of the diploma work

The structure of the diploma work is as follows:

In Section 2 we bring together all the principal results of the paper. In Section 3 we briefly remind the well-known facts about quantum mechanics of an electron on a tube threaded by magnetic field and on a strip. In Section 4 we discuss the scattering of electrons near the Van Hove singularity within the Born approximation. In Section 5 we discuss the non-Born effects for scattering of electrons in the tube and strip geometry and derive the corresponding renormalization of the scattering amplitude for both cases of tube and strip. In the case of attracting potential we find the poles in the scattering matrix that are related to quasistationary states under the bottom of each subband. We also consider the manifestations of the non-Born effects in resistivity, firstly in the single-impurity approximation. In particular we demonstrate that in this approximation the resistivity vanishes exactly at the Van Hove singularity. Then we estimate the effects of interference between scattering events at different impurities in the subsection for a tube. Taking into account these effects resolves the zero-resistivity paradox of the single-impurity approximation and gives estimation for the minimal resistivity. In Section 5.4 we discuss the inhomogeneous broadening of the peaks in the resistivity that arise due to resonant scattering at quasistationary states. In Section 6 we explore the effects that should arise if impurity with different effective scattering amplitudes are present in the system. In Section 7 we summarize the results and outline the direction of future research. In Appendices A and B we evaluate the behaviour of the system in the immediate vicinity of the Van Hove singularity (where the single-impurity approximation breaks down) using the self-consistent approximation. In Appendices C and D we evaluate some integrals that one encounters while analyzing the resistivity of a strip.

2 The principal results

The number of physical scenarios and distinct ranges of parameters considered in this paper is large. Therefore we find it reasonable to start with the list of different regimes and principal results.

2.1 The Born approximation

Away from the Van Hove singularities (at $|\varepsilon| \gg 1$) the applicability of the Born approximation requires only the condition (1.7). Here the system behaves simply as a classic piece of the background two-dimensional material. The density of states, the resistivity and the scattering rate (the latter is being measured in units of E_0) are

$$\nu_0 = \begin{cases} m^* R, & \text{tube} \\ \frac{m^* D}{2\pi}, & \text{strip} \end{cases}, \quad \rho_0 = \frac{1}{e^2 \varepsilon_0} \frac{1}{\tau_0}, \quad \frac{1}{\tau_0} = 2n \left(\frac{\lambda}{\pi} \right)^2 . \tag{2.1}$$

In all cases the main contribution to the current comes from the one-dimensional subbands with labels m that are not very close to N because for $m \approx N$ the longitudinal velocity of electrons with energy E tends to zero. However, the role of the N-band becomes very important near the singularity when $E \to E_N$. Indeed, when the total density of states

$$\nu(\varepsilon) = \nu_0 \left(1 + \frac{\theta(\varepsilon)}{\pi \sqrt{\varepsilon}} \right), \tag{2.2}$$

is dominated by the second term (the contribution of the resonant N-band), the electrons from the current-carrying bands (those with labels $m \sim N/2$) are scattered predominantly to the resonant one (with $m = N$). Near the singularity the properties of electrons in the resonant band differ from the properties of all others. For a general quasi-one-dimensional system there are in principle two distinct scattering amplitudes and corresponding rates: λ_{nonres} and $\tau^{-1}_{\text{nonres}}(\varepsilon)$ describe the scattering from the current-carrying bands to the resonant one while λ_{res} and $\tau^{-1}_{\text{res}}(\varepsilon)$ correspond to scattering within the resonant band. The rate $\tau^{-1}_{\text{nonres}}(\varepsilon)$ directly determines the mean free path and the resistivity

$$\rho(\varepsilon) = \frac{1}{e^2 \varepsilon_0} \frac{1}{\tau_{\text{nonres}}(\varepsilon)} = \frac{1}{e^2 N l(\varepsilon)}, \quad l(\varepsilon) = N \tau_{\text{nonres}}(\varepsilon), \tag{2.3}$$

where we have used the obvious relations $l \sim v_F \tau$ and $v_F \sim E^{1/2} \sim N$. The rate $\tau^{-1}_{\text{res}}(\varepsilon)$ is responsible for smearing of the singularity in the density of states and is relevant only in the immediate vicinity of the singularity. However, we demonstrate in Section 4 that for the case of a tube

$$\tau_{\rm res}(\varepsilon) = \tau_{\rm nonres}(\varepsilon) \equiv \tau. \tag{2.4}$$

However, $\tau_{\rm res}(\varepsilon) \neq \tau_{\rm nonres}(\varepsilon)$ for a strip.

We will show, that close to the singularity the Born approximation remains valid only if the dimensionless concentration n of impurities is relatively high.

Let us first assume that this condition is fulfilled and estimate the width of the smeared Van Hove singularity. The scattering rate is proportional to the density of final states so that

$$\frac{1}{\tau_{\rm nonres}(\varepsilon)} = \frac{1}{\tau_0}\frac{\nu(\varepsilon)}{\nu_0}. \tag{2.5}$$

The width $\Gamma_{\rm B}$ of the peak in the density of states (and in the resistivity at the same time) may be estimated from the condition

$$\tau^{-1}_{\rm nonres}\left(\varepsilon \sim \Gamma_{\rm B}\right) \sim \Gamma_{\rm B}. \tag{2.6}$$

Though $\tau^{-1}_{\rm nonres}(\varepsilon)$ is not necessarily equal to $\tau^{-1}_{\rm res}(\varepsilon)$, within the Born approximation they can only differ in numerical prefactor (see Section). As a result, the Van Hove singularity is smeared on the scale $|\varepsilon| \sim \Gamma_{\rm B}$

$$\Gamma_{\rm B} \sim \left(\frac{n}{\pi}\right)^{2/3}\left(\frac{\lambda}{\pi}\right)^{4/3} \gg \frac{1}{\tau_0}, \tag{2.7}$$

$$\rho_{\rm B}^{\rm max} \sim \frac{1}{e^2\varepsilon_0}\left(\frac{n}{\pi}\right)^{2/3}\left(\frac{\lambda}{\pi}\right)^{4/3} \gg \rho_0. \tag{2.8}$$

2.2 The origin of non-Born effects

The origin of the special importance of non-Born effects in quasi-one-dimensional systems is renormalization of the scattering matrix that is dramatically enhanced near a Van Hove singularity.

2.2.1 The case of tube

In the case of tube this matrix can be effectively reduced to a single complex constant $\Lambda(\varepsilon)$ that can be found from the Dyson equation. As a result

$$\lambda \to \Lambda(\varepsilon) \approx \lambda \left\{ 1 - \frac{\Lambda_{2d}}{\pi\sqrt{\varepsilon}} \right\}^{-1} \tag{2.9}$$

From (2.9) it is clear that the energy scale

$$\varepsilon_{\rm nB} = (\lambda/\pi)^2 \ll 1, \tag{2.10}$$

measures the range near the singularity where the non-Born effects are considerable. In particular, we see that if, due to low concentration of impurities, the Born scattering rate is low enough:

$$\Gamma_{\rm B} < \varepsilon_{\rm nB}, \tag{2.11}$$

then the non-Born effects have chance to come into play in the range $\Gamma < |\varepsilon| < \varepsilon_{\rm nB}$. Substituting the explicit formulas (2.7) and (2.10) to the condition (2.11), we arrive at the criterion

$$n < n_c, \qquad n_c = |\lambda|/\pi. \tag{2.12}$$

of the breakdown of the Born approximation in the vicinity of the singularity. Under the opposite condition the Born approximation is sufficient for all ε.

It is convenient to rewrite (2.9) in the form

$$\lambda \to \Lambda(\epsilon) \approx \frac{\lambda}{1 - [\mathrm{sign}(\lambda) - i|\lambda|](-\varepsilon/\varepsilon_{\rm nB})^{-1/2}}. \tag{2.13}$$

2.2.2 The case of strip

A similar renormalization is present also in the case of strip. However, in this case there are some nice additional effects, absent for a tube. Namely, the renormalization of the scattering amplitude now depends on the position of impurity with respect to wave function nodes.

The bare scattering amplitude reads

$$\Lambda_i = 2\Lambda \sin^2(\pi N \xi_i). \tag{2.14}$$

Here ξ_i is the position of impurity (in units of D). From (2.14) we see that we can distinguish between strong impurities ($\sin^2(\pi N\xi_i) \approx 1$), weak ($\sin^2(\pi N\xi_i) \approx 0$) and typical ones ($\sin^2(\pi N\xi_i) \sim 1$). Renormalization of Λ_i reads:

$$\Lambda_i^{(\mathrm{ren})} = \frac{\Lambda_i}{1 + \Lambda_i/\pi\sqrt{-\varepsilon}}. \tag{2.15}$$

We see from (2.15) that the renormalization of Λ_i is nonlinear - namely, it is stronger for strong impurities. Thus, paradoxically, for $\varepsilon \ll \varepsilon_{\mathrm{nB}}$ scattering predominantly occurs not at strong impurities, but at some 'optimal' ones:

$$\Lambda_i \sim \sqrt{|\varepsilon|}.$$

2.3 The non-Born effects in resistivity: repulsing impurities

At low concentration of impurities $n \ll n_c$ the shape of the $\rho(\varepsilon)$ dependence in the vicinity of Van Hove singularities is strongly modified by non-Born effects in scattering.

A narrow peak at $\varepsilon = 0$ is replaced by a broad one slightly above the bottom – with the maximum at $\varepsilon \sim \varepsilon_{\mathrm{nB}}$ and the width $\Gamma_{\mathrm{nB}}^{(+)} \sim \varepsilon_{\mathrm{nB}}$, independent of the concentration n. The shape of this broad peak can be found explicitly:

$$\frac{1}{\tau(\varepsilon)} = 2\left(\frac{n}{\pi}\right)\left(\frac{\lambda}{\pi}\right) F(\epsilon), \quad \epsilon \equiv \varepsilon/\varepsilon_{\mathrm{nB}}, \tag{2.16}$$

$$F(\epsilon) = \begin{cases} (\epsilon^{1/2} + \epsilon^{-1/2})^{-1}, & \text{tube} \\ \left(\frac{\sqrt{1+4/\epsilon}-1}{2(1+4/\epsilon)}\right)^{1/2}, & \text{strip} \end{cases} \tag{2.17}$$

The maximal (in the range $\varepsilon > 0$) resistivity

$$\rho_{\mathrm{nB}}^{\max(+)} \sim \frac{2}{e^2\varepsilon_0}\left(\frac{n}{\pi}\right)\left(\frac{\lambda}{\pi}\right) F_{\max} \ll \rho_{\mathrm{B}}^{\max}, \tag{2.18}$$

is reached at $\epsilon = \epsilon_{\max}$, where

$$F_{\max} = \begin{cases} 1/2, \\ 1/2\sqrt{2}, \end{cases} \qquad \epsilon_{\max} = \begin{cases} 1, & \text{tube} \\ 4/3, & \text{strip} \end{cases} \tag{2.19}$$

Both variants of function $F(\epsilon)$ are shown in Fig. 2. At $\epsilon \gg 1$ it has asymptotics $F(\epsilon) \approx \epsilon^{-1/2}$ in both cases. It corresponds to the standard Van Hove singularity. The height of the broad peak is much less than it would be within the Born approximation but still is much higher than the background resistivity ρ_0.

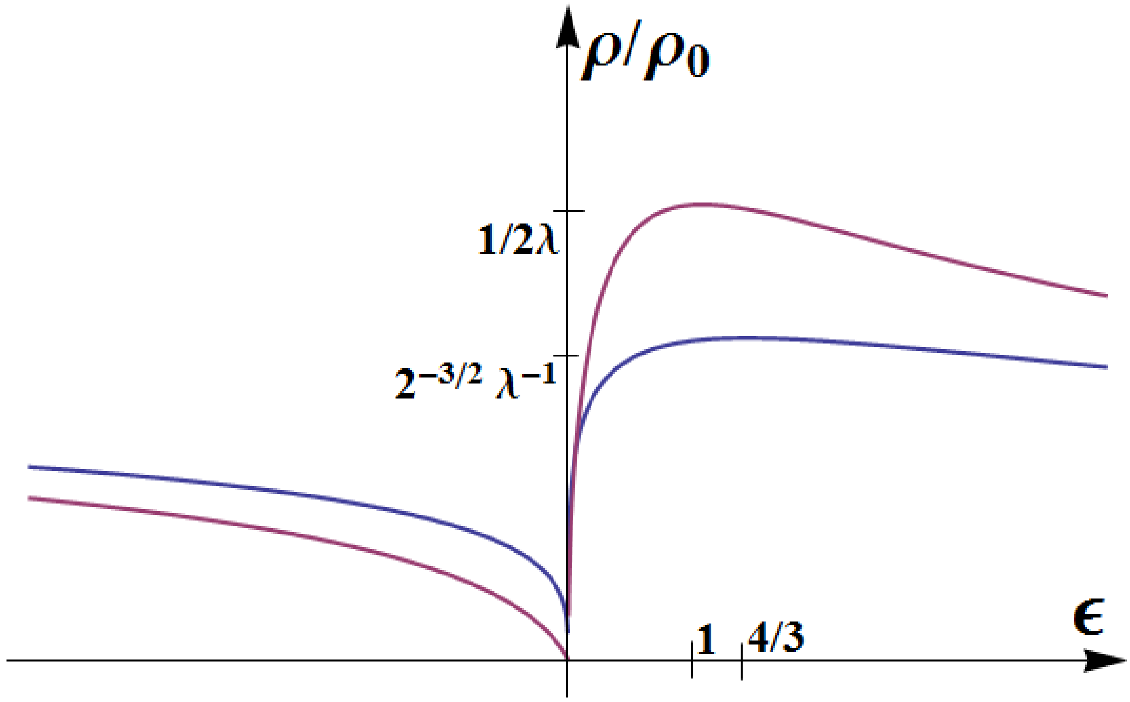

Figure 2. Dependence of the resistivity on the position of the Fermi level for repulsing impurities in the case of low concentration of impurities $n \ll n_c$ (strongly non-Born regime). Red line stands for tube case, blue line – strip case. Note that $\rho(\epsilon)$ vanishes as $\epsilon \to 0$: it is an artefact of the single-impurity approximation that is not applicable in the narrow vicinity of $\epsilon = 0$: for $\epsilon \lesssim \epsilon_{\min} \ll 1$. The horizontal asymptote (dashed line) corresponds to $\rho = \rho_0$.

The behaviour of the resistivity above the Van Hove singularity (for $\varepsilon > 0$), described by (2.16), does not depend on the sign of λ, it is the same for attracting and repulsing impurities. It is not the case for the range $\varepsilon < 0$ below the singularity. For repulsing impurities we obtain

$$\frac{1}{\tau(\varepsilon)} = 2\pi \left(\frac{n}{\pi}\right) \left(\frac{\lambda}{\pi}\right)^2 \tilde{F}(\epsilon), \tag{2.20}$$

$$\tilde{F}(\epsilon) = \begin{cases} [1 + |\epsilon|^{-1/2}]^{-2}, & \epsilon < 0, \quad \text{tube} \\ \frac{|\epsilon|^{1/4}(1+|\epsilon|^{1/2})}{(2+|\epsilon|^{1/2})^{3/2}}, & \epsilon < 0, \quad \text{strip} \end{cases} \tag{2.21}$$

so that $\rho(\varepsilon)$ monotonically increases with $|\varepsilon|$ and saturates at $\rho = \rho_0$.

It is easy to see that, as it formally follows from (2.21), the resistivity $\rho(\varepsilon)$ should vanish for $\varepsilon \to 0$ from either side. Indeed, for $|\epsilon| \ll 1$

$$F(\epsilon) \approx \begin{cases} \epsilon^{1/2}, \\ \frac{\epsilon^{1/4}}{2}, \end{cases} \qquad \tilde{F}(\epsilon) \approx \begin{cases} |\epsilon|, & \text{tube} \\ (|\epsilon|/4)^{3/4}, & \text{strip.} \end{cases} \tag{2.22}$$

Of course we immediately suspect that in reality the decrease of resistivity will be ultimately stopped by some additional effect (and this is indeed so, as we will see). But anyway, a dramatic suppression of resistivity in the narrow vicinity of the Van Hove point is an important phenomenon. Physically it is a result of destructive interference of partial electronic waves with different winding numbers.

2.4 Attracting impurities, quasistationary states and resonant scattering

As we have already mentioned in previous section, the behaviour of resistivity above the singularity is identical for repulsing and attracting impurities. However, below the singularity the attracting impurities introduce some nice additional physics. We will first discuss it for the simpler case of cylinder.

2.4.1 Quasistationary states on a tube

It can be shown that, besides the true bound state with the energy below the bottom of the lowest subband of the electronic spectrum of the cylinder, a weakly attracting short-range impurity produces an infinite series of quasistationary states: one such state below the bottom of each band. In this paper we concentrate on the quasistationary states associated with the quasiclassic subbands (those, with large $N \gg 1$). In particular we show that for $\lambda < 0$ the scattering amplitude (2.13) has a pole at $\epsilon = -1 + 2i|\lambda|$ (or at $\varepsilon = (-1 + 2i|\lambda|)\varepsilon_{\text{nB}}$ in other notation). This pole corresponds to a quasistationary state with a relatively small decay rate. In the case of cylinder these poles are identical for all impurities and, since electrons can be scattered by these resonances, the latter lead to formation of sharp maxima in resistivity for $\varepsilon < 0$ and $\lambda < 0$:

$$\tilde{F}(\epsilon) \approx \begin{cases} \dfrac{1}{(1-|\epsilon|^{-1/2})^2}, & \text{for } |1-|\epsilon|| \gg |\lambda|, \\ \dfrac{4}{(1-|\epsilon|)^2 + 4\lambda^2}, & \text{for } |1-|\epsilon|| \lesssim |\lambda|, \end{cases} \tag{2.23}$$

This result is illustrated by Fig. 3:

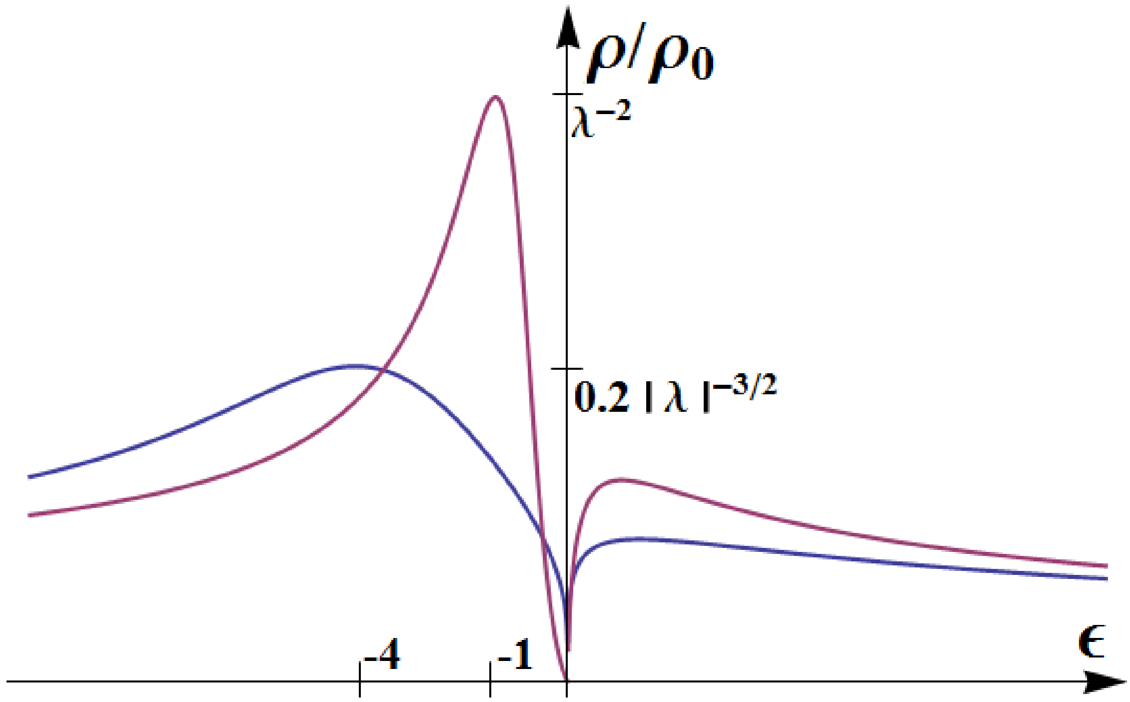

Figure 3. The same as in Fig. 2 but for attracting impurities. The sharp maximum at $\epsilon < 0$ arises due to resonant scattering at quasistationary states.

The maximal (in the range $\epsilon < 0$) resistivity is reached at $\epsilon = -1$,

$$\rho_{\rm nB}^{\max(-)} \sim \frac{1}{e^2 \varepsilon_0} \frac{2n}{\pi^2}, \tag{2.24}$$

the width of this maximum is $\Gamma_{\rm nB}^{(-)} = 4|\lambda|\varepsilon_{\rm nB}$.

The physical origin of the quasistationary states that exist slightly below each of the subbands is as follows. Semiclassical trajectories of electrons with energies near the bottom of subband are almost closed; if an electron with such energy has passed near certain impurity once then it will do so again, and many times. Therefore the attraction to impurity is strongly enhanced and the bound state is formed. An alternative way of thinking is just to neglect in the leading approximation all the transitions from the resonant band to all others. The arising strictly one-dimensional problem grants a bound state for arbitrary weak attraction. Taking the transitions to nonresonant bands into account perturbatively leads to the finite decay rate of the state.

