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Investigation of difference between the phases of the anomalous Green functions and the phase of the order parameter in diffusive Josephson junctions

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МАГИСТЕРСКАЯ ДИССЕРТАЦИЯ

Исследование различия фаз аномальных функций Грина и фазы параметра порядка в диффузных джозефсоновских контактах

Магистерская образовательная программа: Математическая и теоретическая физика

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Investigation of difference between the phases of the anomalous Green functions and the phase of the order parameter in diffusive Josephson junctions

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Abstract

We consider a planar SIS-type Josephson junction between diffusive superconductors (S) through an insulating tunnel interface (I). We construct fully self-consistent perturbation theory with respect to the interface conductance. As a result, we find correction to the first Josephson harmonic and calculate the second Josephson harmonic. At arbitrary temperatures, we correct previous results for the nonsinusoidal current–phase relation in Josephson tunnel junctions, which were obtained with the help of conjectured form of solution. Our perturbation theory also describes the difference between the phases of the order parameter and of the anomalous Green functions.

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1 Introduction

1.1 The statement of the problem

In this work, we consider the SIS Josephson junction between diffusive superconductors at arbitrary temperature T. We consider the tunneling limit but focus on deviations from the sinusoidal current–phase relation due to small but finite conductance of the interface. We develop fully self-consistent perturbation theory taking into account difference between the phases $\varphi(x)$ and $\chi(x,\omega)$. As a result, we find the second harmonic of the Josephson relation, i.e., contribution to $J(\delta\varphi)$ of the form $\sin 2\delta\varphi$. In the limit $T \to T_c$, we reproduce the results by Kupriyanov [1]. At arbitrary temperatures, we revisit the results by Golubov and Kupriyanov GK [2]. In Ref. [2], the authors employed a conjectured form of solution which turns out to be only qualitatively correct. As a result, they obtained parametrically correct answer but with wrong numerical coefficients. Our perturbation theory reproduces their parametrical results and provides exact numerical coefficients.

We also discuss quantitative difference between the phase of the order parameter and the phases of the anomalous Green functions that follows from our perturbation theory.

Throughout the paper, we employ the units with $\hbar = k_B = c = 1$.

1.2 Motivation

One of the key characteristics of a superconductor is the complex-valued order parameter $\Delta(\mathbf{r})$, which is parameterized by its absolute value and phase $\varphi(\mathbf{r})$ [3]. Both parameters are essential for describing current-carrying states of superconductors. While the absolute value of the order parameter determines the density of superconducting electrons, the phase gradient is related to the superconducting condensate velocity. At the same time, more detailed spectral (i.e., energy-resolved) information about superconductivity in a system is contained in the anomalous Green function $F(\mathbf{r},\omega)$ (here ω is the Matsubara frequency), with its own absolute value and phase $\chi(\mathbf{r},\omega)$. The anomalous Green functions and the order parameter (related by the self-consistency equation) fully describe superconductivity inside an equilibrium system [4].

The Josephson effect is a prominent example of the physical role of the superconducting phases [3]. The simplest Josephson system is a planar SIS-type junction (superconductors S separated by an insulating barrier I). All characteristics of the system depend on a single coordinate x (normal to the plain interface). Fully self-consistent treatment of the Josephson effect in the SIS junction requires taking into account difference between the phases of the order parameter and the anomalous Green function, $\varphi(x) \neq \chi(x,\omega)$. Although difference between the two phases is a well-known fact (which is already evident from frequency, or energy, dependence of χ while φ depends only on coordinate) [5–7], it has been taken into account in actual calculations mainly numerically [8]. It is interesting to understand for which systems the phases differ, what is the reason for this, and we also want to understand the quantitative characteristics of this difference.

At the same time, the SIS junction is the fundamental system for which the Josephson effect was originally predicted [9, 10], and it has been considered in many various limiting cases. In the main order with respect to the interface conductance, the Josephson current proportional to the sine of the order-parameter phase difference between the banks arises, $J \propto \sin\delta\varphi$. Next orders with respect to the interface conductance take into account additional effects such as pair-breaking due to current and the proximity effect between the banks (suppression of the order parameter near the interface) [11, 12]. These effects influence basic characteristics of the Josephson current such as the value of the critical current and the current–phase relation in SIS and more complicated types of Josephson junction (including SNS junctions with normal metal N as a weak link) [13–15]. As a result, the current–phase relation $J(\delta\varphi)$ can deviate from the simple sinusoidal form [13, 15].