2.4.2 The quasistationary states on a strip

In the case of strip there are also quasistationary states that are manifested in the pole of renormalized scattering amplitude (2.15): $\epsilon(\xi_i) = 4\sin^4(\pi N\xi_i)(-1+2i\lambda)$. We see that in this case energy of quasistationary state is different at different impurity and its real part is confined in the range $-4 < \epsilon < 0$. Due to uniform distribution of ξ_i between 0 and 1 we can define the following distribution function for ϵ_{qs}:

$$P(\varepsilon_{\text{qs}}) = \int_0^1 d\xi\delta\left[\epsilon_{\text{qs}} + 4\sin^4(\pi N\xi)\right] = \\ = \frac{1}{\pi\sqrt{|\epsilon_{\text{qs}}|(4-|\epsilon_{\text{qs}}|)}}. \tag{2.25}$$

So, we see that for strip case the peak of resistivity is inhomogeneously broadened compared to the case of tube (where all ε_{qs} are identical). Moreover, for $\epsilon < -4$ there are no quasistationary states and as a final states of scattering processes may serve only states of continuos spectrum while for $-4 < \epsilon < 0$, in principle, both kinds of states are likely to do so. However, in this range quasistationary states are dominant everywhere except narrow region $||\epsilon| - 4| \lesssim |\lambda|$. Expression for $\rho(\epsilon)$ reads:

$$\frac{\rho(\epsilon)}{\rho_0} = \begin{cases} 8\sqrt{2}\left(|\epsilon|-4\right)^{-3/2}, & \text{for } 4+\epsilon \to -0, \\ \dfrac{2\sqrt{2}}{|\lambda|}(4-|\epsilon|)^{-1/2}, & \text{for } 4+\epsilon \to +0. \end{cases} \tag{2.26}$$

Thus, at $\epsilon = -4$ the resistivity has an asymmetric peak. If we recall (2.25), we can see that this Van Hove-like(formally divergent) peak is nothing else but the divergency of $P(\epsilon)$ at $\epsilon \to -4$. Here we also note that the main contribution to resistivity in the vicinity of the maximum is given by strong impurities, in the vicinity of $\epsilon = 0$ - by weak ones.

The Van Hove-like singularity at $|\epsilon| \to 4$ is indeed smeared in the range $||\epsilon| - 4| \lesssim |\lambda|$, where the contribution of both types of final states – the continuum and the quasistationary states

– are comparable. As a result of calculations we obtain:

$$\frac{\rho(\epsilon)}{\rho_0} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{a^2+1}-a}{|\lambda|^3(a^2+1)}\right)^{1/2} \approx$$

$$\approx \begin{cases} \dfrac{8\sqrt{2}}{(|\epsilon|-4)^{3/2}}, & \text{for } 8|\lambda| \ll |\epsilon|-4 \ll 1, \\ \dfrac{2\sqrt{2}}{|\lambda|(4-|\epsilon|)^{1/2}}, & \text{for } 8|\lambda| \ll 4-|\epsilon| \ll 1, \end{cases} \tag{2.27}$$

where $a = (|\epsilon|-4)/8|\lambda|$.

From (2.27) we conclude that the maximal resistivity

$$\frac{\rho^{(-)}_{\max}}{\rho_0} = \frac{3^{3/4}}{2\sqrt{2}|\lambda|^{3/2}} \tag{2.28}$$

is attained at $\epsilon = -4\left(1 - \frac{2|\lambda|}{\sqrt{3}}\right)$. The width of this maximum $\Gamma \sim |\lambda| \ll 1$.

Thus, we conclude that the left peak of resistivity (that exists only for attracting impurities) is higher than the right one: its height is proportional to $|\lambda|^{-3/2}$ instead of $|\lambda|^{-1}$. On the other hand, due to the inhomogeneous broadening, it is lower than it would be in the case of cylinder: $|\lambda|^{-3/2}$ instead of $|\lambda|^{-2}$.

The divergency of $P(\epsilon)$ at $\epsilon \to 0$ does not lead to divergency of $\rho(\epsilon)$ at $\epsilon \to 0$: $\rho(\epsilon)$ still goes to zero but for all energies is much larger than in the case of repulsing impurities. The strong scattering at quasistationary states with low binding energies gives additional large factor $|\lambda|^{-1}$ in $\rho(\epsilon)$ dependence at $\epsilon < 0$, $|\epsilon| \ll 1$:

$$\frac{\rho(\epsilon)}{\rho_0} \approx \frac{|\epsilon|^{1/4}}{\sqrt{2}} \begin{cases} 1/|\lambda|, & \text{for } \lambda < 0, \\ 1/2, & \text{for } \lambda > 0. \end{cases} \tag{2.29}$$

while for tube we have

$$\frac{\rho(\epsilon)}{\rho_0} \approx |\epsilon| \tag{2.30}$$

for both signs of λ.

2.5 The minimum of resistivity (tube)

All the effects described above are the single impurity ones. Their origin is the coherent multiple scattering of an electron by the same impurity which fact is manifested in the linear dependence of resistivity on the concentration n. To reveal the mechanism that limits the suppression of resistivity at $\epsilon \to 0$ and to estimate the resistivity at its minimum one has to find the scattering rate $\tau_{\rm res}^{-1}(\varepsilon)$ in the range $|\varepsilon| \ll \varepsilon_{\rm nB}$:

$$\frac{\tau_0}{\tau_{\rm res}(\varepsilon)} = |\varepsilon| \left(1 + \frac{\theta(\varepsilon)}{|\lambda|\sqrt{\varepsilon}}\right) \tag{2.31}$$

The characteristic width $\Gamma_{\rm nB}$ of the feature (namely, the minimum) in the density of states near $\varepsilon = 0$ can be estimated from the condition

$$\Gamma_{\rm nB} \sim \tau_{\rm res}^{-1} \left(\varepsilon \sim +\Gamma_{\rm nB}\right), \tag{2.32}$$

and we get

$$\Gamma_{\rm nB} \sim \varepsilon_{\rm min} \equiv (n/\pi)^2 \ll \varepsilon_{\rm nB} \tag{2.33}$$

for both cases of tube and strip. Note that this width does not depend on λ. At $\varepsilon < 0$ the resonant contribution to the density of states rapidly drops on the same energy scale so that the factor $\nu(\varepsilon)$ becomes of order of ν_0 already at $\varepsilon \sim -\varepsilon_{\rm min}$. As a result, the resistivity has a minimum at $\varepsilon = \varepsilon_{\rm dip}$, where

$$\varepsilon_{\rm dip} < 0, \quad |\varepsilon_{\rm dip}| \sim \varepsilon_{\rm min} \sim (n/\pi)^2. \tag{2.34}$$

The scattering rate $\tau_{\rm nonres}^{-1}$ and the resistivity at minimum are

$$\frac{1}{\tau_{\rm dip}} \sim n^3, \quad \rho_{\rm dip} \sim \frac{n^3}{e^2 \varepsilon_0}, \tag{2.35}$$

and do not depend on the scattering amplitude λ. Thus, there is a deep and narrow minimum of resistivity slightly below the bare Van Hove singularity, the resistivity in the minimum depends on n superlinearly.

3 Ideal system

3.1 Tube case

We consider a tube of radius R threaded by a magnetic flux $\Phi = \pi R^2 H$ (the magnetic field H is oriented along the axis of a cylinder z).

Figure 4. Thin conducting tube, threaded by magnetic field H. Impurities (shown as stars) are embedded in the tube. Electrons live on the surface of the cylinder.

Electrons in the tube have the following spectrum and wave functions:

$$\psi_{mk}(\phi, z) = (2\pi)^{-1/2} \exp\{ikz + im\phi\}, \tag{3.1}$$

$$E_{mk} = \frac{\hbar^2 k^2}{2m^*} + E_m, \tag{3.2}$$

$$E_m = E_0(m + \Phi/2\Phi_0)^2, \quad E_0 = \frac{\hbar^2}{2m^* R^2} \tag{3.3}$$

where $m \in Z$ is the azymuthal quantum number, k is the momentum along the cylinders axis and $\Phi_0 = \pi c\hbar/e = ch/2e$ is the flux quantum. E_m has the meaning of position of the bottom of m-th one-dimensional subband. Actually we have introduced the magnetic field as a tool of easy shifting of the Fermi level in the system but all the physics described below is present already in the case $H = 0$.

The density of states in each subband

$$\nu_m(E) = \int \frac{dk}{2\pi} \delta\left(E - E_m - \frac{k^2}{2m^*}\right) = \\ = \frac{2}{2\pi} \sqrt{\frac{m^*}{2(E - E_m)}} \theta(E - E_m), \tag{3.4}$$

The factor 2 arises because the equation $E - E_m - \frac{k^2}{2m^*} = 0$ has two roots $k = \pm\sqrt{2m^*(E - E_m)}$.

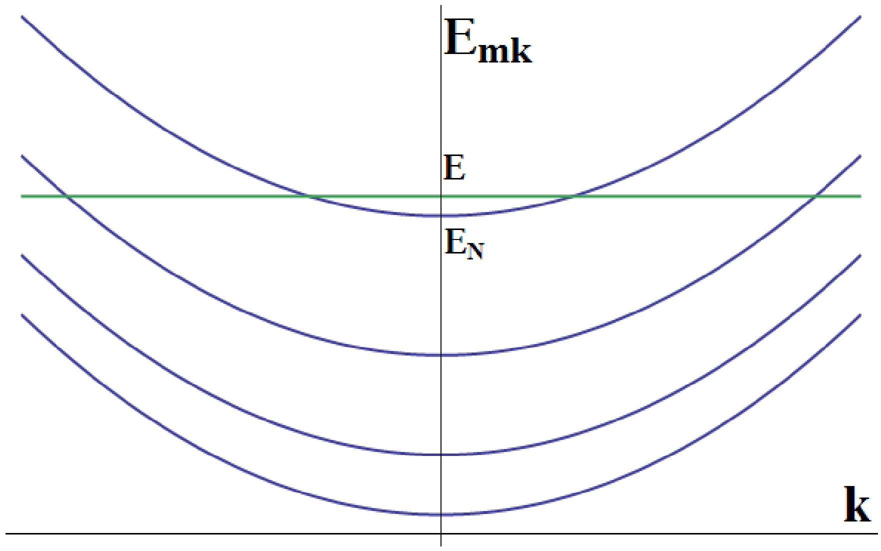

Figure 5. Spectrum of an electron on a surface of an ideal cylinder. Subbands of the transverse quantization are shown. The Fermi level E crosses all the subbands with $m \leq N$.

The total density of states

$$\nu(E) = \sum_m \nu_m(E) = -\frac{1}{\pi}\mathrm{Im}\, g(E), \tag{3.5}$$

$$g(E) \equiv G_E^{(0)}(0,0) = \sum_m \int \frac{dk}{2\pi}\frac{1}{E - E_{km} + i0} =$$
$$= \sum_m \sqrt{\frac{m^*}{2(E_m - E)}}, \tag{3.6}$$

$G_E^{(0)}(0,0)$ being the one-point retarded Green function of an ideal tube. Strictly speaking, the real part of g diverges. The recipe how to deal with this divergency will be discussed somewhat later. Now we just mention that the divergent part is energy-independent and therefore can be removed by a constant shift of the energy.

In the main part of this paper we will measure all energies in the units of E_0 and all distances in the units of $2\pi R$:

$$E \equiv E_0\varepsilon_0, \quad E - E_m \equiv E_0\varepsilon_m, \quad \nu_m(E) \equiv \frac{\nu_m(\varepsilon)}{2\pi R E_0}, \tag{3.7}$$

$$\nu_m(\varepsilon) = \frac{1}{\sqrt{\varepsilon_m}}\theta(\varepsilon_m), \quad g(\varepsilon) = \sum_m \frac{\pi}{\sqrt{-\varepsilon_m}}. \tag{3.8}$$

We are interested in semiclassical case when $E_0 \ll E$ or $\varepsilon_0 \gg 1$. Then, in the leading semiclassical approximation

$$\nu(\varepsilon) = \sum_{m=-\infty}^{\infty} \nu_m(\varepsilon) \approx \nu_0 = \int_0^{\varepsilon_0} \frac{d\varepsilon_m}{\sqrt{\varepsilon_m(\varepsilon_0 - \varepsilon_m)}} = \pi. \tag{3.9}$$

This result is valid for all ε except narrow intervals in the vicinity of points where $\varepsilon_m = 0$ for some m.

The condition of strong magnetic field reads

$$\varepsilon_N - \varepsilon_{-N} \sim N\Phi/\Phi_0 \gg \Gamma, \tag{3.10}$$

where Γ is the broadening of peaks and $N = \sqrt{\varepsilon_0}$ denotes the closest to Fermi level E subband.

In the entire range of variation of ϵ one can write

$$\nu^{(0)}(\varepsilon) \approx \nu_0 \left(1 + \frac{\theta(\varepsilon)}{\pi\sqrt{\varepsilon}}\right), \tag{3.11}$$

where we have introduced $\varepsilon \equiv \varepsilon_N$ for brevity.

Under the semiclassical condition $N \gg 1$ the result (3.9) is not valid in the vicinity of the Van Hove singularity (for $\varepsilon \lesssim 1$) where the second – resonant – term in (3.22) is anomalously large. We see that for

$$\varepsilon > 0, \quad \varepsilon \ll 1 \tag{3.12}$$

the inequality $\nu_N(\varepsilon) \gg \nu_0$ holds: the density of states is indeed dominated by the second term in (3.22) – the contribution of the N-subband. Note that in the semiclassical limit $N \gg 1$ the different peaks in the function $\nu(\varepsilon)$ are strictly identical.

3.2 Case of strip

The eigenfunctions and eigenenergies of electrons in an ideal strip are

$$\psi_{mk}(x,z) = \sqrt{2/D}\sin(\pi(m+1)x/D)\exp\{ikz\}, \tag{3.13}$$

$$E_{mk} = \frac{\hbar^2 k^2}{2m^*} + E_m, \quad E_m = \frac{E_D}{4}(m+1)^2, \tag{3.14}$$

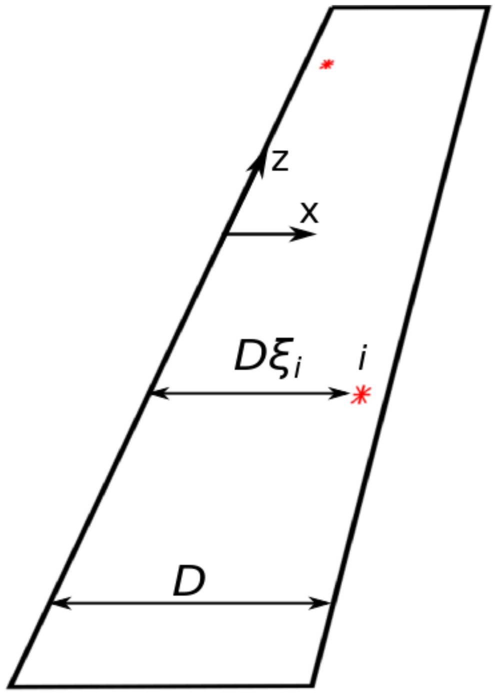

Figure 6. Conducting strip of width D. Impurities (shown as stars) are embedded in the strip, position of ith impurity is denoted as $\xi_i D$. Electrons live within the strip.

$$m = 0, 1, 2 \ldots, \quad E_D = \frac{2\hbar^2\pi^2}{m^* D^2} \tag{3.15}$$

where z is the coordinate along the strip and $0 < x < D$ is the distance from one of the strip's edges. Integer m is the transverse quantum number, k is the momentum along the strip, and E_m has the meaning of position of the bottom of m-th one-dimensional subband. The density of states in each subband

$$\begin{aligned}\nu_m(E) &= \int \frac{dk}{2\pi}\delta\left(E - E_m - \frac{k^2}{2m^*}\right) = \\ &= \frac{2}{2\pi}\sqrt{\frac{m^*}{2(E - E_m)}}\theta(E - E_m),\end{aligned} \tag{3.16}$$

We will measure all energies in the units of E_D and all distances in the units of D:

$$x \equiv D\xi, \quad E - E_m \equiv E_D\varepsilon_m, \tag{3.17}$$

For brevity, we will introduce

$$\varepsilon \equiv \varepsilon_N \tag{3.18}$$

with N being the label of the subband closest to the Fermi level. The partial densities of states in the dimensionless variables

$$\nu_m(E) \equiv \frac{\nu_m(\varepsilon)}{DE_D}, \quad \nu_m(\varepsilon) = \frac{\theta(\varepsilon_m)}{\sqrt{\varepsilon_m}}, \tag{3.19}$$

We are interested in semiclassical case when $E_D \ll E$ or $\varepsilon_0 \gg 1$. Under this condition both the label N of the resonant state and the number N_{ch} of open channels in the system are large:

$$N \approx N_{\text{ch}} \approx 2\sqrt{\varepsilon_0} \gg 1. \tag{3.20}$$

Then, in the leading semiclassical approximation

$$\nu(\varepsilon) = \sum_{m=1}^{\infty} \nu_m(\varepsilon) \approx \nu_0 = \int_0^{\varepsilon_0} \frac{d\varepsilon_m}{\sqrt{\varepsilon_m(\varepsilon_0 - \varepsilon_m)}} = \pi. \tag{3.21}$$

This result is valid for all ε except narrow interval in the vicinity of $\varepsilon = 0$ point. In the entire range of variation of ε one can write

$$\nu(\varepsilon) \approx \nu_0 \left(1 + \frac{\theta(\varepsilon)}{\pi\sqrt{\varepsilon}}\right), \tag{3.22}$$

4 Born scattering by short-range impurities

4.1 Case of tube

Our first step is finding the longitudinal resistivity of the tube using the Drude and Born approximations. We consider weak short range impurities with the hamiltonian

$$\hat{H} = \hat{H}_0 + V\delta(\mathbf{r} - \mathbf{r}_0), \qquad \hat{H}_0 = -\nabla^2/2m^* \tag{4.1}$$

where $\delta(\mathbf{r}) \equiv \frac{1}{R}\delta(z - z_0)\delta(\phi - \phi_0)$ is a two-dimensional delta-function and $\mathbf{r}_0$ denotes the position of the impurity on the wall of the tube. Let us find the self-energy for an electron

$$\Sigma_{km} = \frac{2\pi R n_{\text{imp}}^{(2)}}{E_0} \sum_{m'} \int \frac{dk'}{2\pi} |V_{kk'mm'}|^2 G^{(0)}_{k'm'} \tag{4.2}$$

Since for the short range potential (4.1)

$$V_{kk'mm'} = \frac{V}{2\pi R} \exp\{i(m - m')\phi_0 + i(k - k')z_0\}, \tag{4.3}$$

$|V_{kk'mm'}|^2 \equiv |V|^2$ depends neither on km, nor on $k'm'$, and we conclude that $\Sigma_{km} = \Sigma(E_{km})$ is a function only of the total energy E. In our dimensionless variables we get:

$$\Sigma(\varepsilon) = \frac{g(\varepsilon)}{2\pi\nu_0\tau_0}, \tag{4.4}$$

where τ_0 is the dimensionless scattering time for an electron away from the resonance (i.e., for $\varepsilon \gg 1$):

$$\tau_0^{-1} = \frac{m^* V^2 n_2}{E_0} = 2n(\lambda/\pi)^2, \tag{4.5}$$

the dimensionless Born scattering amplitude

$$\lambda = m^* V/2, \qquad |\lambda| \ll 1, \tag{4.6}$$

is assumed to be small and may have either signs (the positive sign corresponds to repulsion, the negative – to attraction). The dimensionless concentration n is also assumed small.

The Born decay rate

$$\frac{1}{\tau_{k,m}} = \frac{1}{\tau(E_{km})} = -2\mathrm{Im}\,\Sigma = \frac{1}{\tau_0}\frac{\nu(\varepsilon)}{\nu_0}. \tag{4.7}$$

For point impurities the scattering is isotropic and therefore the transport time coincides with the simple decay time.

Thus, if (3.12) is fulfilled, the particle is scattered predominantly (though not completely) to the upper band. In particular, if the particle was already in the upper subband then the scattering event most probably will not remove it from there. It means that in the zero approximation the upper subband is almost decoupled from all others.

Under the condition (3.12) the electrons in the N-subband states have low longitudinal velocity and therefore do not contribute much to the current. The latter is dominated by the states in all other bands. However, the singularity in the N-band is manifested also in the resistivity ρ through the scattering rate that is proportional to the density of the final states on the Fermi surface:

$$\frac{\rho}{\rho_0} = \frac{\nu(\varepsilon)}{\nu_0}, \qquad \rho_0 = \frac{1}{e^2\varepsilon_0\tau_0}. \tag{4.8}$$

These final states predominantly belong to the N-band.

4.2 Case of a strip

In the case of strip the scattering matrix elements depend both on the quantum numbers of scattering states and on the position of the impurity $\mathbf{r}_i$:

$$\begin{aligned} V^{(i)}_{kk'mm'}(\xi_i, z_i) = {} & \frac{2V}{D}\exp\{i(k-k')z_i\} \\ & \times \sin(\pi(m+1)\xi_i)\sin(\pi(m'+1)\xi_i), \end{aligned} \tag{4.9}$$

where z_i and ξ_i characterize the position of i-th impurity.

The averaged over the positions of impurities decay rate for a general state m:

$$\frac{1}{\tau_m(\varepsilon)} = n_{\rm imp} \left\langle \int dk' \sum_{m'} |V^{(i)}_{kk'mm'}(\xi)|^2 \delta(E_{m'k'} - E_{mk}) \right\rangle_\xi =$$

$$= \frac{1}{\tau_0} \int_0^1 d\xi \sum_{m'} (2\sin^2(\pi(m+1)\xi))(2\sin^2(\pi(m'+1)\xi)) \frac{\theta(\varepsilon_{m'})}{\pi\sqrt{\varepsilon_{m'}}} = \frac{1}{\nu_0\tau_0}\left(\sum_{m'} \nu_{m'} + \frac{1}{2}\nu_m\right) \approx$$

$$\approx \frac{1}{\tau_0}\left(1 + \frac{\theta(\varepsilon)}{\pi\sqrt{\varepsilon}} \left\{ \begin{array}{ll} 1, & \text{for } m \neq N, \\ 3/2, & \text{for } m = N. \end{array} \right\} \right)$$

where we have used

$$\int_0^1 d\xi (2\sin^2(\pi(m+1)\xi))(2\sin^2(\pi(m'+1)\xi)) = \left\{ \begin{array}{ll} 1, & m' \neq m, \\ 3/2, & m' = m. \end{array} \right.$$

and the fact that for large $N \gg 1$ each individual non-resonant contribution to the sum is relatively small, while the resonant one may be large, provided $\varepsilon_N \equiv \varepsilon \ll 1$. The rate of scattering away from resonance τ_0^{-1} coincides with that in an infinite plane:

$$\frac{1}{\tau_0} = 2n\left(\frac{\lambda}{\pi}\right)^2. \tag{4.10}$$

where λ is a dimensionless scattering amplitude and n is dimensionless concentration of impurities (both λ and n are assumed to be small throughout this paper

$$n = n^{(2)}_{\rm imp} D^2 \ll 1, \quad \lambda = m^* V/2, \qquad |\lambda| \ll 1, \tag{4.11}$$

So, we have shown that, within the Born approximation the scattering rate is the same for all the nonresonant states $m \neq N$:

$$\frac{\tau_0}{\tau_m(\varepsilon)} = \frac{\tau_0}{\tau_{\rm nonres}(\varepsilon)} \approx 1 + \frac{\theta(\varepsilon)}{\pi\sqrt{\varepsilon}} = \frac{\nu(\varepsilon)}{\nu_0}, \tag{4.12}$$

while for the resonant state $m = N$

$$\frac{\tau_0}{\tau_N(\varepsilon)} = \frac{\tau_0}{\tau_{\rm res}(\varepsilon)} \approx 1 + \frac{3\theta(\varepsilon)}{2\pi\sqrt{\varepsilon}}, \tag{4.13}$$

4.3 The conductivity

To evaluate the conductivity of both systems (per one spin projection) we can use the Kubo formula:

$$\sigma = \frac{e^2}{2\pi}\mathrm{Tr}[\hat{v}_z \hat{G}^R \hat{v}_z \hat{G}^A] = \frac{e^2}{2\pi}\int \frac{dk}{2\pi}\sum_m \frac{(v^z_{km})^2}{(\varepsilon - E_{km})^2 + 1/4\tau^2_m(\varepsilon)} \approx$$
$$\approx e^2 \int \frac{dk}{2\pi}\sum_m (v^z_{km})^2 \delta(\varepsilon - E_{km})\tau_m(\varepsilon), \qquad (4.14)$$

From (4.14) we immediately see that only non-resonant states are expected to be current carrying: the resonant state contribution to the current is suppressed by the factor $(v^z_N)^2 \propto \varepsilon_N \ll 1$. Hence, we can write

$$\sigma \approx e^2 D(\varepsilon)\nu_{\mathrm{tr}}(\varepsilon), \quad D(\varepsilon) = \frac{1}{2}v_F^2 \tau_{\mathrm{nonres}} = D_0 \frac{\nu_0}{\nu(\varepsilon)} \qquad (4.15)$$

where $D_0 = \frac{1}{2}v_F^2\tau_0$ is the two-dimensional diffusion coefficient, and

$$\nu_{\mathrm{tr}}(\varepsilon) = 2\int \frac{dk}{2\pi}\sum_m \left(\frac{v^z_{km}}{v_F}\right)^2 \delta(\varepsilon - E_{km}), \qquad (4.16)$$

is the "transport density of states". In contrast with the standard density of states, the transport one does not exhibit any Van Hove singularity at $\varepsilon \to 0$: the latter is suppressed by the factor $\left(\frac{v^z_{km}}{v_F}\right)^2$. As a result, under the semiclassical condition $\varepsilon_0 \gg 1$ we can always substitute $\nu_{\mathrm{tr}}(\varepsilon) \equiv \nu_0$, even at $\varepsilon \to 0$.

Thus, in the Born domain for the resistivity $\rho \equiv 1/\sigma$ we get a simple result:

$$\frac{\rho(\varepsilon)}{\rho_0} = \frac{\tau_0}{\tau_{\mathrm{nonres}}(\varepsilon)} = \frac{\nu(\varepsilon)}{\nu_0}, \qquad (4.17)$$

where

$$\rho_0 = \frac{1}{e^2\varepsilon_0\tau_0}. \qquad (4.18)$$

is the resistivity away from the resonance, coinciding with the resistivity of an infinite two-dimensional sample.

As we see, for $\varepsilon \to 0$ the resistivity $\rho(\varepsilon)$ diverges. This divergency is nothing else but Van Hove singularity.

So, we conclude that in the range of ε, where the perturbation theory is applicable (i.e., neither the single-impurity non-born effects, nor the interference of scattering at different impurities are relevant) the resistivity of a conducting strip is described by exactly the same formulae, as the resistivity of a conducting tube. One should only replace the unique scattering time τ of the tube theory by $\tau_{\rm nonres}$ of the strip theory.

4.4 Smearing of the Van Hove singularity within Born approximation

It is instructing to distinguish two groups of effects nonlinear in the scattering amplitude:

1. Single-impurity non-Born effects, arising due to more accurate treatment of individual scattering acts;

2. The multi-impurity ones, coming from the interference of scattering acts at different impurities.

Upon approaching the Van Hove singularity the nonlinear effects of both types become stronger. However, if the concentration of impurities is relatively high,

$$n \gg \lambda, \tag{4.19}$$

then the multi-impurity effects come into play earlier than the non-Born single-impurity effects, so that the latter do not have chance to show up and effectively can be neglected (the Born regime). In this section we will be dealing only with this Born regime.

4.4.1 Shift of the singularity

The strongest of the multi-impurity effects, that comes into play at $\varepsilon \sim \lambda n$, is quite simple. It is just the shift of the resonant subband by the average potential of impurities:

$$\overline{U} = \left\langle V \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \right\rangle_{\mathbf{r}_i} = \frac{\lambda n}{\pi^2}, \tag{4.20}$$

It is important to note that an introduction of this shift makes sense only under condition (4.19). Indeed, an effective self-averaging of the potential takes place if the electronic wave function does not change much on the scale of an inter-impurity distance n^{-1}, which means $n^{-1}(m\overline{U})^{1/2} \sim (\lambda/n)^{1/2} \ll 1$. The latter condition is equivalent to (4.19). Thus, if (4.19) is fulfilled, one should first of all renormalize the position of the Van Hove singularity:

$$\varepsilon \to \tilde{\varepsilon} = \varepsilon - \overline{U}, \tag{4.21}$$

and substitute $\tilde{\varepsilon}$ instead of ε in the results of the preceding section.

4.4.2 Smoothing of the singularity: qualitative description

The next multi-impurity effect is the smearing of the singularity due to scattering. This effect becomes essential at still smaller energy scales $\tilde{\varepsilon} \lesssim \tilde{\varepsilon}_{\rm min}$, where the perturbation theory breaks down. The scale $\tilde{\varepsilon}_{\rm min}$ can be extracted from the condition

$$\tau_{\rm res}^{-1}(\tilde{\varepsilon}_{\rm min}) \sim \tilde{\varepsilon}_{\rm min}, \tag{4.22}$$

when the resonant state becomes smeared. Note that the current carrying nonresonant states become smeared at the same scale, since, as it follows from (4.12) and (4.13), $\tau_{\rm res} \approx (2/3)\tau_{\rm nonres}$. The divergencies of both $\tau_{\rm nonres}^{-1}(\varepsilon)$ and $\tau_{\rm res}^{-1}(\varepsilon)$ are due to the divergency of the density of final states in the scattering processes.

4.4.3 Smoothing of the singularity: a link to strictly one-dimensional systems

Electrons with energies $|\tilde{\varepsilon}| \ll |\overline{U}|$ are effectively scattered not by individual impurities, but by fluctuations of the density of impurities. Typically such fluctuations are constituted by many impurities and, therefore, their distribution is essentially gaussian. It is important to note that these gaussian fluctuations are universal, in particular, they do not depend on the character (repulsing or attracting) of individual impurities.

The latter is not true for rare very large non-gaussian fluctuations with $|\tilde{\varepsilon}| \gtrsim |\overline{U}|$. However, these large fluctuations are not relevant, since the corresponding part of the spectrum is likely to be dominated not by the far tail of the resonant band, but by the non-resonant ones (see below).

Combining the formulas (4.13) and (4.22) we get an estimate for the width of smeared singularity

$$\tilde{\varepsilon}_{\rm min} \sim (n\lambda^2)^{2/3}, \quad \frac{\tilde{\varepsilon}_{\rm min}}{\overline{U}} \sim \left(\frac{\lambda}{n}\right)^{1/3} \ll 1, \tag{4.23}$$

so, indeed, under condition (4.19) the smearing occurs on the energy scale that is much smaller than the shift of the band.

For energies $|\tilde{\varepsilon}| \lesssim \tilde{\varepsilon}_{\min}$ plane waves $\exp(ikz)$ do not provide any good approximation to the eigenfunctions of an electron in the resonant band: they should be substituted by a set of certain nontrivial wave functions $\psi_\alpha(z)$, depending of concrete realization of disorder. At the same time, the plane waves remain valid eigenfunctions for electrons in the current-carrying nonresonant bands. Then, if we, as before, neglect the contribution of the resonant band to the current, the conductivity still can be written in a form (4.14), the only modification occurs in the expression for $\tau_m(\varepsilon)$ for $m \neq N$:

$$\frac{1}{\tau_m(\varepsilon)} = n_{\text{imp}} \left\langle \int dk' \sum_{m' \neq N} |V^{(i)}_{kk'mm'}(\xi)|^2 \delta(E_{m'k'} - E_{mk}) \right\rangle_\xi + \\ + n_{\text{imp}} \left\langle \sum_\alpha |V^{(i)}_{k\alpha mN}(\xi, z)|^2 \delta(E_{N\alpha} - E_{mk}) \right\rangle_{\xi, z} \tag{4.24}$$

$$V^{(i)}_{k\alpha mN}(\xi_i, z_i) = \frac{2V}{D} \exp\{ikz_i\} \psi^*_\alpha(z_i) \sin(\pi(m+1)\xi_i) \sin(\pi N)\xi_i), \tag{4.25}$$

So, the second term in (4.24) can be rewritten in terms of a density of states $\nu_{\text{res}}(\tilde{\varepsilon})$ for strictly one-dimensional system

$$\frac{1}{\tau_0} \int_0^1 d\xi (2\sin^2(\pi(m+1)\xi))(2\sin^2(\pi N\xi)) \int dz \sum_\alpha |\psi_\alpha(z)|^2 \delta(E_{N\alpha} - E_{mk}) = \frac{\nu_{\text{res}}(\tilde{\varepsilon})}{\pi\tau_0} \tag{4.26}$$

We would like to stress that in our quasi-one-dimensional problem the conductivity is expressed through the exact average density of states of a purely one-dimensional problem, which is the average of one-particle Green-function (involving two ψ-operators). On the other hand, it is well known that the conductivity should be expressed through the exact average two-particle Green function (four ψ-operators), which is a much more sophisticated object that the one-particle one.

The explanation for this paradox is as follows: There are two distinct types of ψ-operators in our quasi-one-dimensional problem: ψ_{nonres} for electrons in non-resonant bands and ψ_{res} – for electrons in the resonant band. Since in our problem the resonant band does not contribute to the current directly, each term in the conductivity should necessarily contain at least two ψ_{nonres}-operators. Remaining two ψ-operators may be either both of ψ_{nonres} type (that leads to the first term in (4.24)), or both of ψ_{res}-type (the second term in (4.24)). In this term the ψ_{res}-operators enter through the density of final states in the scattering process (c.f. (4.12)). There are no terms containing four $\psi_{\rm res}$-operators since the purely one dimensional contribution to the current is strongly suppressed.

So, in the Born regime we end up with the formula (4.12) for the scattering rate of nonresonant electrons with

$$\nu(\tilde{\varepsilon}) \approx \nu_{\rm nonres}(\tilde{\varepsilon}) + \nu_{\rm res}(\tilde{\varepsilon}), \tag{4.27}$$

where $\nu_{\rm nonres}(\tilde{\varepsilon}) \approx \nu_0$, while the relation $\nu_{\rm res}(\tilde{\varepsilon}) = \theta(\tilde{\varepsilon})(\tilde{\varepsilon})^{-1/2}$ is true only for $|\tilde{\varepsilon}| \gg \tilde{\varepsilon}_{\rm min}$. At $|\tilde{\varepsilon}| \lesssim \tilde{\varepsilon}_{\rm min}$ one should use exact solutions from the theory of strictly one dimensional disordered systems.

4.4.4 Correction to the density of states due to hybridization of bands

Besides the nontrivial and strong modification of $\nu_{\rm res}(\tilde{\varepsilon})$ by disorder, there is an additional effect – hybridization between resonant and nonresonant bands due to presence of impurities. As we will see in the next subsection, the corresponding correction to the nonresonant density of states $\nu_{\rm nonres}$ is relatively small in the relevant range of energies and can be evaluated perturbatively:

$$\nu_{\rm nonres}(\tilde{\varepsilon}) = \nu_0 + \delta\nu(\tilde{\varepsilon}), \quad \delta\nu(\tilde{\varepsilon}) = \nu_0 \frac{d}{d\tilde{\varepsilon}} \delta\varepsilon(\tilde{\varepsilon}), \tag{4.28}$$

where $\delta\varepsilon(\tilde{\varepsilon})$ is the second order (in V) correction to the energy $\tilde{\varepsilon}$ of certain nonresonant state arising due to scattering

$$\delta\varepsilon(\tilde{\varepsilon}) = \frac{n\lambda^2}{\pi^4} \ \text{v.p.} \int \frac{\nu(\tilde{\varepsilon}')d\tilde{\varepsilon}'}{\tilde{\varepsilon} - \tilde{\varepsilon}'}, \tag{4.29}$$

For $\tilde{\varepsilon} < 0$ and $|\tilde{\varepsilon}| \gg \tilde{\varepsilon}_{\rm min}^{\rm (t)}$ the principal contribution to the integral in (4.29) comes from the states in the resonant band with energies $\tilde{\varepsilon}' > 0$ and $\tilde{\varepsilon}' \sim |\tilde{\varepsilon}|$, so that the correction can be estimated as

$$\delta\nu(\tilde{\varepsilon}) = \frac{n\lambda^2}{\pi^4} \int_0^\infty \frac{d\tilde{\varepsilon}'}{(\tilde{\varepsilon} - \tilde{\varepsilon}')^2 \sqrt{\tilde{\varepsilon}'}} \sim \nu_0 \left(\frac{\tilde{\varepsilon}_{\rm min}}{|\tilde{\varepsilon}|} \right)^{3/2}. \tag{4.30}$$

Thus, we conclude that for $|\tilde{\varepsilon}| \gg \tilde{\varepsilon}_{\rm min}$ the relative correction to the density of states is indeed small.

4.4.5 Non-matching asymptotes and SCBA

At this point one should also note that the correction (4.30) is relatively small already for $|\varepsilon| \lesssim \varepsilon_{\min}$. For $\tilde{\varepsilon} > 0$ it seems natural that the value of $\nu(\tilde{\varepsilon})$ can not change considerably for $\tilde{\varepsilon} < \tilde{\varepsilon}_{\min}$ and therefore

$$\nu(\tilde{\varepsilon}) \sim \nu_{\max}, \quad \tau_{\rm res}(\tilde{\varepsilon}) \sim \tau_{\min}, \quad \text{for } |\tilde{\varepsilon}| \lesssim \tilde{\varepsilon}_{\min} \tag{4.31}$$

Also, (4.30) means that

$$\nu(-\tilde{\varepsilon}_{\min}) \sim \nu_0 \ll \nu(\tilde{\varepsilon}_{\min}) \tag{4.32}$$

and direct smooth matching of (4.32) and (4.31) is impossible!

To resolve this paradox one should in principle go beyond the estimates made above, and accurately solve the problem in the range $|\tilde{\varepsilon}| \lesssim \tilde{\varepsilon}_{\min}$. This will be done in following sections in 2 ways: (i) self-consistent Born approximation and (ii) asymptotical mapping to exactly soluable one-dimensional problem. Although the self-consistent Born approximation in our case can not be justified analytically, we will see by comparison with more robust second approach that it gives qualitatively reasonable results. For a qualitative understanding of paradox resolution it is enough to note that there is practically only one scenario for such a giant drop in the density of states: a "quasifold" – an inflection point with almost vertical slope, see Fig. 7.

In the dependence $\nu(\tilde{\varepsilon})$ at some point $\tilde{\varepsilon}_{\rm bi}$ there should be

(i) very large positive first derivative $\nu'(\tilde{\varepsilon}_{\rm bi}) \gg \nu(\tilde{\varepsilon}_{\min})/\tilde{\varepsilon}_{\min}$,

(ii) zero second derivative, and

(iii) rather small third derivative.

An example of such a behaviour is provided by the results of the self-consistent Born approximation. One can write(see, e.g., [6, 8])

$$\langle \nu(\varepsilon) \rangle = \nu^{(0)}[\varepsilon - \Sigma(\varepsilon)] \tag{4.33}$$

where $\Sigma(\varepsilon)$ is given by (4.4). This equation is solved in Appendix A. Although all the calculations in this Appendix are carried out for the case of tube, $\tau_{\rm res}^{(s)}$ and $\tau_{\rm res}^{(t)}$ differs only by the factor 3/2 so that all the formulas hold true up to replacement $\tau_0^{-1} \to 3/2\tau_0^{-1}$.

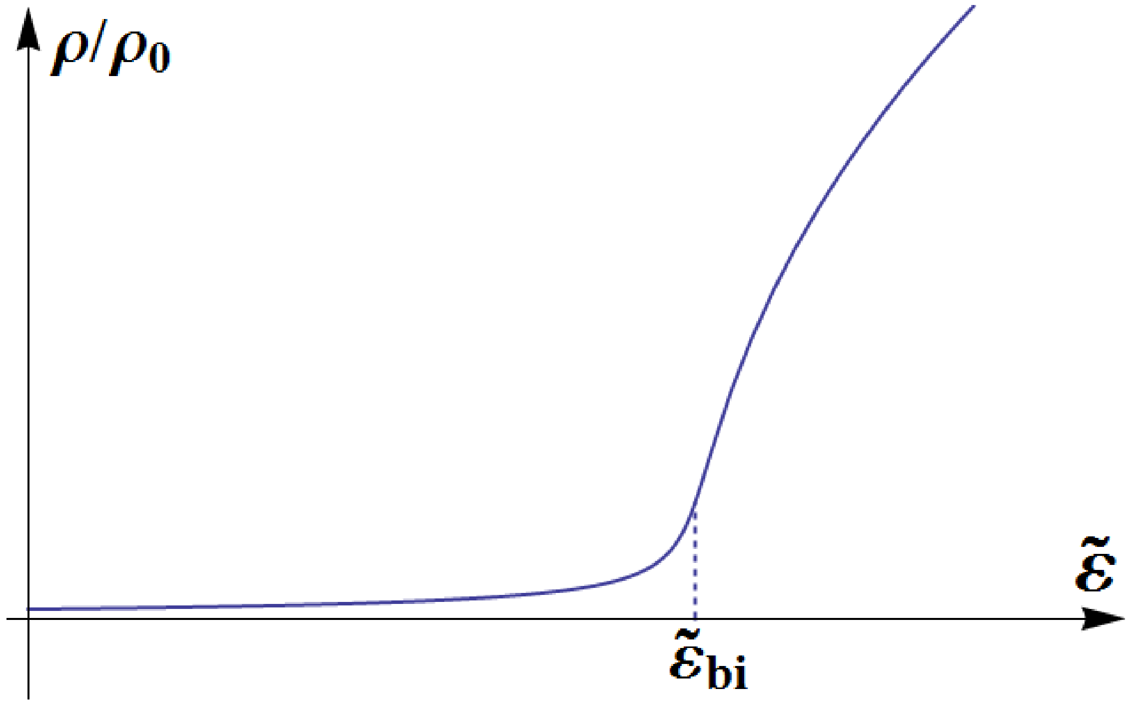

Figure 7. The "quasifold": in the vicinity of the bifurcation point $\tilde{\varepsilon} = \tilde{\varepsilon}_{\rm bi}$ the slope of the curve $\rho(\tilde{\varepsilon})$ is anomalously steep.