Anharmonic (nonsinusoidal) Josephson current is also possible in the case of pair-breaking interfaces [16, 17]. SIS junctions with arbitrary interface transparency have been considered in the limit of temperature close to the critical one [18, 19]. Therefore, taking an ordinary SIS contact as the object of study, which is the simplest from theoretical consideration, developing the apparatus for solving the superconductivity equations, we can find corrections to the sinusoidal behavior of the current in the Josephson relation.

1.3 The Usadel equations

In this section we write down the basic equations describing the superconductivity of a planar SIS junction.

SIS-type junction is a system of two superconductors separated by a thin insulating layer Fig. 1.1.

In the diffusive, or so-called dirty, limit, when the superconducting coherence length ξ is much larger than the mean free path l, superconductors can be described by the Usadel equations [20].

In the planar SIS junction all characteristics depend only on the x coordinate (the direction perpendicular to the boundary (I)). We can eliminate the vector potential by a gauge transformation, so that all currentcarrying properties of the system are encoded in the phase gradients. Finally, with the use of Eq. (A.43) we obtain the system of equations for SIS

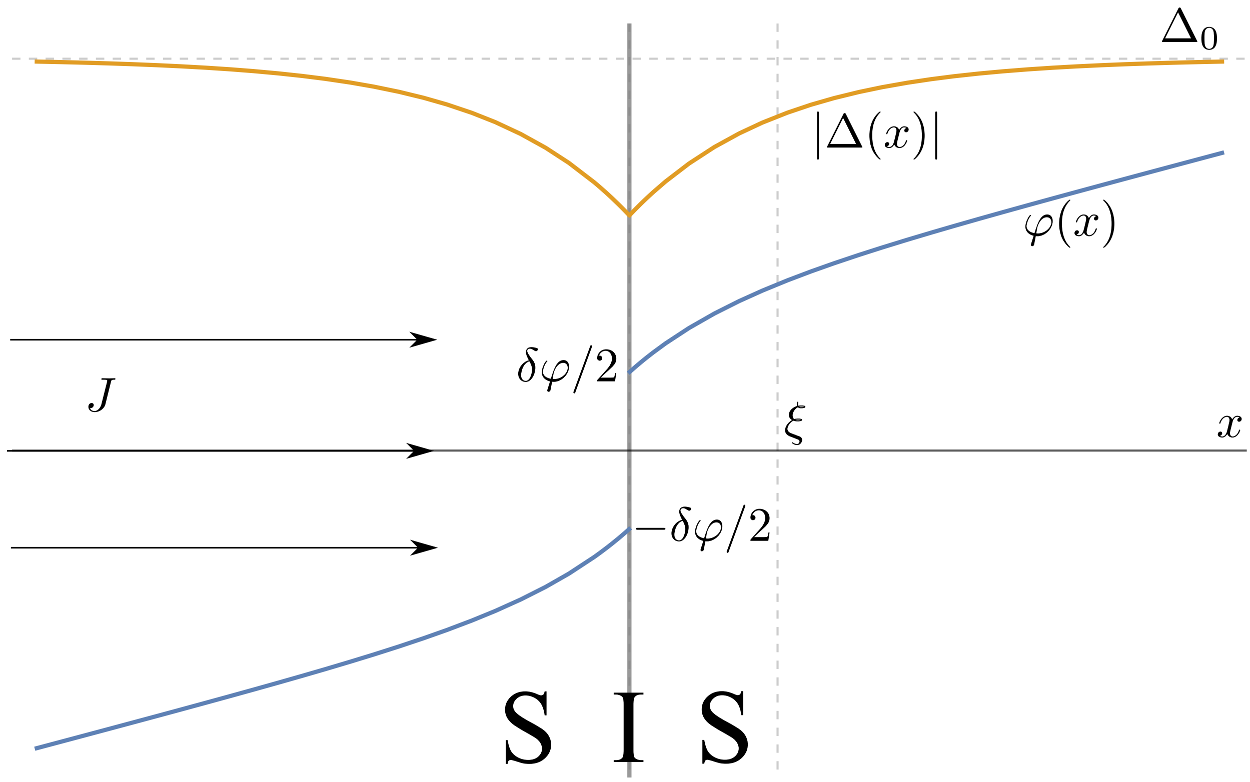

Figure 1.1: Sketch of a planar SIS junction. Two superconductors (S) are separated by a thin insulating layer (I). At the interface, the order-parameter phase φ is discontinuous. The absolute value of the order parameter, $|\Delta|$, is suppressed at $x = 0$, while at the bulk it reaches the value Δ_0. Both plots are schematic. Due to the spatial symmetry of the problem, the coordinate dependence $|\Delta(x)|$ is even while $\varphi(x)$ can be chosen odd. We parameterize the Josephson current J in the junction by the phase difference $\delta\varphi \equiv \varphi(+0) - \varphi(-0)$ at the interface.

junction in the angular parameterisation (for more details see Appendix A):

$$\frac{D}{2}\frac{d^2\theta}{dx^2} + |\Delta|\cos(\chi - \varphi)\cos\theta - \omega\sin\theta - \frac{D}{2}\sin\theta\cos\theta\left(\frac{d\chi}{dx}\right)^2 = 0, \tag{1.1}$$

$$\frac{D}{2}\frac{d}{dx}\left(\frac{d\chi}{dx}\sin^2\theta\right) = |\Delta|\sin(\chi - \varphi)\sin\theta, \tag{1.2}$$

$$|\Delta| = \pi\lambda T \sum_{|\omega|<\omega_D} e^{i(\chi-\varphi)}\sin\theta, \tag{1.3}$$