Despite the fact that these results can not be taken too seriously (since the self-consistent Born approximation is not rigorous), as we see below, the main message happens to be true: the entire domain $|\tilde{\varepsilon}| \sim \tilde{\varepsilon}_{\rm min}$ is split into two basic subdomains: $\tilde{\varepsilon} < \tilde{\varepsilon}_{\rm bi}$ where $\nu \sim \nu_0$ and $\tilde{\varepsilon} > \tilde{\varepsilon}_{\rm bi}$ where $\nu \sim \nu(+\tilde{\varepsilon}_{\rm min}) \gg \nu_0$. Between these two subdomains there is a narrow intermediate layer around $\tilde{\varepsilon}_{\rm bi}$ in which $\nu(\tilde{\varepsilon})$ undergoes a dramatic change.

The results of self-consistent Born approximation can be roughly summarized as follows:

$$\frac{\rho(\tilde{\varepsilon})}{\rho_0} = \begin{cases} 1 + \dfrac{1}{\pi\sqrt{\tilde{\varepsilon}}}, & \text{for } \tilde{\varepsilon}_{\rm min} \ll \tilde{\varepsilon},\, \tilde{\varepsilon} > 0, \\ \sim \lambda^{-2/3} n^{-1/3}, & \text{for } \tilde{\varepsilon}_{\rm bi} < \tilde{\varepsilon} \lesssim \tilde{\varepsilon}_{\rm min}, \\ \sim 1, & \text{for } \tilde{\varepsilon} < \tilde{\varepsilon}_{\rm bi}, \end{cases} \tag{4.34}$$

with certain $\tilde{\varepsilon}_{\rm bi} < 0$, $|\tilde{\varepsilon}_{\rm bi}| \sim \tilde{\varepsilon}_{\rm min}$. $\tilde{\varepsilon}_{\rm bi}^{(s)}$ and $\tilde{\varepsilon}_{\rm min}^{(s)}$ are suppressed by factor $2^{-2/3}$ compared to that of a tube. A schematic plot of (4.34) is shown in FIG.8.

4.4.6 Exact results: the case of tube

Now we start from the case of the tube with a more accurate approach exploring the exact solutions known for the strictly one-dimensional systems. Under the condition (4.19) the one-dimensional model with identical point-like scatterers randomly distributed on a line, was exhaustively studied in [31]. It was shown[30] that the random potential is effectively

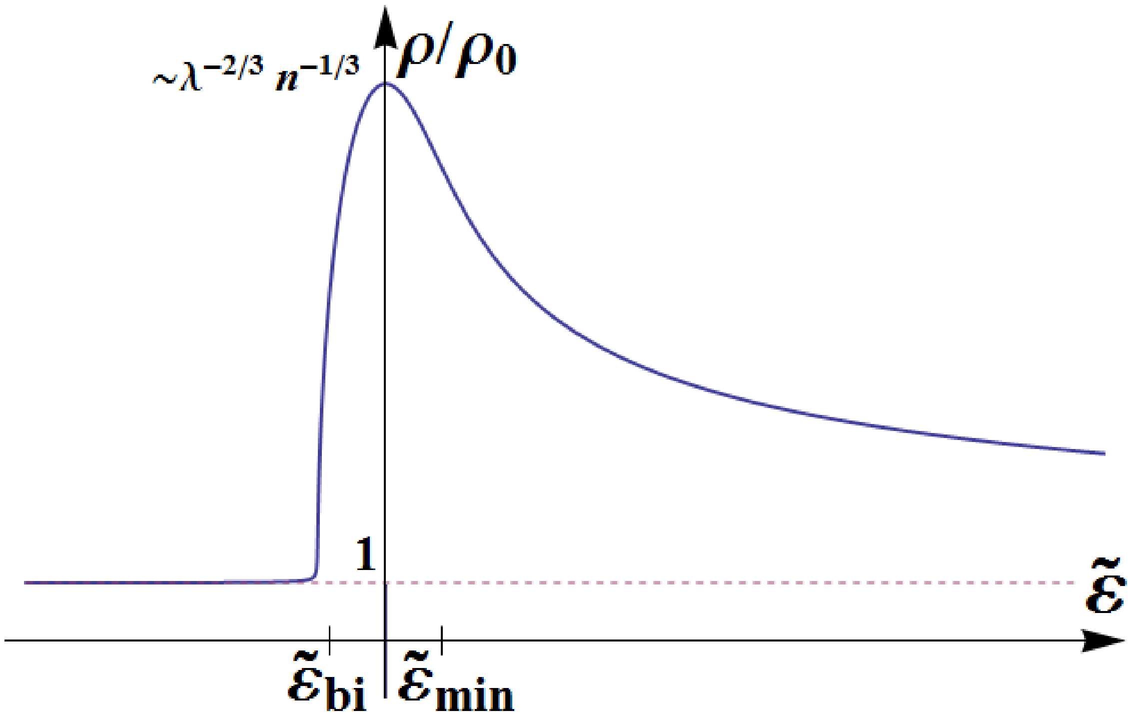

Figure 8. The shape of a smeared Van Hove singularity within the self-consistent Born approximation.

gaussian and the density of states may be evaluated with the help of Fokker-Planck equation. As a result

$$\nu_{\rm res}^{(\rm t)}(\tilde{\varepsilon}) = \nu_0 \left(\tilde{\varepsilon}_{\rm min}^{(\rm t)}\right)^{-1/2} Y\left(\tilde{\varepsilon}/\tilde{\varepsilon}_{\rm min}^{(\rm t)}\right), \tag{4.35}$$

where

$$\tilde{\varepsilon}_{\rm min}^{(\rm t)} = (2\pi\tau_0)^{-2/3} = \left(\frac{n}{\pi}\right)^{2/3} \left(\frac{\lambda}{\pi}\right)^{4/3}, \tag{4.36}$$

$$Y(q) = \frac{2}{\sqrt{\pi}} \frac{\partial}{\partial q} \left(\int_0^\infty \frac{dx}{\sqrt{x}} \exp\left\{ -xq - \frac{x^3}{12} \right\} \right)^{-1}. \tag{4.37}$$

The asymptotics of (4.37) at $q > 0$, $q \gg 1$,

$$Y(q) \approx \frac{1}{\pi\sqrt{q}}, \tag{4.38}$$

corresponds to the trivial perturbative result, while the asymptotics for $q < 0$, $|q| \gg 1$

$$Y(q) \approx \frac{4|q|}{\pi} \exp\left\{-\frac{4}{3}|q|^{3/2}\right\}, \tag{4.39}$$

describes the well-known Lifshits tail of the density of states in one dimensional system with effectively gaussian disorder. It should be noted that (4.39) is indeed only an intermediate asymptotics[31], valid in the range $1 \ll |q| \ll \sqrt{n/\lambda}$, where the random potential is effectively gaussian.

As it was argued in Section 4.4.5, there should be certain bifurcation energy $\tilde{\varepsilon}_{\rm bi}^{(\rm t)}$, such that for all energies $\tilde{\varepsilon}_{\rm bi}^{(\rm t)} < \tilde{\varepsilon} \ll 1$ the principal contribution to the density of states comes from the resonant subband N: $\nu_{\rm nonres}(\tilde{\varepsilon}) \ll \nu_{\rm res}(\tilde{\varepsilon})$. Let us demonstrate that this statement is valid also for the exact solution.

The bifurcation point $\tilde{\varepsilon}_{\rm bi}^{(\rm t)}$ can be roughly defined as the energy, at which the contribution to the density of states coming from the resonant band becomes equal to that of the nonresonant ones:

$$\nu_{\rm nonres}^{(\rm t)}(\tilde{\varepsilon}_{\rm bi}^{(\rm t)}) = \nu_{\rm res}^{(\rm t)}(\tilde{\varepsilon}_{\rm bi}^{(\rm t)}). \tag{4.40}$$

As a first step, let's suppose that $|\tilde{\varepsilon}_{\rm bi}^{(\rm t)}| \gg \tilde{\varepsilon}_{\rm min}^{(\rm t)}$. Then, according to (4.30), $\nu_{\rm nonres}^{(\rm t)}(\tilde{\varepsilon})$ differs from ν_0 only slightly, and (4.40) takes the form

$$\nu_0 = \nu_0 \left(\tilde{\varepsilon}_{\rm min}^{(\rm t)}\right)^{-1/2} Y\left(q_{\rm bi}^{(\rm t)}\right), \tag{4.41}$$

$$\tilde{\varepsilon}_{\rm bi}^{(\rm t)} = \tilde{\varepsilon}_{\rm min}^{(\rm t)} q_{\rm bi}^{(\rm t)}, \quad q_{\rm bi}^{(\rm t)} \approx -\left(\frac{3}{8}\right)^{2/3} \ln^{2/3}\left(1/\tilde{\varepsilon}_{\rm min}^{(\rm t)}\right). \tag{4.42}$$

We want to remind here again that the result (4.42) (as well as (4.39)) is valid under condition $\tilde{\varepsilon}_{\rm min}^{(\rm t)} \ll |\tilde{\varepsilon}_{\rm bi}^{(\rm t)}| \ll \overline{U}$, which is equivalent to

$$1 \ll \ln\left(1/n\lambda^2\right) \ll \left(\frac{n}{\lambda}\right)^{1/2}. \tag{4.43}$$

In particular, the first inequality in (4.43) justifies our assumption $|\tilde{\varepsilon}_{\rm bi}^{(\rm t)}| \gg \tilde{\varepsilon}_{\rm min}^{(\rm t)}$.

So, we conclude that the contribution of the nonresonant bands is essentially unperturbed in the relevant domain $|\tilde{\varepsilon}| > |\tilde{\varepsilon}_{\rm bi}^{(\rm t)}|$. As a result the total density of states and the resistivity of a tube can be written as

$$\frac{\nu^{(\mathrm{t})}(\tilde{\varepsilon})}{\nu_0} = \frac{\rho^{(\mathrm{t})}(\tilde{\varepsilon})}{\rho_0} \approx 1 + \left(\tilde{\varepsilon}_{\mathrm{min}}^{(t)}\right)^{-1/2} Y\left(\tilde{\varepsilon}/\tilde{\varepsilon}_{\mathrm{min}}^{(\mathrm{t})}\right), \tag{4.44}$$

with high accuracy in the entire range of energies $\tilde{\varepsilon}$.

4.4.7 Exact results: the case of the strip

Evaluation of the density of states in the case of strip is very similar to that in the case of tube. For the energies above the bifurcation point the density of states is dominated by the states from the resonant subband and its smearing is also controlled by scattering processes in which both initial and final states belong to the resonant subband. It means that the smearing depends on $\tau_{\mathrm{res}}^{(\mathrm{s})}(\tilde{\varepsilon})$, but not on $\tau_{\mathrm{nonres}}^{(\mathrm{s})}(\tilde{\varepsilon})$. In this sence the problem is very similar to that of the tube, the only difference is an additional factor 2/3 in the definition (4.13) of $\tau_{\mathrm{res}}^{(\mathrm{s})}$, as compared to $\tau_{\mathrm{res}}^{(\mathrm{t})}$. This difference, however, can be removed by the redefinition of the energy scale:

$$\tilde{\varepsilon}_{\mathrm{min}}^{(\mathrm{t})} = (2\pi\tau_0)^{-2/3} \longrightarrow \tilde{\varepsilon}_{\mathrm{min}}^{(s)} = (4\pi\tau_0/3)^{-2/3} \tag{4.45}$$

After the rescaling the scattering rate and the density of states can be expressed in terms of the very same function $Y(q)$, which appeared in the results for the tube (see (4.37), (4.38), (4.39)). It also can easily be demonstrated that, exactly as in the case of tube, the nonresonant contribution to the density of states remains equal to ν_0 for all $|\tilde{\varepsilon}| > |\tilde{\varepsilon}_{\mathrm{bi}}|$. As a result

$$\frac{\nu^{(\mathrm{s})}(\tilde{\varepsilon})}{\nu_0} = \frac{\rho^{(\mathrm{s})}(\tilde{\varepsilon})}{\rho_0} \approx 1 + \left(\tilde{\varepsilon}_{\mathrm{min}}^{(\mathrm{s})}\right)^{-1/2} Y\left(\tilde{\varepsilon}/\tilde{\varepsilon}_{\mathrm{min}}^{(\mathrm{s})}\right), \tag{4.46}$$

$$\tilde{\varepsilon}_{\mathrm{min}}^{(\mathrm{s})} = (4\pi\tau_0/3)^{-2/3} = \left(\frac{3n}{2\pi}\right)^{2/3} \left(\frac{\lambda}{\pi}\right)^{4/3}. \tag{4.47}$$

So, the difference in the resistivities of a tube and a strip is only in different numerical factors entering characteristic energy scales $\tilde{\varepsilon}_{\mathrm{min}}^{(\mathrm{t})}$ and $\tilde{\varepsilon}_{\mathrm{min}}^{(\mathrm{s})}$

4.4.8 General features of resistivity in the Born case

In general, the energy profile of the resistivity of a quasi-one-dimensional system with "relatively high" concentration (that is, for $|\lambda| \ll n \ll 1$) of weak short-range impurities consists of a set of shifted and smeared Van Hove singularities (see Fig.9).

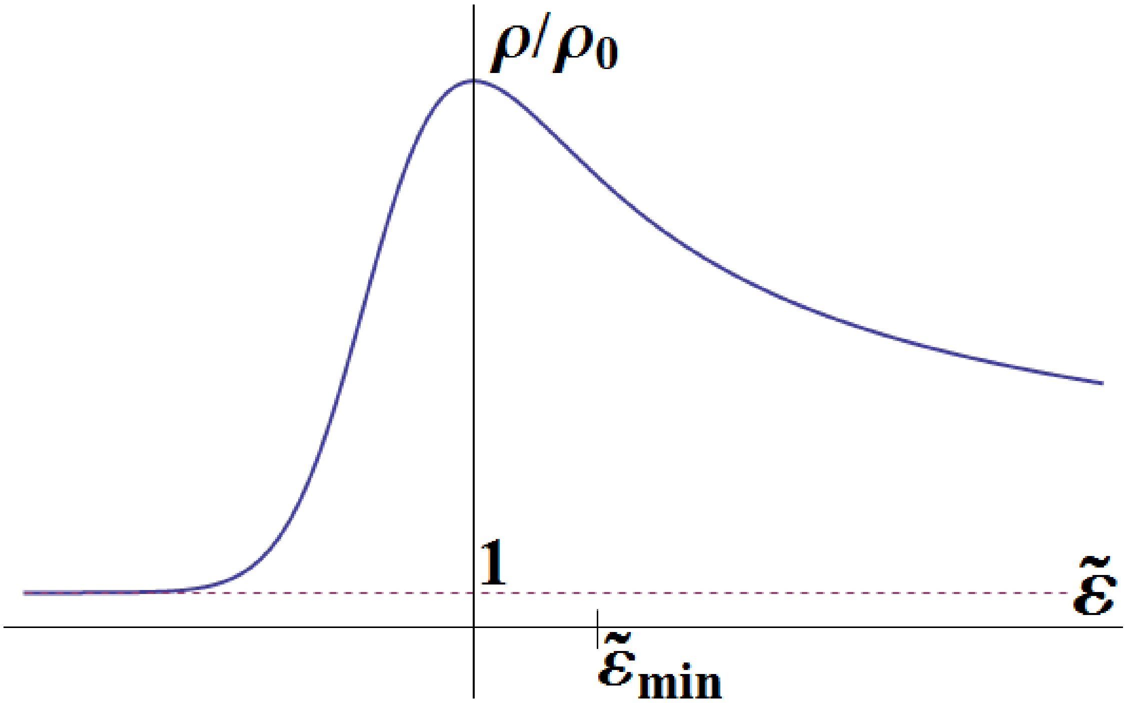

Figure 9. The shape of a smeared Van Hove singularity in Born approximation (results from asymptotical mapping to 1D exactly soluable chain).

Each singularity is characterized by four distinct ranges:

1. Relatively smooth right slope of a shifted singularity:

$$\rho(\varepsilon) \approx \frac{\rho_0}{\pi(\varepsilon - \overline{U})^{1/2}}, \quad \varepsilon - \overline{U} > 0, \quad \frac{|\varepsilon - \overline{U}|}{\tilde{\varepsilon}_{\min}} \gg 1, \tag{4.48}$$

2. Smeared core of the singularity:

$$\rho(\varepsilon) \sim \rho_{\max} \sim \frac{\rho_0}{\pi \tilde{\varepsilon}_{\min}^{1/2}}, \quad \frac{|\varepsilon - \overline{U}|}{\tilde{\varepsilon}_{\min}} \lesssim 1, \tag{4.49}$$

3. Exponentially steep left slope of a shifted singularity:

$$\rho(\varepsilon) \approx \frac{4\rho_0}{\pi(\varepsilon - \overline{U})^{1/2}} \exp\left\{ -\frac{4}{3} \left| \frac{\varepsilon - \overline{U}}{\tilde{\varepsilon}_{\min}} \right|^{3/2} \right\}, \tag{4.50}$$

$$\varepsilon - \overline{U} < 0, \quad 1 \ll \frac{|\varepsilon - \overline{U}|}{\tilde{\varepsilon}_{\min}} < |q_{\rm bi}|, \tag{4.51}$$

4. Left plateau:

$$\rho(\varepsilon) \approx \rho_0. \tag{4.52}$$

The relevant energy scales $\overline{U}$ (shift of the peak) and $\tilde{\varepsilon}_{\min}$ (its width) are given by (4.20), (4.36) and (4.47). For the tube and the strip cases these scales differ only in numerical prefactor. Logarithmically large parameter q_{bi} is defined in (4.42).

4.4.9 Comparison of exact and SCBA results

We see that self-consistent Born approximation is in qualitative agreement with the exact results. Namely, its prediction has the same main feature: $\rho(\tilde{\varepsilon})$ changes very fast (has a bifurcation point $\tilde{\varepsilon}_{\mathrm{bi}}$) under the position of shifted bottom of a subband. In the exact solution the rapid decrease of $\rho(\tilde{\varepsilon})$ is the manifestation of exponential Lifshits tail in 1D density of states of resonant subband.

5 Beyond the Born approximation

The above considerations seem plausible and straightforward. However, the analysis below shows that they are only applicable if the concentration of impurities is high enough, i.e. for $n \gg n_c \sim |\lambda|$. For $n \ll n_c$ the scattering that determines the form of smeared Van Hove singularities is strongly modified by the single-impurity non-Born effects that dramatically grow upon approaching the singularity. We start our discussion from the properties of an exact amplitude of scattering by a single short-range impurity, placed on the infinite 2D plane.

5.1 A single impurity problem in two dimensions: non-Born effects

Properties of short-range impurities or defects in two-dimensional systems are well studied. In this subsection we briefly remind the main facts.

In particular, it is known that a weakly-attracting short-range impurity always forms a bound state [24]. Writing the hamiltonian of the system in the form (4.1) with $\lambda < 0$ one finds that there is a single bound state with small binding energy

$$E_{\text{bound}}^{(2d)} \approx -\frac{\hbar^2}{m^* a_0^2} \exp\left(-\frac{\pi}{|\lambda|}\right), \tag{5.1}$$

where a_0 is the ultraviolet cutoff ("radius of the delta-function") The wave-function of the ground state

$$\psi_0(r) \sim \exp(-r/a^{(2d)}), \qquad a^{(2d)} = (2m^*|E_{\text{bound}}^{(2d)}|)^{-1/2}, \tag{5.2}$$

being the radius of the ground state wave function.

A scattering of a particle with positive energy $E \ll \frac{\hbar^2}{m^* a_0^2}$ is isotropic. For $r \gg a_0$ one can write the "scattering wave-function" in the form [24]

$$\psi_{\mathbf{p}}(r) = \exp\{i(\mathbf{p}\cdot\mathbf{r})\} - i\lambda H_0^{(1)}(pr), \quad E = p^2/2m^*, \tag{5.3}$$

where $H_0^{(1)}(x)$ is the Hankel function. Moreover, for $pr \gg 1$ one can use the asymptotics of the Hankel function:

$$\psi_{\mathbf{p}}(r) \approx \exp\{i(\mathbf{p}\cdot\mathbf{r})\} - \lambda\sqrt{\frac{2}{-i\pi pr}} \exp(ipr), \tag{5.4}$$

The above results should be modified if one wants to go beyond the Born approximation. If the condition $E \ll \frac{\hbar^2}{m^* a_0^2}$ (or $ka_0 \ll 1$) is fulfilled then the scattering remains isotropic even beyond the Born approximation; it means that the scattering amplitude is still characterised by a single dimensionless constant: small real λ in the result (5.4) should be replaced by not necessarily small complex Λ – the nonperturbative dimensionless scattering amplitude. The latter should obey the optical theorem:

$$\mathrm{Im}\Lambda = -|\Lambda|^2, \tag{5.5}$$

hence the scattering amplitude can be parametrised by a single real constant λ

$$\Lambda = \lambda e^{-i \arcsin \lambda} \equiv \lambda \left(\sqrt{1-\lambda^2} - i\lambda \right). \tag{5.6}$$

In particular, for weak interaction ($|\lambda| \ll 1$)

$$\Lambda \approx \lambda - i\lambda^2. \tag{5.7}$$

Note that parameter λ in (5.6) is related to the potential amplitude V by formula (4.6) only for $|\lambda| \ll 1$. In general case it is not true and λ is just a convenient parameter for expressing the phenomenological scattering amplitude.

5.2 Non-Born effects in a tube

5.2.1 A single impurity problem on a tube: semiclassical treatment of non Born effects

Let us place a single weakly attracting impurity on the surface of the cylinder. Clearly, there are two distinct cases with respect to the bound state of an electron:

(i) Wide cylinder or strong scattering: $R \gg a^{(2d)}$. In this case the bound state will not differ much from the purely two-dimensional case and the formula (5.1) applies.

(ii) Narrow cylinder or weak scattering: $R \ll a^{(2d)}$. This is an effectively one-dimensional case, the bound state can also be studied easily.

In this paper we will be interested, however, not in the ground state but in the scattering matrix for an electron with an essentially positive energy $E > 0$ in the range

$$E_0, E_{\mathrm{bound}} \ll E \ll \frac{\hbar^2}{m^* a_0^2}. \tag{5.8}$$

Under this condition the scattering process can be conveniently described in semiclassical terms. To find the scattering amplitude beyond the Born approximation one has to solve the Dyson equation

$$G(\mathbf{r_1}, \mathbf{r_2}) = G_0(\mathbf{r_1}, \mathbf{r_2}) + G_0(\mathbf{r_1}, \mathbf{r_0}) V G(\mathbf{r_0}, \mathbf{r_2}) \tag{5.9}$$

for the retarded Green function defined as

$$G = \left\{E - \hat{H} + i0\right\}^{-1} \quad G_0 = \left\{E - \hat{H}_0 + i0\right\}^{-1} \tag{5.10}$$

where $\mathbf{r_0}$ is the position of the impurity. In particular, putting $\mathbf{r_1} = \mathbf{r_2} = \mathbf{r_0}$ we arrive at the equation

$$g = g_0 + g_0 V g, \qquad g \equiv G(\mathbf{r_0}, \mathbf{r_0}) = \frac{g_0}{1 - V g_0} \tag{5.11}$$

where

$$g_0 \equiv G_0(\mathbf{r_0}, \mathbf{r_0}) \tag{5.12}$$

One can also write

$$G(\mathbf{r_1}, \mathbf{r_2}) = G_0(\mathbf{r_1}, \mathbf{r_2}) + G_0(\mathbf{r_1}, \mathbf{r_0}) V_{\mathrm{ren}} G_0(\mathbf{r_0}, \mathbf{r_2}) \tag{5.13}$$

with the renormalized scattering amplitude

$$V_{\mathrm{ren}} = \frac{V}{1 - V g_0} \tag{5.14}$$

First of all we have to find the single-site $g_0 \equiv G_E(0, 0)$. For our nontrivial topology one can write in the semiclassical approximation

$$g_0 = \sum_{n=-\infty}^{\infty} e^{\pi i n \Phi/\Phi_0} \mathcal{G}_E(2\pi n R), \tag{5.15}$$

$\mathcal{G}_E(\mathbf{r})$ being the retarded Green function in an infinite two-dimensional metal. For $n \neq 0$

one can use the semiclassical approximation:

$$\mathcal{G}_E(\mathbf{r}) \approx \sqrt{\frac{2}{\pi p r}} e^{i(pr+\pi/4)}. \tag{5.16}$$

For the $n = 0$ term we have

$$\mathcal{G}_E(0) = -\frac{im^*}{2} + C \tag{5.17}$$

where C is a formally infinite real constant. This divergency is well known – it means that the perturbation theory does not work well in spatial dimensions $d \geq 2$ when applied to point-like impurities. This phenomenon is not specific for the cylinder geometry – it is present in an infinite two-dimensional metal as well. Special methods to deal with this divergence were developed already long ago. It was shown that in the case of isotropic scattering, accurate calculations lead to the substitution of the bare coupling constant λ by the exact complex amplitude Λ of scattering by the same impurity in the infinite two-dimensional metal. Thus, for the fully renormalized scattering amplitude $\Lambda^{(\mathrm{ren})}$ in the case of cylinder we get

$$\Lambda^{(\mathrm{ren})}(\varepsilon) = \frac{\Lambda}{1 + \Lambda g(\varepsilon)/\pi\nu_0}, \tag{5.18}$$

where $g(\varepsilon) \equiv g_0$ is given by the formulas (5.15), (5.52), (5.17) where the infinite constant C is discarded. As a result, we arrive at the expression (3.8). Consequently, the scattering rate is also renormalized:

$$\frac{1}{\tau(\varepsilon)} = \frac{2n}{\pi^2} \left| \frac{\Lambda}{1 + \Lambda g(\varepsilon)/\pi\nu_0} \right|^2 \frac{\nu(\varepsilon)}{\nu_0}. \tag{5.19}$$

For small λ the discussed renormalization is only essential in the vicinity of some Van Hove singularity so that we can use asymptotics $g(\varepsilon)/\pi\nu_0 \approx \pi^{-1}(-\varepsilon)^{-1/2}$ and for small $\lambda \ll 1$ one can write

$$\Lambda^{(\mathrm{ren})}(\varepsilon) \approx \frac{\lambda - i\lambda^2}{1 + (\lambda - i\lambda^2)(-\varepsilon)^{-1/2}/\pi}. \tag{5.20}$$

The importance of the renormalization of the scattering matrix in the systems with the singularity in the density of states (e.g., superconductors) that can even lead to formation of bound states was discovered and explored in details already in 60-ies (see [25, 26, 27, 28, 29]).

It is clear that the non-Born effects first come into play for $\varepsilon \lesssim \varepsilon_{\rm nB}$, where

$$\varepsilon_{\rm nB} = (\lambda/\pi)^2, \tag{5.21}$$

so that it is sometimes convenient to use the "normalized" energy:

$$\epsilon \equiv \varepsilon/\varepsilon_{\rm nB}. \tag{5.22}$$

Note that for $\varepsilon \ll \varepsilon_{\rm nB}$ the scattering amplitude formally vanishes: $\Lambda^{(\rm ren)} \approx \pi(-\varepsilon)^{1/2}$. It means, in particular, that exactly at the van Hove singularity a quasi-one-dimensional system tends to become an ideal conductor with zero resistivity. In the following section we will demonstrate that for finite concentration of impurities the resistivity remains finite, though very small: it is proportional not to n, but to n^3.

5.2.2 Single-impurity Non-Born effects in resistivity

Physically the effect of renormalization is manifested in the scattering time $\tau(\varepsilon)$ in which λ should be replaced by $\Lambda^{(\rm ren)}(\varepsilon)$. Similar to the Born case, for

$$\tau^{-1}(\varepsilon) \ll \varepsilon \tag{5.23}$$

the scattering is effectively weak (though non-Born!) so that only the single impurity effects should be taken into account and one can use the standard Drude formula with properly renormalized scattering time. In this Section we concentrate on this "weak non-Born scattering" regime. We will consider the cases of repulsing and attracting impurities separately.

Certainly, there were some theoretical approaches to the non-Born effects in quasi-one-dimensional systems in the past. S. Hügle and R. Egger [8] studied the smearing of Van Hove singularities within the self-consistent Born approximation. In contrast with our work, instead of the quadratic spectrum of electrons they considered more realistic linear spectrum, characteristic of carbon nanotubes. This difference, however, is not essential, as far as one is interested only in the shape of the Van Hove singularities: it may actually be reduced to redefinition of some constants. What is much more important, instead of considering individual impurities, the Authors of [8] introduced the disorder in the form of gaussian white noise. Such an approach does not allow to find the single-impurity non-Born effects which, as we have seen, are crucial at low concentrations $n_2 \ll n_2^{(c)}$. So, their results are applicable to impurities only at high concentration $n_2 \gg n_2^{(c)}$.

5.2.3 Repulsing impurities

For weak repulsive impurities ($\lambda > 0$, $|\lambda| \ll 1$) the imaginary part of Λ can be neglected and we get

$$\frac{\rho(\varepsilon)}{\rho_0} = \frac{\tau_0}{\tau} = \frac{|\Lambda^{(\mathrm{ren})}|^2}{\lambda^2}\left(1 + \frac{1}{\pi\sqrt{\varepsilon}}\theta(\varepsilon)\right) = \\ = \begin{cases} \dfrac{1}{\lambda}\dfrac{1}{\epsilon^{1/2}+\epsilon^{-1/2}}, & \text{for } \epsilon > 0, \\ \dfrac{1}{(1+|\epsilon|^{-1/2})^2}, & \text{for } \epsilon < 0, \end{cases} \tag{5.24}$$

This dependence is plotted in Fig. 2:

So, for $\varepsilon > 0$ both the scattering rate and the resistivity have smooth maxima at $\varepsilon = \varepsilon_{\mathrm{nB}}$) with the value at maximum

$$\frac{1}{\tau^{(+)}_{\min}} = \frac{1}{2\lambda\tau_0} = \frac{n\lambda}{\pi^2}, \tag{5.25}$$

or, in dimensional variables

$$\frac{1}{\tau^{(+)}_{\min}} = \frac{2n_2}{m^*}\lambda, \quad \frac{\rho^{(+)}_{\max}}{\rho_0} = \frac{1}{\lambda} \gg 1. \tag{5.26}$$

For $\varepsilon < 0$ the scattering rate grows monotonically with growing $|\varepsilon|$ and saturates at $\tau^{-1} = \tau_0^{-1}$ for $|\varepsilon| \gg \varepsilon_{\mathrm{nB}}$.

The non-Born effects somewhat suppress the resistivity, compared to the Born results. For repulsing impurities this is true for all ε but the strongest effect is expected for $|\varepsilon| \lesssim \varepsilon_{\mathrm{nB}}$.

5.2.4 Attracting impurities

For attracting impurities the renormalized scattering amplitude has a pole in the complex plane of ε at

$$\varepsilon = \varepsilon_{\mathrm{nB}}(-1 + 2i\lambda), \tag{5.27}$$

close to the real axis. This fact indicates the existence of a quasistationary state. We have to take into account the imaginary part of Λ that keeps trace of the decay of this state: otherwise the pole would move to the real axis and there will be a nonphysical divergence of amplitude. However, this is only necessary in the narrow vicinity of the resonance at

$|\varepsilon| = \varepsilon_{\mathrm{nB}}$. So we can write

$$\frac{\rho(\varepsilon)}{\rho_0} = \frac{\tau_0}{\tau} =$$

$$= \begin{cases} \dfrac{1}{|\lambda|} \dfrac{1}{\epsilon^{1/2} + \epsilon^{-1/2}}, & \text{for } \epsilon > 0, \\ \dfrac{1}{(1 - |\epsilon|^{-1/2})^2}, & \text{for } \epsilon < 0,\ |1 - |\epsilon|| \gg |\lambda|, \\ \dfrac{4}{(1 - |\epsilon|)^2 + 4\lambda^2}, & \text{for } \epsilon < 0,\ |1 - |\epsilon|| \lesssim |\lambda|, \end{cases} \tag{5.28}$$

This result is plotted Fig. 3.

Thus, for $\varepsilon > 0$ (and also for $\varepsilon < 0$ but $|\varepsilon| \ll \varepsilon_{\mathrm{nB}}$) the behaviour of the renormalized scattering rate for attracting impurities is identical to that of repulsing ones. Their behaviours are very different, however, for $\varepsilon < 0$ (and not small $|\varepsilon|$ compared to $\varepsilon_{\mathrm{nB}}$). While for repulsive impurities both the rate τ^{-1} and the resistivity ρ smoothly and monotonically increase with $|\varepsilon|$, for attracting impurities they first grow, reach sharp maxima at $\varepsilon = -\varepsilon_{\mathrm{nB}}$ and only then decrease, saturating at $\tau^{-1} = \tau_0^{-1}$ and $\rho = \rho_0$ for $|\varepsilon| \gg \varepsilon_{\mathrm{nB}}$. The maximum has a Lorenzian shape:

$$\rho(\varepsilon) = \rho_{\mathrm{max}}^{(-)} \frac{\pi \Gamma_{\mathrm{hom}}}{2} L(\varepsilon + \varepsilon_{\mathrm{nB}}, \Gamma_{\mathrm{hom}}), \tag{5.29}$$

$$L(x, \gamma) \equiv \frac{\gamma/2}{\pi\,(x^2 + (\gamma/2)^2)}. \tag{5.30}$$

The width of maximum (homogeneous broadening)

$$\Gamma_{\mathrm{hom}} \sim 4|\lambda|\varepsilon_{\mathrm{nB}} = \frac{4|\lambda|^3}{\pi^2} \ll \varepsilon_{\mathrm{nB}}, \tag{5.31}$$

is relatively small. This decay is due to small (but finite) probability of scattering to the bands other than the N-band. The height of the maximum is universal – it does not depend on the strength of impurities λ. In dimensional variables:

$$\rho_{\mathrm{max}}^{(-)} = \frac{4n_2}{e^2 m^* R E}. \tag{5.32}$$

The scattering rate at maximum is even more universal:

$$\frac{1}{\tau_{\mathrm{min}}^{(-)}} = \frac{1}{\lambda^2 \tau_0} = \frac{2n}{\pi^2} = \frac{4n_2}{m^*}, \tag{5.33}$$

it depends neither on λ, nor on R or E.

5.2.5 Multi-impurity effects. The central dip in resistivity

In the previous section we have implicitly assumed the concentration of impurities n to be so low that scattering amplitude at certain impurity could not be affected by the presence of all the others: $\tau^{-1}(\varepsilon) \ll \varepsilon$. Let us first derive the condition that would justify this assumption. We have found that the non-Born effects are negligible for $\varepsilon \gtrsim \varepsilon_{\rm nB}$. On the other hand, if one totally neglects the non-Born effects, then, as it follows from (4.22), the scattering effects lead to the saturation of both the density of states and the conductivity for $\varepsilon \lesssim \varepsilon_{\rm min}$. These two facts taken together mean that for $\varepsilon_{\rm nB} \ll \varepsilon_{\rm min}$ the non-Born effects do not have chance to show up at all. On the contrary, for $\varepsilon_{\rm min} \ll \varepsilon_{\rm nB}$ the scattering only comes into play at $\varepsilon \ll \varepsilon_{\rm nB}$ where the non-Born effects are already huge. Thus, looking at the expressions (4.36) for $\varepsilon_{\rm min}$ and (5.21) for $\varepsilon_{\rm nB}$ we conclude that the non-Born effects are relevant for $n < n_c$, where

$$n_c \sim |\lambda|, \tag{5.34}$$

while for $n > n_c$ the Born approximation is justified for all ε and the results of section 4 are applicable.

In this Section we are going to study the effect of scattering at low concentration $n \ll n_c$ but also at very low $|\varepsilon|$ at the same time. We will show that the presence of other impurities ultimately becomes essential in the narrow vicinity of the Van Hove singularity – at certain energy scale $\varepsilon_{\rm min}^{\rm (nB)} \ll \varepsilon_{\rm nB}$.

In the case of developed non-Born regime, for $\varepsilon \ll \varepsilon_{\rm nB}$ we have $\Lambda g \gg 1$, so that

$$\left|\Lambda^{\rm (ren)}(\varepsilon)\right|^2 \approx \pi^2|\varepsilon|. \tag{5.35}$$

We see that the rate $1/\tau$ ceases to depend on λ and becomes universal: independent on the characteristics of impurities:

$$\tau^{-1}(\varepsilon) = 2|\varepsilon|n\left(1 + \frac{1}{\pi\sqrt{\varepsilon}}\theta(\varepsilon)\right). \tag{5.36}$$

It should be stressed that the scattering rate decreases as the Fermi level approaches the Van Hove singularity from either side and formally vanishes at $\varepsilon = 0$. Taken seriously, it would mean that exactly at singularity the system has zero residual resistivity. Of course, we expect that taking scattering in account will remove this paradox.

To demonstrate this, we have to incorporate the scattering in the result (5.36). Again, as in Section 4.4.2 we notice that the above calculations only make sense for $\tau^{-1}(\varepsilon) \ll \varepsilon$, so that the dip in the resistivity predicted by (5.36) will be rounded at certain $\varepsilon \sim \varepsilon_{\min}^{(\mathrm{nB})}$, where $\varepsilon_{\min}^{(\mathrm{nB})}$, however, is not given by (4.36) any more because the expression for the scattering time (5.36) differs from (4.7): it has been changed by the non-Born effects. So, the self-consistency condition $\tau^{-1}(\varepsilon) \sim \varepsilon$ for $\varepsilon_{\min}^{(\mathrm{nB})}$ reads

$$\tau^{-1}\left(\varepsilon = +\varepsilon_{\min}^{(\mathrm{nB})}\right) = \frac{2n}{\pi}\sqrt{\varepsilon_{\min}^{(\mathrm{nB})}} \sim \varepsilon_{\min}^{(\mathrm{nB})}, \tag{5.37}$$

from where immediately follows

$$\varepsilon_{\min}^{(\mathrm{nB})} = (n/\pi)^2 . \tag{5.38}$$

Comparing (5.38) to (5.21) we see that, indeed, the scattering effects bring the renormalization of the amplitude $\Lambda^{(\mathrm{ren})}(\varepsilon)$ to stop at some small, but nonzero value.

The results (5.35) and (5.38) were obtained under the assumption $\varepsilon > 0$ so we need yet to discuss the scattering effects for $\varepsilon < 0$. Here we get

$$\tau^{-1}(\varepsilon) = 2n|\varepsilon| \ll |\varepsilon|, \tag{5.39}$$

which formally means that for negative ε the scattering does not affect the result (5.36) for all values of $|\varepsilon|$, down to $\varepsilon = 0$! This is, of course, not quite true because, due to scattering effects, the discontinuity in the density of states at $\varepsilon = 0$ should be smoothed and $1/\tau(\varepsilon)$ should remain of the order $1/\tau_{\max}$ also for $\varepsilon < 0$ in the range $|\varepsilon| \lesssim \varepsilon_{\min}^{(\mathrm{nB})}$.

Thus, in the strongly non-Born domain $n \ll n^{(\mathrm{nB})}$ we encounter the similar paradox as in the Born case at $n \gg n^{(\mathrm{nB})}$. Namely, the above consideration gives nonmatching estimates on the opposite sides of the interval $|\varepsilon| \lesssim \varepsilon_{\min}^{(\mathrm{nB})}$:

$$\tau^{-1} \sim \begin{cases} n^2, & \text{for } \varepsilon > 0,\ \varepsilon \sim n^2, \\ n^3, & \text{for } \varepsilon < 0,\ |\varepsilon| \sim n^2. \end{cases} \tag{5.40}$$

The resolution of this paradox is also similar to that in the Born case: there is a quasifold at certain $\varepsilon = \varepsilon_{\mathrm{bi}}^{(\mathrm{nB})} \equiv q_{\mathrm{bi}}\varepsilon_{\min}^{(\mathrm{nB})}$, (with $q_{\mathrm{bi}} < 0$, $|q_{\mathrm{bi}}| \sim 1$) where the scattering rate undergoes a dramatic drop, so that

$$\tau^{-1}(\varepsilon) \sim \begin{cases} 2n|\varepsilon|, & \text{for } \varepsilon < \varepsilon_{\text{bi}}^{(\text{nB})}, \\ n^2, & \text{for } \varepsilon > \varepsilon_{\text{bi}}^{(\text{nB})},\ \varepsilon \lesssim \varepsilon_{\text{min}}^{(\text{nB})}, \\ 2n\sqrt{\varepsilon}/\pi, & \text{for } \varepsilon \gg \varepsilon_{\text{min}}^{(\text{nB})}, \end{cases} \tag{5.41}$$

and the weakest scattering is realized at some $\varepsilon = \varepsilon_{\text{dip}}^{(\text{nB})}$ below $\varepsilon_{\text{bi}}^{(\text{nB})}$:

$$\frac{1}{\tau_{\text{max}}} \approx \frac{1}{\tau(\varepsilon = \varepsilon_{\text{dip}}^{(\text{nB})})} \sim n\varepsilon_{\text{min}}^{(\text{nB})} \sim n^3, \tag{5.42}$$

or, in dimensional variables

$$\frac{1}{\tau_{\text{max}}} \sim \frac{(2\pi R)^4 [n_2]^3}{m^*}. \tag{5.43}$$

This result is supported by the calculations within the "self-consistent non-Born approximation", given in Appendix B. Thus, we conclude that the minimal value of the scattering rate and, consequently, the minimal value of resistivity is attained a little bit to the left from the initial (nonrenormalized) position of the Van Hove singularity, at $\varepsilon = \varepsilon_{\text{dip}}^{(\text{nB})} \sim -n^2$ and

$$\rho_{\text{min}} = \frac{1}{e^2 RE} \frac{1}{\tau_{\text{max}}} \sim \frac{(2\pi R)^4 [n_2]^3}{e^2 m^* RE} \tag{5.44}$$

This minimal value depends neither on sign, nor on magnitude of λ and is much less than the standard resistivity:

$$\frac{\rho_{\text{min}}}{\rho_0} \sim \frac{n^2}{\lambda^2} = \left(\frac{n}{n_c}\right)^2 \ll 1. \tag{5.45}$$

The dependence $\rho(\varepsilon)$ near the minimum is shown in Fig 10.

5.3 Non-Born effects in a strip

5.3.1 General results

From the results of our study of the scattering of electrons in a tube we know that non-Born effect strongly modify the above picture for the energies, sufficiently close to the Van Hove singularities. We expect quite similar phenomena also for the case of a strip. We will see, however, that in the case of a strip there is important specifics, absent for a tube.