$$J = 2\pi\nu_0 SDTe \sum_{|\omega|<\omega_D} \sin^2\theta \frac{d\chi}{dx} = \text{const}, \tag{1.4}$$

where $\Delta(x) = |\Delta(x)|\, e^{i\varphi(x)}$ is the order parameter, $f(x,\omega) = \sin\theta(x,\omega)e^{i\chi(x,\omega)}$ is the anomalous Green function, $\omega = \pi T(2n+1)$ is the Matsubara frequency (at temperature T), $D = v_F l/3$ is the diffusion constant, v_F is the Fermi velocity, λ is the BCS coupling constant, ω_D is the Debye frequency of the superconducting material, e is the charge of electron, ν_0 is the density of states at the Fermi level in the normal state, J is the total current flowing through the junction, and S is the interface area, . The equation (A.43) leads to current conservation in this system.

Note that the resulting equations (1.1) and (1.2) are invariant under the following change of variables:

$$\theta(x) \to \theta(-x) \tag{1.5}$$

$$\chi(x) \to -\chi(-x) + \text{const} \tag{1.6}$$

1.3.1 Equality of the phases

In the bulk of the superconductor the angular variable θ and the velocity of the Cooper pairs $d\chi/dx$ do not change, which allows us to find the solution in the form:

$$\theta(x, \omega) = \text{const}(\omega) \tag{1.7}$$

$$\varphi(x) = \chi(x, \omega) = a \operatorname{sgn} x + bx \tag{1.8}$$

As we can see, deep in the superconductor, the phases of the anomalous Green's functions and the phase of the order parameter simply coincide. Thus, the question arises whether this equality holds identically for any superconducting system? If not, is it possible to provide quantitative characteristics of this difference.

In 1D, in the case of the presence of the boundary the phases χ and φ cannot coincide. Indeed, we use the continuity equation (1.2) with a source, assuming that $\chi = \varphi$:

$$\frac{d\varphi(x)}{dx} = \frac{J_\omega(\omega)}{\sin^2 \theta(x, \omega)}, \tag{1.9}$$

where J_ω depends only on ω. From this formula, we see a rigid relationship between the condensate velocity $d\varphi/dx$ and the value of the angular variable $\theta(x)$. In the case when theta gradients $\nabla\theta$ have a nonzero component in the direction of the current $\mathbf{j}$ (i.e. $\mathbf{j} \cdot \nabla\theta \neq 0$), a situation may arise that φ is not equal to χ. It is most pronounced in one dimension, and therefore it is interesting to study the SIS contact.