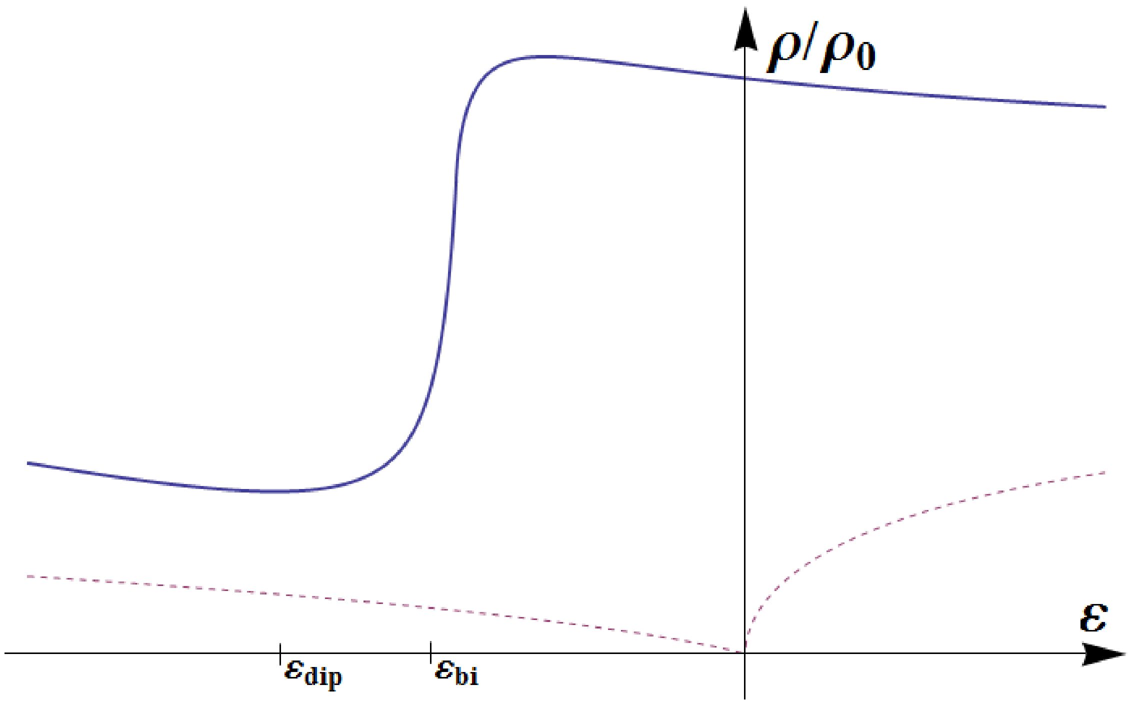

Figure 10. The energy-dependence of resistivity near the minimum. Dashed line – for $n \to 0$, solid line – for finite n.

Namely, the renormalization of the scattering amplitude depends on the spatial position of the impurity with respect to the edges of the strip. It introduces nonequivalence of different impurities and, therefore, new nontrivial (and quite unexpected!) physics.

Solving the standard Dyson equation

$$G(\mathbf{r_1}, \mathbf{r_2}) = G_0(\mathbf{r}_1, \mathbf{r}_2) + G_0(\mathbf{r}_1, \mathbf{r}_i) V G(\mathbf{r}_i, \mathbf{r}_2) \tag{5.46}$$

where $\mathbf{r_i}$ is the position of the impurity and G_0 is the free Green function. Then we arrive at the following expression for the renormalized scattering amplitude:

$$\hat{V}_{\mathrm{ren}} = \frac{\hat{V}}{1 - V G_0(\mathbf{r}_i, \mathbf{r}_i)} \tag{5.47}$$

With the help of expressions (3.13) and (3.14), we can easily write

$$G_E^{(R)}(\mathbf{r}_i, \mathbf{r}_i) = -\sum_{m=1}^{\infty} \frac{2\pi \sin^2(\pi(m+1)\xi)}{\sqrt{-\varepsilon_m}}. \tag{5.48}$$

The imaginary part of (5.48) is finite, it is related to the density of states

$$\nu(\varepsilon) = -\frac{1}{\pi} \mathrm{Im} \int G_E^{(R)}(\mathbf{r}, \mathbf{r}) d\mathbf{r}. \tag{5.49}$$

The problem is that, due to contribution of terms with large m, the real part of (5.48) diverges. To understand the reason for this ultraviolet divergency and to find a proper way of its regularization, it is instructive to rewrite $G(\mathbf{r}_i, \mathbf{r}_i)$ in a form

$$G(\boldsymbol{r}_i, \boldsymbol{r}_i) = \mathcal{G}(0) + g(\varepsilon, \xi_i), \tag{5.50}$$

$$g(\varepsilon, \xi_i) = -\mathcal{G}(2\xi_i) + \\ + \sum_{n=1}^{\infty} \left\{2\mathcal{G}(2n) - \mathcal{G}(2n + 2\xi_i) - \mathcal{G}(2n - 2\xi_i)\right\}, \tag{5.51}$$

where $\mathcal{G}$ is the free Green-functions on an infinite plane. This result can be obtained by the method of images. In the quasiclassical limit $N \gg 1$ all the terms entering $g(\varepsilon, \xi_i)$ are quasiclassical and can be written in the form

$$\mathcal{G}(\mathbf{r}) \approx -\frac{m^*}{2}\sqrt{\frac{2i}{\pi p_F r}}e^{ip_F r}, \quad p_F = 2\pi\sqrt{\varepsilon_0} \gg 1. \tag{5.52}$$

In particular, near the Van Hove singularity, where $\varepsilon_0 = (N/2)^2 + \varepsilon$ with $\varepsilon \ll 1$,

$$p_F \approx \pi\left(N + \frac{2\varepsilon}{N}\right), \tag{5.53}$$

$$\mathcal{G}(\mathbf{r}) \approx -\pi\sqrt{\frac{2i}{Nr}} \exp\left\{\pi i r\left(N + \frac{2\varepsilon}{N}\right)\right\} \tag{5.54}$$

Using the language of the Feynman path integrals, one can interpret each term in (5.51) as a contribution of a bunch of paths close to certain classical trajectory that starts and ends at the same point $\mathbf{r}_i$. We are interested in the results valid in the vicinity of the Van Hove singularity, therefore $|\varepsilon| \ll 1$ and we expect that the sum in (5.51) is dominated by large n, so that the summation can be replaced by integration. As a result:

$$g(\varepsilon, \xi_i) \approx -4\pi\sin^2(\pi N\xi_i)\sqrt{\frac{i}{N}}\int_0^\infty \frac{dn}{\sqrt{n}} \exp\left\{\frac{4\pi i\varepsilon n}{N}\right\} = \\ = -\frac{2\pi\sin^2(\pi N\xi_i)}{\sqrt{-\varepsilon}} \tag{5.55}$$

Thus, we conclude, that (at least near the Van Hove singularity) the quasiclassical part of $G_0(\mathbf{r}_i, \mathbf{r}_i)$ corresponds to the resonant term in the sum (5.48), while the divergent contribution of terms with $m \to \infty$ should be associated with the $\mathcal{G}(0)$ term in (5.50).

The latter term is by no means quasiclassical: it is a contribution of short purely quantum loops (their length being $\sim p_F^{-1} \ll D$), that typically have no chance to reach any boundary of the strip. Formally this contribution diverges for the case of infinitely-short-range potential of impurity, so that to accurately evaluate this contribution one has to introduce short but finite range $r_0 \ll p_F^{-1}$ of potential and then tend it to zero only at the end of calculations. An explicit treating of this problem can be avoided if one notes that taking into account only the first term in (5.51), when solving the Dyson equation (5.46) gives just the renormalization effect in the infinite 2D plane. Therefore, from the phenomenological point of view we can simply discard this term in the sum (5.50) and simultaneously replace everywhere the Born scattering amplitude λ by the exact complex scattering amplitude Λ for a lonely impurity on an infinite 2D plane. It should satisfy the unitarity condition $\mathrm{Im}\Lambda = -|\Lambda|^2$ so that for our case (weak scattering $|\lambda| \ll 1$) one can write

$$\Lambda \approx \lambda - i\lambda^2, \quad \lambda = m^* V/2, \tag{5.56}$$

Although for $|\lambda| \ll 1$ the exact scattering amplitude Λ only slightly differs from the Born one λ, we will see that the difference is essential in some cases.

Now we are prepared to write down the scattering operator $\hat{V}_{\rm ren}$. Combining the effects, coming from the $\mathcal{G}(0)$-term (i.e., $\lambda \to \Lambda$) and those, coming from the g-term (i.e., the quasiclassical corrections due to presence of the strips boundary) we arrive at

$$\hat{V}_i^{(\rm ren)} = \frac{\hat{V}}{1 + \Lambda_i/\pi\sqrt{-\varepsilon}}, \quad \Lambda_i = 2\Lambda \sin^2(\pi N \xi_i). \tag{5.57}$$

5.3.2 Non-Born scattering rates

With the help of (5.57) we can easily find matrix elements of operator $\hat{V}_{\rm ren}$, the scattering rates τ_m^{-1} and the resistivity:

$$V_{kk'mm'}^{(\rm ren)}(\xi_i, z_i) \approx \frac{2V}{D\left\{1 + 2\Lambda \sin^2(\pi N \xi_i)/\pi\sqrt{-\varepsilon}\right\}} \exp\{i(k-k')z_i\} \sin(\pi(m+1)\xi_i) \sin(\pi(m'+1)\xi_i), \tag{5.58}$$

$$\begin{aligned} \frac{\tau_0}{\tau_m(\varepsilon)} &= \sum_{m'} \int_0^1 d\xi \frac{[1-\cos(2\pi(m+1)\xi)][1-\cos(2\pi(m'+1)\xi)]}{|1 + 2\Lambda \sin^2(\pi N \xi)/\pi\sqrt{-\varepsilon}|^2} \frac{\theta(\varepsilon_{m'})}{\pi\sqrt{\varepsilon_{m'}}} \approx \\ &\approx \int_0^1 d\xi \frac{1-\cos(2\pi(m+1)\xi)}{|1 + 2\Lambda \sin^2(\pi N \xi)/\pi\sqrt{-\varepsilon}|^2} \left(1 + 2\sin^2(\pi N \xi) \frac{\theta(\varepsilon)}{\pi\sqrt{\varepsilon}}\right) \end{aligned} \tag{5.59}$$

Compared to the Born results (4.9),(4.12) we have in (5.58) an additional renormalization factor in the denominator depending on the distance ξ_i between impurity and the edge of the strip. The integration over ξ in (5.59) represents the averaging over the positions of impurities.

The expression for the resistivity involves summation over all the current carrying (non-resonant) states m. This summation suppresses the rapidly oscillating term, proportional to $\cos(2\pi(m+1)\xi)$, and therefore all the nonresonant are characterized by the same decay rate: $\tau_m \equiv \tau_{\rm nonres}(\varepsilon)$ for $m \neq N$, where

$$\frac{\tau_0}{\tau_{\rm nonres}(\varepsilon)} \approx \int_0^1 d\xi \frac{1+2\sin^2(\pi N\xi)\frac{\theta(\varepsilon)}{\pi\sqrt{\varepsilon}}}{\left|1+2\Lambda\sin^2(\pi N\xi)/\pi\sqrt{-\varepsilon}\right|^2}, \tag{5.60}$$

The decay rate for the resonant state $\tau_N = \tau_{\rm res}$, where

$$\frac{\tau_0}{\tau_{\rm res}(\varepsilon)} \approx \int_0^1 d\xi \frac{\left(1+2\sin^2(\pi N\xi)\frac{\theta(\varepsilon)}{\pi\sqrt{\varepsilon}}\right)2\sin^2(\pi N\xi)}{\left|1+2\Lambda\sin^2(\pi N\xi)/\pi\sqrt{-\varepsilon}\right|^2}, \tag{5.61}$$

The obtained results imply that $p_F r_0 \ll 1$ (i.e., the Fermi wave-length $\lambda_F \equiv p_F^{-1}$ is much larger that the radius of potential r_0). It is quite possible for isoelectronic impurities in semiconductors with low concentration of charge carriers.

It is convenient to introduce the characteristic energy scale for non-Born effects:

$$\varepsilon_{\rm nB} = (\lambda/\pi)^2, \qquad \epsilon = \varepsilon/\varepsilon_{\rm nB} \tag{5.62}$$

Then, using, instead of ξ, a new variable $t = \sin^2(\pi N\xi)$, we arrive at the following results for the resistivity

$$\frac{\rho(\varepsilon)}{\rho_0} = \frac{\tau_0}{\tau_{\rm nonres}(\varepsilon)} = \frac{\nu(\varepsilon)}{\nu_0}\begin{cases} F(\epsilon), & \varepsilon > 0, \\ \tilde{F}(\epsilon), & \varepsilon < 0, \end{cases} \tag{5.63}$$

where

$$F(\epsilon,\lambda) = \frac{2}{\pi}\int_0^1 \frac{t^{1/2}(1-t)^{-1/2}dt}{|1+2t\,{\rm sign}(\lambda)(1+i|\lambda|)/\sqrt{-\epsilon}|^2}, \tag{5.64}$$

$$\tilde{F}(\epsilon,\lambda) = \frac{1}{\pi}\int_0^1 \frac{t^{-1/2}(1-t)^{-1/2}dt}{|1+2t\,{\rm sign}(\lambda)(1+i|\lambda|)/\sqrt{-\epsilon}|^2}, \tag{5.65}$$

The functions $F(\epsilon)$ and $\tilde{F}(\epsilon,\lambda)$ are evaluated in Appendices C and D. It is convenient to continue the discussion of the resistivity separately for the cases of repulsing ($\lambda > 0$) and attracting ($\lambda < 0$) impurities.

5.3.3 Non-Born resistivity: repulsing impurities

For repulsing impurities ($\lambda > 0$) using Eq(5.63) and the results (C.1) for the function $F(\epsilon)$ we find that for $\epsilon > 0$

$$\frac{\rho(\epsilon)}{\rho_0} = \frac{1}{\lambda}\left(\frac{\sqrt{1+4/\epsilon}-1}{2(1+4/\epsilon)}\right)^{1/2} \approx$$
$$\approx \frac{1}{\lambda}\begin{cases} \epsilon^{1/4}/2, & \text{for } \epsilon \ll 1, \\ 1/\sqrt{\epsilon}, & \text{for } \epsilon \gg 1, \end{cases} \tag{5.66}$$

while for $\epsilon < 0$, with the help of (D.1) we obtain

$$\frac{\rho(\epsilon)}{\rho_0} =$$
$$= \frac{|\epsilon|^{1/4}(1+|\epsilon|^{1/2})}{(2+|\epsilon|^{1/2})^{3/2}} \approx \begin{cases} \frac{1}{2\sqrt{2}}|\epsilon|^{1/4}, & \text{for } |\epsilon| \ll 1, \\ 1, & \text{for } |\epsilon| \gg 1, \end{cases} \tag{5.67}$$

The maximum

$$\frac{\rho^{(+)}_{\max}}{\rho_0} = \frac{1}{2\sqrt{2}\lambda} \tag{5.68}$$

is reached at $\varepsilon = \frac{4}{3}\varepsilon_{\rm nB}$. Thus, the maximum of the resistivity in the case of strip is somewhat broadened, compared to that in the case of cylinder.

5.3.4 Paradox: at $|\varepsilon| \ll \varepsilon_{\rm nB}$ weak impurities scatter more effectively than strong ones!

It is important to note a different (compared to the case of cylinder) law $\rho \propto |\epsilon|^{1/4}$ (5.66), (5.67) of vanishing $\rho(\epsilon)$ at $\epsilon \to 0$. For the cylinder the analogous law isTo elucidate the reason for this difference let's analyze the integral over ξ in (5.64) and (5.65). While for $|\epsilon| \gtrsim 1$ the entire interval $0 < \xi < 1$ (or $t \sim 1$) contributes to this integral, for $|\epsilon| \ll 1$ the main contribution comes from small $t = \sin^2(\pi N\xi) \sim \sqrt{|\epsilon|} \ll 1$. It means that scattering at "weak" impurities, situated close to nodes of the transversal wave-function of the resonant band, turn out to be more effective in scattering, than the strong ones, sitting close to antinodes. How it can possibly be?

The reason is that the strong "non-Born self-screening" of the scattering amplitudes $\Lambda_i^{(\mathrm{ren})}(\varepsilon)$, occurring at $\varepsilon \ll \varepsilon_{\mathrm{nB}}$, is nonlinear: it is stronger for the impurities i with larger bare amplitudes Λ_i. As a result, scattering by the impurities with large bare $\Lambda_i \gg \sqrt{|\epsilon|}$ turns out to be suppressed stronger than scattering by those with some moderately small optimal $\Lambda_i \sim \sqrt{|\epsilon|}$.

$$\Lambda_i \sim \Lambda_{\mathrm{opt}}(\epsilon) = \sqrt{|\epsilon|} \ll 1. \tag{5.69}$$

Thus, we arrive at paradoxical and exciting conclusion: though for small $|\epsilon| \ll 1$ the scattering is generally suppressed, the residual weak scattering is dominated by presumably ineffective impurities, that sit relatively close to the nodes (at distances $\xi \sim \lambda_F |\epsilon|^{1/4} \ll \lambda_F$) and have, therefore, anomalously small bare scattering amplitudes. As one of the consequences, the resistivity of a strip vanish with $|\epsilon| \to 0$ slower than the resistivity of a cylinder.

5.3.5 Non-Born resistivity for a strip: attracting impurities

Above the Van Hove singularity, for $\varepsilon > 0$ the scattering rate depends only on λ^2, so that the case of attracting impurities does not differ from that of the repulsing ones and the resistivity for $\varepsilon > 0$ is described by the formula (5.66). Below the Van Hove singularity, for $\varepsilon > 0$, however, there are some impressive effects, specific for the attractive impurities. They are mostly due to the presence of quasistationary states.

5.3.6 Quasistationary states

As we have shown in [34] in a quasi-one-dimensional system each attracting impurity forms a quasistationary state below each subband of transverse quantization. These states arise for arbitrary weak attraction, without a threshold. Moreover, for weak attraction the quasistationary states are even better defined, than for strong one: the quality factor (i.e., the ratio of the energy to the decay rate) increases with decreasing strength of attraction. The quasistationary states are manifested as poles of the renormalized scattering amplitude

$$\Lambda^{(\mathrm{ren})}(\varepsilon) = \frac{\Lambda}{1 + \Lambda[2\sin^2(\pi N \xi)]/\pi\sqrt{-\varepsilon}}, \tag{5.70}$$

in the complex ε plane.

In contrast to the case of cylinder, in a strip to each impurity i corresponds its own value of the scattering amplitude $\Lambda_i \approx (\lambda - i\lambda^2) 2\sin^2(\pi N\xi_i)$, so that energies of the quasistationary states are different at different impurities:

$$\epsilon_{\rm qs}(\xi_i) = 4\sin^4(\pi N\xi_i)(-1+2i\lambda), \quad \epsilon_{\rm qs} = \varepsilon_{\rm qs}/\varepsilon_{\rm nB}. \tag{5.71}$$

Let's forget for a while about small imaginary part of $\epsilon_{\rm qs}$; we will easily restore it in a due time. We see that values of $\epsilon_{\rm qs}$ are confined in an interval $-4 < \epsilon_{\rm qs} < 0$. Since ξ_i is a random variable homogeneously distributed between 0 and 1, the distribution function for $\varepsilon_{\rm qs}$ is

$$\begin{aligned} P(\varepsilon_{\rm qs}) = \int_0^1 d\xi \delta\left[\epsilon_{\rm qs} + 4\sin^4(\pi N\xi)\right] = \\ = \frac{1}{\pi\sqrt{|\epsilon_{\rm qs}|(4-|\epsilon_{\rm qs}|)}}. \end{aligned} \tag{5.72}$$

Thus, for $\epsilon < -4$ the scattering is only possible to usual states of continuous spectrum, while for $-4 < \epsilon < 0$, in principle, both continuum and the quasistationary states may serve as final states of scattering processes. In fact, we will see that quasistationary states dominate everywhere in this range, except narrow interval at the boundary $\epsilon = -4$, with a width being of order $\mathrm{Im}\,\epsilon_{\rm qs} \sim |\lambda|$ – the decay rate of the quasistationary states.