1.4 Boundary conditions

The presence of boundary (I) imposes the requirement to set boundary conditions on the Usadel equations. In our work, we will consider the limit of the tunnel boundary, i.e. when the transparency of the barrier is low. The boundary conditions in this limit were investigated in the work of Kupriyanov and Lukichev [21]. For two superconductors (1– left, 2– right) with Green functions $\hat{g}_{0,i}$ [for more details see Appendix A] (we will use the notation G_i for brevity), Eq. (A.19), mean free path l_i, and Fermi momentum $p_{F,i}$ respectively one can write [7] :

$$\left(p_{F_1}^2 l_1 G_1 i\hat{P} G_1\right)_x = \left(p_{F_2}^2 l_2 G_2 i\hat{P} G_2\right)_x, \tag{1.10}$$

$$\left(l_2 G_2 i\hat{P} G_2\right)_x = t\left[G_1, G_2\right], \tag{1.11}$$

$$t \equiv \left\langle \frac{(p_F)_{2,x} T}{(p_F)_2 R} \right\rangle \ll 1, \tag{1.12}$$

where $\mathbf{n}$ is the vector, perpendicular to the (I) boundary and t is an effective transparency of the boundary. For the same reasons as in the Usadel equations, among the boundary conditions there are repeating ones. Excluding them, we obtain:

$$\left[l_{1,2} g_{1,2}^2 \left(\nabla - 2ie\mathbf{A}\right)\left(\frac{f_{1,2}}{g_{1,2}}\right)\right]_x = 2t \cdot g_1 g_2 \left(\frac{f_2}{g_2} - \frac{f_1}{g_1}\right). \tag{1.13}$$

In angular parameterisation we can rewritten boundary conditions in the following way:

$$l_1 \nabla_x \theta_1 = -2t\left(\sin\theta_1 \cos\theta_2 - \sin\theta_2 \cos\theta_1 \cos\left(\chi_2 - \chi_1\right)\right) \tag{1.14}$$

$$l_2 \nabla_x \theta_2 = 2t\left(\sin\theta_2 \cos\theta_1 - \sin\theta_1 \cos\theta_2 \cos\left(\chi_2 - \chi_1\right)\right) \tag{1.15}$$

$$l_{1,2} \sin^2\theta_{1,2}\left(\nabla\chi_{1,2} - 2e\mathbf{A}\right)_x = 2t \sin\theta_1 \sin\theta_2 \sin\left(\chi_2 - \chi_1\right) \tag{1.16}$$

In SIS contact we suppose the both sides to be from one material, thus $l_1 = l_2$ and $p_{F,1} = p_{F,2}$. As we have mentioned before, we can eliminate the vector potential $\mathbf{A}$ with the use of gauge transform. In addition, boundary conditions are invariant under the change of variables Eqs. (1.5)–(1.6), which allows us to find the solution to the Usadel Eqs. (1.1)–(1.2) in the following form:

$$\chi(x) = -\chi(-x), \tag{1.17}$$

$$\theta(x) = \theta(-x), \tag{1.18}$$

which means that χ is an odd function, while θ is even. Moreover, by the physical meaning $\sin^2\theta$ does play the role the spectral density of the Cooper pairs on the frequency ω and must be a continuous function of x, therefore $\theta(x)$ is also continuous. At the same time, $\chi(x)$ can be discontinuous at the boundary $x = 0$, which leads to non-zero derivatives at $x = 0$ Eqs. (1.14)–(1.16). Finally, we obtain:

$$\frac{d}{dx}\theta(x = \pm 0) = \pm\frac{2t}{l} \sin\theta_{x=0} \cos\theta_{x=0}\left(1 - \cos\delta\chi\right), \tag{1.19}$$

$$\frac{d\chi}{dx}(x = \pm 0) = \frac{2t}{l} \sin\delta\chi, \tag{1.20}$$

$$\delta\chi \equiv \chi(+0) - \chi(-0), \tag{1.21}$$

where $\delta\chi$ is called a phase jump. In some cases it is convinient to use the alternative form of the expression $2t/l$:

$$\frac{2t}{l} = \frac{g_N}{\sigma}, \tag{1.22}$$

where g_N is the conductance of the interface per unit area and σ is the normal-state conductivity of the superconductor material.