5.3.7 Nonresonant scattering

Since there are no quasistationary states in the energy range $\epsilon < -4$, here we can simply put $\lambda = 0$ in (5.65). The corresponding integral is evaluated in (D.5) and we get

$$\frac{\rho(\epsilon)}{\rho_0} = \tilde{F}(\epsilon, 0) = \frac{|\epsilon|^{1/4}\left(\sqrt{|\epsilon|}-1\right)}{\left(\sqrt{|\epsilon|}-2\right)^{3/2}} \approx \begin{cases} 8\sqrt{2}\left(|\epsilon|-4\right)^{-3/2}, & \text{for } |\epsilon|-4 \ll 1 \\ 1, & \text{for } |\epsilon| \gg 1 \end{cases} \tag{5.73}$$

5.3.8 Resonant scattering

For any given energy in the range $-4 < \epsilon < 0$ the leading contribution to the resistivity comes from the scattering on resonant impurities with such ξ_i that $\epsilon_{\rm qs}(\xi_i) \approx \epsilon$. In contrast with the previous case, to avoid divergency, here we have to take into account the imaginary part of $\epsilon_{\rm qs}(\xi_i)$. The corresponding calculations are presented in Appendix D, resulting in (D.4).

$$\frac{\rho(\epsilon)}{\rho_0} = \tilde{F}(\epsilon, \lambda) = \frac{1}{|\lambda|}\left(\frac{\sqrt{|\epsilon|}}{2-\sqrt{|\epsilon|}}\right)^{1/2} \approx \frac{1}{|\lambda|}\begin{cases} |\epsilon|^{1/4}, & \text{for } |\epsilon| \ll 1, \\ 2\sqrt{2}(4-|\epsilon|)^{-1/2}, & \text{for } 4-|\epsilon| \ll 1. \end{cases} \tag{5.74}$$

5.3.9 Van Hove-like feature in resistivity: resonant scattering on strongest impurities

Combining (5.73) and (5.74), we arrive at

$$\frac{\rho(\epsilon)}{\rho_0} = \begin{cases} 8\sqrt{2}\,(|\epsilon|-4)^{-3/2}, & \text{for } 4+\epsilon \to -0, \\ \dfrac{2\sqrt{2}}{|\lambda|}(4-|\epsilon|)^{-1/2}, & \text{for } 4+\epsilon \to +0. \end{cases} \tag{5.75}$$

Thus, at $\epsilon = -4$ the resistivity has an asymmetric (formally divergent) peak, somewhat similar to Van Hove singularity.

This entire feature is nothing else, but the inhomogeneously broadened (due to the dispersion of scattering amplitudes λ_i for different impurities) peak of the resonant scattering, that in the case of cylinder (where all λ_i are identical) was manifested as a sharp line. The square-root divergency in (5.75) reflects the fact that the values of $\epsilon_{\rm qs}$ are confined in the range $-4 < \epsilon_{\rm qs} < 0$, the lower boundary corresponding to "strongest" impurities – those, sitting in the antinodes of the transversal wave-function of the resonant band. The density of quasistationary states (5.72) indeed diverges at $\epsilon = 0$ and at $\epsilon = -4$. It is just the divergency of $P(\epsilon)$ at $\epsilon \to -4$, that is manifested as a Van Hove-like singularity in the resistivity $\rho(\epsilon)$. Let us stress again, that scattering near the peak is dominated by strongest impurities.

The Van Hove-like singularity at $|\epsilon| \to 4$ is indeed smeared in the range $||\epsilon|-4| \lesssim |\lambda|$, where the contributions of both types of final states – the continuum and the quasistationary states – are comparable. To elucidate this mixing one should accurately treat the Eq(D.2) directly, without using an approximate formula (D.3). As a result of calculations (see Appendix D), we obtain

$$\frac{\rho(\epsilon)}{\rho_0} = \tilde{F}(\tilde{\epsilon},\lambda) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{a^2+1}-a}{|\lambda|^3(a^2+1)}\right)^{1/2} \approx$$

$$\approx \begin{cases} \dfrac{8\sqrt{2}}{(|\epsilon|-4)^{3/2}}, & \text{for } 8|\lambda| \ll |\epsilon|-4 \ll 1, \\ \dfrac{2\sqrt{2}}{|\lambda|(4-|\epsilon|)^{1/2}}, & \text{for } 8|\lambda| \ll 4-|\epsilon| \ll 1, \end{cases} \tag{5.76}$$

where $a = (|\epsilon|-4)/8|\lambda|$. Naturally, the asymptotics of (5.75) and (5.76) overlap at $|\lambda| \ll ||\epsilon|-4| \ll 1$. The function F reaches its maximum $F^{\max} = \frac{3^{3/4}}{2\sqrt{2}|\lambda|^{3/2}}$ at $a = -3^{-1/2}$, so that the maximal resistivity

$$\frac{\rho_{\max}^{(-)}}{\rho_0} = \frac{3^{3/4}}{2\sqrt{2}|\lambda|^{3/2}} \tag{5.77}$$

is reached at $\epsilon = -4\left(1 - \frac{2|\lambda|}{\sqrt{3}}\right)$. The width of this maximum $\Gamma \sim |\lambda| \ll 1$.

Thus, we conclude that the left peak of resistivity (that exists only for attracting impurities) is higher, than the right one: its height is proportional to $|\lambda|^{-3/2}$ instead of $|\lambda|^{-1}$. On the other hand, due to the inhomogeneous broadening, it is lower than it would be in the case of cylinder: $|\lambda|^{-3/2}$ instead of $|\lambda|^{-2}$.

5.3.10 Low energy resonant scattering on weak impurities

The divergency of $P(\epsilon)$ at $\epsilon \to 0$ does not lead to divergency of $\rho(\epsilon)$ at $\epsilon \to 0$: $\rho(\epsilon)$ still goes to zero but for all energies is much larger than in the case of repulsing impurities. The strong scattering at quasistationary states with low binding energies gives additional large factor $|\lambda|^{-1}$ in $\rho(\epsilon)$ dependence at $\epsilon < 0$, $|\epsilon| \ll 1$:

$$\frac{\rho(\epsilon)}{\rho_0} = \frac{\tau_0}{\tau_{\text{nonres}}(\epsilon)} \approx \frac{|\epsilon|^{1/4}}{\sqrt{2}} \begin{cases} 1/|\lambda|, & \text{for } \lambda < 0, \\ 1/2, & \text{for } \lambda > 0. \end{cases} \tag{5.78}$$

5.4 Inhomogeneous contribution to broadening of the resonant peak

In this section we discuss possible contribution of other impurities to the width of the $\rho(\epsilon)$ peak under the Van Hove singularity. This peak is present for $\lambda < 0$ in both cases of tube and strip. Generally speaking, in tube case the energy of a quasistationary state does not depend on the position of impurity which is not the case for a strip. Thus, we start with the simplier tube case and then move to the case of strip

One could expect that in the case of attracting impurities the scattering would lead also to broadening of the narrow resonant peak at $\varepsilon = -\varepsilon_{\text{nB}}$, so that $\Gamma \to \Gamma_{\text{hom}} + \tau^{-1}$. But this idea is wrong since the corresponding electrons are localized at resonant states of certain individual impurities and, at low concentration, have no chance to be scattered by some other impurity. This statement is justified if $na_{\text{loc}} \ll 1$, where $a_{\text{loc}} = (2\varepsilon_{\text{nB}})^{-1/2} = \pi|\lambda|^{-1}$ is the radius of the localized state. So, $na_{\text{loc}} \sim n/|\lambda| \sim n/n_c$ and, under condition $n \ll n_c$, the influence of other impurities *typically* is exponentially small. However, this influence may be large in some rare non-typical configurations and we will estimate their contribution.

Due to a rare local fluctuation two impurities may occur at non-typically small distance

$r \lesssim a_{\rm loc}$ from each other, resulting in a considerable splitting $\Delta(r) \sim \varepsilon_{\rm nB}$ of a pair of initially degenerate localized states. It leads to inhomogeneous broadening

$$\Gamma_{\rm inhom} \sim (n a_{\rm loc})\varepsilon_{\rm nB} \sim \frac{n}{n_c}\varepsilon_{\rm nB} \tag{5.79}$$

that prevails in the intermediate range of concentrations: $|\lambda|^2 \ll n \ll |\lambda|$, while for lowest $n \ll |\lambda|^2$ the homogeneous broadening is stronger. We should stress that the inhomogeneous broadening (5.79) exists already in the system where all impurities are identical (have the same λ). Naturally, the systems with dispersion of λ demonstrate much stronger inhomogeneous broadening. We will briefly discuss such systems in Section 6.

For the case of strip the criterion of 'single-impurity approximation' at the energy near resonance is the same: $n a_{\rm loc} \ll 1$. Here $a_{\rm loc} \sim |\varepsilon|^{-1/2}$ where ε is the energy of a quasistationary state and is in the range $-4\varepsilon_{\rm nB} < \epsilon < 0$. Thus, the criterion may be rewritten as $|\varepsilon| \gg n^2$. Note that it coincides with the threshold of perturbation theory applicability (5.38). However, the $\rho(\epsilon)$ peak is positioned at $\epsilon = -4$ because of the inhomogeneous distribution of quasistationary states energies (5.72). Therefore, inhomogeneous broadening for a strip is identical to that for a tube.

6 Systems with different sorts of impurities (tube)

In realistic physical systems the impurities are not necessarily identical. They may be of different types and they may be situated not directly in the wall of the tube, but at some distance from it. As a result the effective scattering amplitudes Λ_i of different impurities may be different and random, with some distribution function $P(\lambda)$ for real parameter λ_i (see (5.6)). The most important characteristic of this distribution is

$$\overline{\lambda} \equiv \sqrt{\langle \lambda^2 \rangle} \tag{6.1}$$

What may be the consequences of such disorder? In the Drude approximation the only dependence of the resistivity $\rho(\varepsilon)$ on Λ comes from the factor $\tau^{-1}(\varepsilon)$. Since the contributions of different impurities to the resistivity are additive, one can write

$$\langle \rho(\varepsilon) \rangle \propto \left\langle \frac{1}{\tau(\varepsilon, \lambda)} \right\rangle_\lambda \propto \int \left| \frac{\Lambda}{1 + \Lambda g(\varepsilon)/\pi\nu_0} \right|^2 P(\lambda) d\lambda \tag{6.2}$$

For $n \gg \overline{\lambda}$ the expression (6.2) can be expanded in small Λ, the non-Born effects are small and we return to the results of Section 4 where one should substitute $\lambda \to \overline{\lambda}$.

For $n \ll \overline{\lambda}$ the scattering rate does not depend on λ in the range $|\varepsilon| \ll \varepsilon_{\text{nB}} \equiv (\overline{\lambda}/\pi)^2$, therefore all the results of Section 5.2.5 also apply to the case of random λ in this range. The case $|\varepsilon| \sim \varepsilon_{\text{nB}}$ is non-universal, here the result of averaging may depend on explicit shape of the function $P(\lambda)$. In particular, the contribution of the inhomogeneously broadened resonant peak can be evaluated with the help of expressions (5.31), (5.29). Assuming that the Lorenzian peak in (5.29) is much sharper than the distribution $P(\lambda)$, we obtain

$$\begin{aligned} \langle \rho^{(\text{res})}(\varepsilon) \rangle &= \int d\lambda P(\lambda) \rho^{(-)}_{\max} \pi \Gamma_{\text{hom}}(\lambda) \delta(\varepsilon + (\lambda/\pi)^2) = \\ &= \rho^{(-)}_{\max} \pi^3 |\varepsilon| P\left(-\pi\sqrt{|\varepsilon|}\right), \quad \text{for } \varepsilon < 0. \end{aligned} \tag{6.3}$$

7 Summary and discussion

In this work we have found the shape of the Van Hove singularity manifested in the resistivity of 2 distinct quasi-one-dimensional systems: a clean metallic tube of radius R and a conducting strip with width D with low concentration n_2 of weak short-range impurities (either repulsing or attracting). Below we measure all lengths in units of $2\pi R$ or D and all energies in units of $E_0^{(s)} = \frac{\hbar^2}{2mR^2}$ or $E_0^{(t)} = \frac{2\pi^2\hbar^2}{mD^2}$ respectively. We have shown that there is certain crossover concentration

$$n^{(c)} = |\lambda|/\pi. \tag{7.1}$$

For $n \gg n^{(c)}$ the Van Hove singularities are smoothed peaks at $|E - E_N| \sim \Gamma_{\rm B}$ with the width

$$\Gamma_{\rm B} \sim \left(n\lambda^2\right)^{2/3}. \tag{7.2}$$

The smoothing happens due to interference of scattering events at different impurities, while the amplitude of scattering at each individual impurity is not affected. The structure of the Van Hove singularity for $n_2 \gg n_2^{(c)}$ remains simple: "plateau–maximum–plateau" (see Fig. 8).

In the most interesting regime at $n_2 \ll n_2^{(c)}$, the non-Born renormalization of individual scattering amplitudes happens already at $|E - E_N| \sim E_{\rm nB}$ where the interference effects are still negligible:

$$E_{\rm nB} = \left(\frac{\lambda}{\pi}\right)^2 \gg \Gamma_{\rm B}. \tag{7.3}$$

Note that the energy scale $E_{\rm nB}$ does not depend on the concentration of impurities. The interference of scattering events at different impurities comes into play only at $|E - E_N| \sim \Gamma_{\rm nB}$, where

$$\Gamma_{\rm nB} \sim |E_{\rm dip}| \sim n^2 \ll E_{\rm nB}. \tag{7.4}$$

In this energy range the individual scattering amplitudes are already strongly renormalized (suppressed) and take universal value (only for a tube!):

$$\lambda \to \lambda_0(E) = \pi\sqrt{|E - E_N|} \tag{7.5}$$

which does not depend on the initial "bare" λ. As a result, instead of maximum $\rho(E)$ demonstrates a deep and narrow minimum at $E - E_N = E_{\rm dip} < 0$ with a width $\Gamma_{\rm nB}$.
For the case of strip the "bare" amplitudes depend on the position of impurities with respect to wave function nodes so that one can distinguish between "strong" (sitting in the antinodes of wave function) and "weak" (sitting near the nodes of wave function) impurities. Moreover, scattering amplitude renormalization is nonlinear – scattering at the stronger impurities is stronger suppressed. Therefore, at low enough energy $|E| \ll E_{\rm nB}$ the scattering predominantly occurs at "weak" impurities.

The structure of the Van Hove singularity for $n_2 \ll n_2^{(c)}$ depends on the sign of the scattering amplitude. For repulsive interaction it is "plateau–minimum–maximum–plateau" (Fig. 2), while for attractive interaction it is "plateau–maximum–minimum–maximum–plateau" (Fig. 3). Additional maximum below the Van Hove singularity in the case of attractive interaction is due to the presence of quasistationary states. For a strip, the scattering near this maximum occurs predominantly at "strong" impurities.

The structures "plateau–maximum–plateau" and "plateau–minimum–maximum–plateau" can be (very roughly) simulated by the Fano formula (1.1) with $q \to \infty$ and $q \sim 1$, correspondingly (see Fig.11).

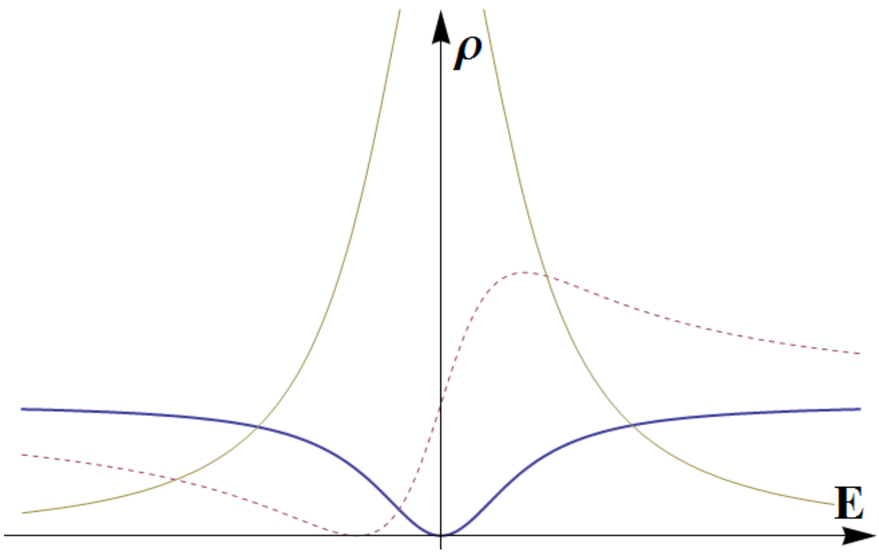

Figure 11. Different shapes of $\rho(E)$, produced by the Fano formula (1.1). Thin solid line: $q = 100$, broken line $q = 1$, thick solid line $q = 0$.

The correspondence between formula (1.1) and the results of the present study is, however, by no means quantitative. We should also stress that, contrary to the prediction of the Fano theory, an asymmetric splitting of the Van Hove singularity arises at low concentration of impurities even for positive sign of the scattering amplitude, when the resonant state does not exist and the Fano scenario is inapplicable. Moreover, in the presence of resonant states due to attracting impurities the obtained structure "plateau–maximum–minimum–maximum–plateau" can not be reproduced by the Fano formula (1.1) even on a qualitative level.

In the leading approximation in small parameter $n_2/n_2^{(c)}$ (that corresponds to independent scattering at different impurities) the resistivity an minimum $\rho_{\min}$ vanishes, as it is shown in Figs. 2 and 3. The value of $\rho_{\min}$ becomes nonzero (tube case – $\rho_{\min} \propto n_2^3$, see Fig. 10) only if one takes into account the interference of scattering events at different impurities.

In the future publication we are going to consider a more robust approach to the calculation of resistivity behavior near the minimum provided by the interference of different acts of scattering. In this work this was done only for tube case with the help with self-consistent non-Born approximation which is not rigorous. The more robust approach, as in Born case, will be based on the corresponding one-dimensional problem that was solved in [31, 32].

In Conclusion, our study shows that at low concentration of impurities the single-impurity non-Born effects lead to splitting of the Van Hove singularities in resistivity of a tube (or, in general, any other quasi-one-dimensional conductor) and this effect can not be described in terms of the Fano formula (1.1). The character of the splitting depends on whether the impurities are attracting or repulsing.

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A Self-consistent calculations: strong Born scattering

Strictly speaking, the concept of the self energy is relevant only in the weak scattering domain where $|\varepsilon| \gg \varepsilon_{\min}$ (for both $\varepsilon > 0$ and $\varepsilon < 0$). However, using the perturbative expressions (4.33) and (4.4) also in the strong scattering domain $|\varepsilon| \ll \varepsilon_{\min}$ can be helpful for qualitative understanding of the behaviour of the density of states and resolving the paradox mentioned in the subsection 4.4.5.

For a qualitative description of the density of states at strong scattering the self-consistent Born approximation can be used. The selfconsistency equation for Σ reads

$$\Sigma(\varepsilon) = -\frac{i}{2\tau_0}\left(1 + \frac{1}{\pi\sqrt{\varepsilon - \Sigma(\varepsilon)}}\right), \tag{A.1}$$

so that

$$\frac{\nu(\varepsilon)}{\nu_0} = 1 + \mathrm{Re}\left\{\frac{1}{\pi\sqrt{\varepsilon - \Sigma(\varepsilon)}}\right\}, \tag{A.2}$$

$$\Sigma(\varepsilon) = -\frac{i}{2\tau_0} - \varepsilon_{\min} Y\left[\frac{1}{\varepsilon_{\min}}\left(\varepsilon + \frac{i}{2\tau_0}\right)\right], \tag{A.3}$$

where $Y(q)$ is the solution of cubic equation

$$Y^2(Y+q) + 1 = 0. \tag{A.4}$$

There is a bifurcation point $q = q_{\mathrm{bi}}$ such that for real $q < q_{\mathrm{bi}}$ all three solutions of (A.4) are real while for $q > q_{\mathrm{bi}}$ there is one purely real solution and two conjugated complex solutions (only the latter ones are physically relevant). Near the point $q = q_{\mathrm{bi}}$ one can write

$$Y \approx Y_{\mathrm{bi}} \pm iA\sqrt{q - q_{\mathrm{bi}}} \tag{A.5}$$

$$Y_{\mathrm{bi}} = 2^{1/3}, \quad A = 2^{2/3}3^{-1/2}, \quad q_{\mathrm{bi}} = -3\cdot 2^{-2/3}. \tag{A.6}$$

Thus, if the parameter q were purely real then $\mathrm{Im}\,\Sigma$ would vanish for $\varepsilon < \varepsilon_{\mathrm{bi}} \equiv q_{\mathrm{bi}}\,\varepsilon_{\min}$. In our case, however, q has small but finite imaginary part

$$\mathrm{Im}\,q = \pi\sqrt{\varepsilon_{\min}} \ll 1. \tag{A.7}$$

For $\varepsilon > \varepsilon_{\rm bi}$ and $|\varepsilon - \varepsilon_{\rm bi}| \gg \operatorname{Im} q$ this imaginary part can be totally neglected and

$$\Sigma(\varepsilon) \approx -\varepsilon_{\rm min} Y(\varepsilon/\varepsilon_{\rm min}), \tag{A.8}$$

$$\frac{\nu(\varepsilon)}{\nu_0} = 1 + \frac{1}{\pi\sqrt{\varepsilon_{\rm min}}} \frac{1}{\sqrt{(\varepsilon/\varepsilon_{\rm min}) + Y(\varepsilon/\varepsilon_{\rm min})}}. \tag{A.9}$$

On the other side of the bifurcation point, for $\varepsilon < \varepsilon_{\rm bi}$ and $|\varepsilon - \varepsilon_{\rm bi}| \gg \operatorname{Im} q$ the $\operatorname{Im} q$-term may be taken into account perturbatively:

$$\Sigma(\varepsilon) \approx -\varepsilon_{\rm min} Y\left(\varepsilon/\varepsilon_{\rm min}\right) - \frac{i}{2\tau_0}\left[1 + Y'\left(\varepsilon/\varepsilon_{\rm min}\right)\right], \tag{A.10}$$

$$\frac{\nu(\varepsilon)}{\nu_0} = 1 + \frac{1 + Y'\left(\varepsilon/\varepsilon_{\rm min}\right)}{2\left[(\varepsilon/\varepsilon_{\rm min}) + Y(\varepsilon/\varepsilon_{\rm min})\right]^{3/2}}, \tag{A.11}$$

where $Y'(q) \equiv dY(q)/dq$.

In the narrow vicinity of the bifurcation point, for $|\varepsilon - \varepsilon_{\rm bi}| \lesssim \operatorname{Im} q$ one should keep $\operatorname{Im} q$ but, on the other hand, one can use expansion (A.6) for $Y(q)$. As a result, in this range we obtain

$$\operatorname{Re}\Sigma(\varepsilon) \approx -\varepsilon_{\rm min} Y_{\rm bi}, \tag{A.12}$$

and

$$\operatorname{Im}\Sigma(\varepsilon) \approx -\frac{A}{2\tau_0}\left[\frac{\left(Q^2+1\right)^{1/2} + Q}{\operatorname{Im} q}\right]^{1/2} \approx$$

$$\approx -\frac{A}{2\tau_0\sqrt{\operatorname{Im} q}} \begin{cases} (2Q)^{1/2}, & Q > 0, \;\; 1 \ll Q \ll (\operatorname{Im} q)^{-1}, \\ (-2Q)^{-1/2}, & Q < 0, \;\; 1 \ll |Q| \ll (\operatorname{Im} q)^{-1}. \end{cases} \tag{A.13}$$

$$\frac{\nu(\varepsilon)}{\nu_0} \approx \frac{1}{\pi\sqrt{\varepsilon_{\rm min}}} \frac{A \operatorname{Im} q}{2\left(q_{\rm bi} + Y_{\rm bi}\right)^{3/2}} \left[\frac{\left(Q^2+1\right)^{1/2} + Q}{\operatorname{Im} q}\right]^{1/2} \approx$$

$$\approx \frac{1}{\pi\sqrt{\varepsilon_{\rm min}}} \frac{A\sqrt{\operatorname{Im} q}}{2\left(q_{\rm bi} + Y_{\rm bi}\right)^{3/2}} \begin{cases} (2Q)^{1/2}, & Q > 0, \;\; 1 \ll Q \ll (\operatorname{Im} q)^{-1}, \\ (-2Q)^{-1/2}, & Q < 0, \;\; 1 \ll |Q| \ll (\operatorname{Im} q)^{-1}. \end{cases} \tag{A.14}$$

Here

$$Q(\varepsilon) = 2\tau_0\left(\varepsilon - \varepsilon_{\rm bi}\right). \tag{A.15}$$

So, as it is easy to check, for $|Q| \gg 1$ the asymptotics (A.14) overlaps with the results (A.9) and (A.11).

The above results should not be taken too seriously: the self-consistency equation (A.3) can not be justified rigorously. However, the qualitative behaviour of the decay rate and the density of states predicted by (A.13) and (A.14) gives us a reasonable pattern of matching conflicting results (4.32) and (4.31). Namely, there is a narrow interval $|\varepsilon - \varepsilon_{\rm bi}| \lesssim 1/2\tau_0$ around certain bifurcation point $\varepsilon_{\rm bi}$ ($\varepsilon_{\rm bi} < 0$, $|\varepsilon_{\rm bi}| \sim \varepsilon_{\rm min}$) where both $\nu(\varepsilon)$ and $\tau^{-1}(\varepsilon)$ increase with ε very rapidly, and just this increase explains the parametrically large difference between the results (4.32) and (4.31).

B Self-consistent calculations: strong non-Born scattering

The general (with an account for the non-Born renormalization of the scattering amplitude) self-consistency equation for the self-energy Σ reads

$$\Sigma(\varepsilon) = \frac{n}{\pi^2} \left|\Lambda^{(\mathrm{ren})}(\varepsilon - \Sigma(\varepsilon))\right|^2 \frac{g(\varepsilon - \Sigma(\varepsilon))}{\pi\nu_0} \tag{B.1}$$

where $\Lambda^{(\mathrm{ren})}$ is given by (5.20) and the density of states is determined by formula (A.2). In the case of strong non-Born effect one can use an asymptotic expression (5.35) and get

$$\Sigma(\varepsilon) = -in|\varepsilon - \Sigma| \left(1 + \frac{1}{\pi\sqrt{\varepsilon - \Sigma(\varepsilon)}}\right), \tag{B.2}$$

Let us first neglect the constant term in g, then we get

$$\Sigma(\varepsilon) = -\frac{in}{\pi}\sqrt{\varepsilon - \Sigma^*}, \tag{B.3}$$

or

$$\Sigma = -\varepsilon_{\min}^{(\mathrm{nB})} Y, \quad q \equiv \frac{\varepsilon}{\varepsilon_{\min}^{(\mathrm{nB})}}, \tag{B.4}$$

$$Y^2 + q + Y^* = 0. \tag{B.5}$$

For real q the general structure of solutions for equation (B.5) is as follows:

For $q > 1/4$ there are two complex conjugated solutions:

$$Y_{1,2} = \frac{1}{2} \mp i\sqrt{\frac{3}{4} + q} \tag{B.6}$$

For $q < -3/4$ there are two real solutions:

$$Y_{3,4} = -\frac{1}{2} \mp \sqrt{\frac{1}{4} - q} \tag{B.7}$$

For $-3/4 < q < 1/4$ all four solutions $Y_{1,2,3,4}$ are acceptable.

There is, however, always only one physically relevant solution:

$$Y(q) = \begin{cases} Y_4, & \text{for } q < q_{\rm bi}, \\ Y_2, & \text{for } q > q_{\rm bi} \end{cases} \qquad q_{\rm bi} = -3/4. \tag{B.8}$$

Thus, the bifurcation energy

$$\varepsilon_{\rm bi}^{\rm (nB)} = \varepsilon_{\rm min}^{\rm (nB)} q_{\rm bi}, \tag{B.9}$$

and for $\varepsilon_{\rm bi}^{\rm (nB)} < \varepsilon \ll \varepsilon_{\rm nB}$ we have

$$\frac{1}{\tau(\varepsilon)} = -2{\rm Im}\,\Sigma(\varepsilon) = 2\sqrt{\varepsilon_{\rm min}^{\rm (nB)}\left(\varepsilon - \varepsilon_{\rm bi}^{\rm (nB)}\right)}, \tag{B.10}$$

$$\frac{\nu(\varepsilon)}{\nu_0} = 1 + {\rm Re}\left\{\frac{1}{\pi\sqrt{\varepsilon + \varepsilon_{\rm min}^{\rm (nB)} Y_2\left(\varepsilon/\varepsilon_{\rm min}^{\rm (nB)}\right)}}\right\} = $$

$$= 1 + \frac{\sqrt{\varepsilon - \varepsilon_{\rm bi}^{\rm (nB)}}}{\pi\left(\varepsilon + \varepsilon_{\rm min}^{\rm (nB)}\right)}, \tag{B.11}$$

where we have used the formula

$$\frac{1}{\sqrt{a+ib}} = \sqrt{\frac{\sqrt{a^2+b^2}+a}{2(a^2+b^2)}} - i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2(a^2+b^2)}}. \tag{B.12}$$

So, the approximate equation (B.3) leads to the result $1/\tau(\varepsilon) \equiv -2{\rm Im}\,\Sigma(\varepsilon) = 0$ for all $\varepsilon < \varepsilon_{\rm bi}$. To determine finite scattering rate in this range we should go beyond and take into account the first term $-in|\varepsilon - \Sigma|$ on the right hand side of equation (B.2). When doing so we can, however, substitute the found zero-approximation solution to this correction term. Then, instead of (B.5), we arrive at

$$[Y + in(Y_4(q) + q)]^2 + q + Y^* = 0, \tag{B.13}$$

where we have noted that in the range $q < q_{\rm bi}$ both q and $Y_4(q)$ are real, and also $q - Y_4(q) \equiv$

$Y_4^2(q)$ is real and negative so that we could write $|\varepsilon - \Sigma| = \varepsilon_{\min}^{(\mathrm{nB})} Y_4^2(q)$. Then

$$\Sigma = -\varepsilon_{\min}^{(\mathrm{nB})} [inY_4^2(q) + Y(\tilde{q})], \tag{B.14}$$

where $Y(\tilde{q})$ is the solution of (B.5) with complex

$$\tilde{q} \equiv q - inY_4^2(q). \tag{B.15}$$

For $q < q_{\mathrm{bi}}$ the imaginary part of $\tilde{q}$ can be treated perturbatively:

$$Y(\tilde{q}) \approx Y_4(q) + in\frac{Y_4^2(q)}{2Y_4(q) - 1}, \tag{B.16}$$

and

$$\Sigma(q) \approx -\varepsilon_{\min}^{(\mathrm{nB})} Y_4(q) \left\{ 1 + in\frac{Y_4^2(q)}{Y_4(q) - 1/2} \right\} \tag{B.17}$$

$$\frac{1}{\tau(\varepsilon)} = -2\mathrm{Im}\,\Sigma(q) = n\varepsilon_{\min}^{(\mathrm{nB})} \frac{(\sqrt{1 + 4|q|} - 1)^3(\sqrt{1 + 4|q|} + 2)}{8(|q| - |q_{\mathrm{bi}}|)}, \quad q \equiv \frac{\varepsilon}{\varepsilon_{\min}^{(\mathrm{nB})}} < q_{\mathrm{bi}} \equiv -\frac{3}{4}, \tag{B.18}$$

$$\frac{\nu(\varepsilon)}{\nu_0} = 1 + \frac{1}{2[Y_4(q) - 1/2]} = 1 + \frac{\sqrt{\frac{1}{4} + |q|} + 1}{2(|q| - |q_{\mathrm{bi}}|)}, \tag{B.19}$$

where we have used

$$Y_4(q) - 1/2 = \sqrt{\frac{1}{4} + |q|} - 1 = \frac{|q| - |q_{\mathrm{bi}}|}{\sqrt{\frac{1}{4} + |q|} + 1}. \tag{B.20}$$

In particular, for $|q| \gg 1$ the scattering rate grows with $|\varepsilon|$ while $\nu(\varepsilon)$ saturates:

$$\Sigma(\epsilon) \approx -\sqrt{\varepsilon_{\min}^{(\mathrm{nB})}|\varepsilon|} - in|\varepsilon|, \quad \nu(\varepsilon) \approx \nu_0 \tag{B.21}$$

which is in agreement with (5.39). When q approaches q_{bi} (i.e., $\varepsilon \to \varepsilon_{\mathrm{bi}}^{(\mathrm{nB})}$ from below), both the scattering rate and the density of states grow:

$$\Sigma \approx -\frac{\varepsilon_{\rm min}^{(\rm nB)}}{2}\left\{1+\frac{in}{2}\frac{\varepsilon_{\rm min}^{(\rm nB)}}{\varepsilon_{\rm bi}^{(\rm nB)}-\varepsilon}\right\}, \quad \frac{\nu(\varepsilon)}{\nu_0} \approx \frac{\varepsilon_{\rm min}^{(\rm nB)}}{\varepsilon_{\rm bi}^{(\rm nB)}-\varepsilon}. \tag{B.22}$$

So, the scattering rate reaches its minimum at some $\varepsilon = \varepsilon_{\rm dip}^{(\rm nB)} \equiv \varepsilon_{\rm min}^{(\rm nB)} q_{\rm dip}$, where

$$q_{\rm dip} = -\frac{21}{16}, \quad \frac{1}{\tau(\varepsilon_{\rm dip})} = \frac{27}{8} n\varepsilon_{\rm min}^{(\rm nB)} \tag{B.23}$$

C Evaluation of function $F(\epsilon)$

C.1 Case $\epsilon > 0,\ \lambda \lessgtr 0$

For $\epsilon > 0$ we have

$$F(\epsilon) = \frac{2}{\pi}\int_0^1 \frac{t^{1/2}(1-t)^{-1/2}dt}{1+4t^2/\epsilon} = \frac{1}{\pi}\oint_{\text{cut }[0,1]} \frac{z^{1/2}(1-z)^{-1/2}dz}{1+4z^2/\epsilon} = \text{Re}\left\{\frac{z_0^{1/2}(1-z_0)^{-1/2}}{2z_0/\epsilon}\right\} =$$
$$= \frac{\sqrt{\epsilon}}{2}\text{Im}\left\{(1+4i/\sqrt{\epsilon})^{-1/2}\right\} = \sqrt{\epsilon}\left(\frac{\sqrt{1+4/\epsilon}-1}{2(1+4/\epsilon)}\right)^{1/2} \approx \begin{cases} 1, & \text{for } \epsilon \gg 1, \\ \epsilon^{3/4}/2, & \text{for } \epsilon \ll 1, \end{cases} \tag{C.1}$$

where $z_0 = i\sqrt{\epsilon}/2$ is the position of the integrand's pole in the upper half-plane of complex z. Note that the result (C.1) is valid for either signs of λ, the only necessary requirement being $\epsilon > 0$.

C.2 Case $\epsilon < 0,\ \lambda > 0$ (repulsion)

$$F(\epsilon) = \frac{1}{\pi}\oint_{\text{cut }[0,1]} \frac{dz}{(1+2z/\sqrt{|\epsilon|})^2}\sqrt{\frac{z}{1-z}} = 2i\frac{|\epsilon|}{4}\left(\frac{d}{dz}\sqrt{\frac{z}{1-z}}\right)_{z=-\sqrt{|\epsilon|}/2} =$$
$$= \frac{i|\epsilon|}{4}\left(\frac{1}{\sqrt{z(1-z)^3}}\right)_{z=-\sqrt{|\epsilon|}/2} = (2/\sqrt{|\epsilon|}+1)^{-3/2} \approx \begin{cases} 1, & \text{for } |\epsilon| \gg 1, \\ 2^{-3/2}|\epsilon|^{3/4}, & \text{for } |\epsilon| \ll 1, \end{cases} \tag{C.2}$$

C.3 Case $\epsilon < 0,\ \lambda < 0$ (attraction)

As in the case of cylinder, to avoid divergency, for general $\epsilon < 0$ we have to take into account the imaginary part of $\varepsilon_{\text{qs}}(\xi_i)$, so that for $\lambda < 0$ and $\varepsilon < 0$ F is a function of two dimensionless variables. It can be rewritten in a form

$$F(\epsilon,\lambda) = \frac{1}{\pi}\int_0^1 \frac{t^{1/2}(1-t)^{-1/2}dt}{(1-2t/\sqrt{|\epsilon|})^2 + 4t^2\lambda^2/|\epsilon|} \approx$$
$$\approx \frac{1}{\pi}\frac{|\epsilon|}{4}\int_0^1 \frac{t^{1/2}(1-t)^{-1/2}dt}{(\sqrt{|\epsilon|}/2-t)^2 + |\epsilon||\lambda|^2/4}, \tag{C.3}$$

where the second term in the denominator was substituted by its value at resonance: $4t^2\lambda^2/|\epsilon| \to \lambda^2$. Since $|\lambda| \ll 1$ it is possible to write

$$\frac{1}{(\sqrt{|\epsilon|}/2 - t)^2 + |\epsilon|\lambda^2/4} \approx \frac{2\pi}{\sqrt{|\epsilon|}|\lambda|}\delta(\sqrt{|\epsilon|}/2 - t), \tag{C.4}$$

and get for $-4 < \epsilon < 0$

$$F(\epsilon, \lambda) \approx \frac{\sqrt{|\epsilon|}}{|\lambda|}\left(\frac{\sqrt{|\epsilon|}}{2 - \sqrt{|\epsilon|}}\right)^{1/2}. \tag{C.5}$$

On the other hand, for $\epsilon < -4$ we can simply put $\lambda = 0$ and obtain

$$\begin{aligned} F(\epsilon, 0) = \frac{1}{2\pi}\oint \frac{dz}{(1 - 2z/\sqrt{|\epsilon|})^2}\sqrt{\frac{z}{1-z}} = \\ = \frac{|\epsilon|^{3/4}}{(\sqrt{|\epsilon|} - 2)^{3/2}} \end{aligned} \tag{C.6}$$

Both the results (C.5) and (C.6) are not applicable in the narrow vicinity of $\epsilon = -4$, namely, for $|\epsilon + 4| \lesssim |\lambda|$. In this range we should write

$$\begin{aligned} F(\epsilon) \approx \frac{u}{2\pi}\int_0^1 \frac{dt}{\left(1 + \frac{u-4}{8} - t\right)^2 + \lambda^2}\frac{1}{\sqrt{1-t}} = \\ = \frac{u}{\pi\lambda^2}\int_0^1 \frac{dz}{1 + \left(\frac{z^2}{|\lambda|} + \frac{u-4}{8|\lambda|}\right)^2} \approx \frac{u}{\pi|\lambda|^{3/2}}\Phi\left(\frac{u-4}{8|\lambda|}\right), \end{aligned} \tag{C.7}$$

D Evaluation of function $\tilde{F}(\epsilon, \lambda)$

D.1 Case $\epsilon < 0$, $\lambda > 0$ (repulsion)

Here

$$\tilde{F}(\epsilon) = \frac{1}{2\pi} \oint \frac{z^{-1/2}(1-z)^{-1/2} dz}{(1 + 2z/\sqrt{|\epsilon|})^2} = i\frac{|\epsilon|}{4} \left(\frac{d}{dz} \sqrt{\frac{1}{z(1-z)}} \right)_{z=-\sqrt{|\epsilon|}/2} =$$

$$= \frac{i|\epsilon|}{4} \left(-\frac{1}{2}\right) \left(\frac{1-2z}{\sqrt{z(1-z)^3}} \right)_{z=-\sqrt{|\epsilon|}/2} = \frac{|\epsilon|^{1/4}(1+\sqrt{|\epsilon|})}{(2+\sqrt{|\epsilon|})^{3/2}} \approx \begin{cases} 1, & \text{for } |\epsilon| \gg 1, \\ 2^{-3/2}|\epsilon|^{1/4}, & \text{for } |\epsilon| \ll 1, \end{cases} \tag{D.1}$$

D.2 Case $\epsilon < 0$, $\lambda < 0$ (attraction)

As in the case of cylinder, to avoid divergency, for general $\epsilon < 0$ we have to take into account the imaginary part of $\varepsilon_{\text{qs}}(\xi_i)$, so that for $\lambda < 0$ and $\varepsilon < 0$ $\tilde{F}$ in (**??**) is a function of two dimensionless variables (see (5.65)). It can be rewritten in a form

$$\begin{aligned} \tilde{F}(\epsilon, \lambda) &= \frac{1}{\pi} \int_0^1 \frac{t^{-1/2}(1-t)^{-1/2} dt}{(1 - 2t/\sqrt{|\epsilon|})^2 + 4t^2\lambda^2/|\epsilon|} \approx \\ &\approx \frac{1}{\pi} \frac{|\epsilon|}{4} \int_0^1 \frac{t^{-1/2}(1-t)^{-1/2} dt}{(\sqrt{|\epsilon|}/2 - t)^2 + |\epsilon||\lambda|^2/4}, \end{aligned} \tag{D.2}$$

where the second term in the denominator was substituted by its value at resonance: $4t^2\lambda^2/|\epsilon| \to \lambda^2$. Since $|\lambda| \ll 1$ it is possible to write

$$\frac{1}{(\sqrt{|\epsilon|}/2 - t)^2 + |\epsilon|\lambda^2/4} \approx \frac{2\pi}{\sqrt{|\epsilon|}|\lambda|} \delta(\sqrt{|\epsilon|}/2 - t), \tag{D.3}$$

and get for $-4 < \epsilon < 0$

$$\tilde{F}(\epsilon, \lambda) \approx \frac{1}{|\lambda|} \sqrt{\frac{\sqrt{|\epsilon|}}{2 - \sqrt{|\epsilon|}}}, \tag{D.4}$$

On the other hand, for $\epsilon < -4$ we can simply put $\lambda = 0$ and obtain

$$\begin{aligned} \tilde{F}(\epsilon, 0) &= \frac{1}{2\pi} \oint \frac{dz}{(1 - 2z/\sqrt{|\epsilon|})^2} \sqrt{\frac{1}{z(1-z)}} = \\ &= \frac{|\epsilon|^{1/4}(\sqrt{|\epsilon|} - 1)}{(\sqrt{|\epsilon|} - 2)^{3/2}} \end{aligned} \tag{D.5}$$

The results (D.4) and (D.5) are not applicable in the narrow vicinity of $\varepsilon = -4$, namely, for $|\varepsilon + 4| \lesssim |\lambda|$. In this range we should write

$$\tilde{F}(\epsilon, \lambda) \approx \frac{1}{\pi} \int_0^1 \frac{t^{-1/2}(1-t)^{-1/2} dt}{(1 - t + (|\epsilon| - 4)/8)^2 + \lambda^2} \approx \frac{1}{\pi |\lambda|^{3/2}} \Phi\left(\frac{|\epsilon| - 4}{8|\lambda|}\right), \tag{D.6}$$

$$\Phi(a) = \int_{\infty}^{\infty} \frac{d\phi}{(\phi^2 + a)^2 + 1} = \mathrm{Im}\left\{\frac{\pi}{\sqrt{a+i}}\right\} = \pi \left(\frac{\sqrt{a^2+1} - a}{2(a^2+1)}\right)^{1/2} \approx$$

$$\approx \begin{cases} \dfrac{\pi}{2a^{3/2}}, & \text{for } a > 0,\ a \gg 1, \\ \dfrac{\pi}{|a|^{1/2}}, & \text{for } a < 0,\ |a| \gg 1, \end{cases} \tag{D.7}$$

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