In this work we obtain $\Delta(x)$, $\varphi(x)$, J from the Usadel equations (1.1)–(1.4) with boundary conditions (1.19)–(1.20). In our work, we assume that the value of the phase jump $\delta\varphi \equiv \varphi(+0) - \varphi(-0)$ of the order parameter phase $\varphi(x)$ at the boundary is a fixed value of the model.

1.5 Tunnelling limit

The self-consistent Usadel equations cannot be solved analytically for arbitrary transparencies, but in some limiting cases this can be done approximately. In this work, we solve the Usadel equations by the perturbation theory with respect to the interface conductance.

In a superconductor, the natural energy scale is the bulk temperature-dependent value of the order parameter $\Delta_0(T)$. It determines the coherence length, which can be written (in the diffusive limit) as:

$$\xi(T) \equiv \sqrt{\frac{D}{2\Delta_0(T)}}. \tag{1.23}$$

This characteristic length follows from the Usadel equations. Indeed, if we divide the equation (1.1) by the value Δ_0, then the multiplier at the second derivative has the meaning of the square of the length, the scale on which the angular variable and the phase of the order parameter change. However, it turns out to be indeed the relevant spatial scale on which the superconducting properties vary, only at temperatures not too close to the superconducting critical temperature T_c. In the vicinity of T_c, the full set of the Usadel equations reduces to the Ginzburg–Landau (GL) equation (for more details see Appendix B) written for the order parameter only. In the course of this reduction, the Matsubara summation in the self-consistency equation generates a different coherence length, which can be written as:

$$\xi_{\mathrm{GL}}(T) \equiv \sqrt{\frac{\pi D}{8\,(T_c - T)}}. \tag{1.24}$$

Although this GL coherence length arises when considering the $T \to T_c$ limit, the resulting expression can be used at any T. From this point of view, we can say that at T not too close to T_c, the GL coherence length Eq. (1.24) is of the same order[1] as the Usadel coherence length Eq. (1.23). However, at $T \to T_c$, they are parametrically different since $\Delta_0(T) \propto \sqrt{T_c - T}$, and ξ_{GL} is turns out to be the actual scale for $\Delta(x)$ variation.

[1]In particular, at $T = 0$ we have $\xi(0)/\xi_{\mathrm{GL}}(0) = 2e^{C/2}/\pi \approx 0.85$, where $C \approx 0.577$ is Euler's constant

The boundary conditions (1.19)–(1.20) can be rewritten in terms of the dimensionless variable $z = x/\xi$ as:

$$\frac{d\theta(\pm 0,\omega)}{dz} = \pm\frac{\alpha}{2}\sin 2\theta(0,\omega)\left[1-\cos\delta\chi(\omega)\right], \tag{1.25}$$

$$\frac{d\chi(0,\omega)}{dz} = \alpha\sin\delta\chi(\omega), \tag{1.26}$$

where we have defined the dimensionless conductance parameter:

$$\alpha \equiv \frac{2\xi t}{l} = \frac{\xi g_N}{\sigma}. \tag{1.27}$$

In the tunnelling limit we suppose t, the average transparency of the barrier, to be small.

Due to finite value of α, the proximity effect between the two sides of the Josephson junction leads to suppression of $|\Delta(x)|$ in the vicinity of the interface (at nonzero phase difference). We standardly define the tunneling limit as the regime in which the proximity effect [i.e., suppression of $|\Delta(x)|$] is weak. This condition implies that α must be small. The exact condition for the smallness of α will be discussed below in Sec. 2.5.

One more point regarding various interface parameters should be commented here. The KL boundary conditions (1.19) and (1.20) are valid in the limit of small transparencies of interface conducting channels, which may be formulated as $t \ll 1$. They can be obtained in the first order with respect to t from the more general Nazarov boundary conditions [22]. We plan to do the perturbation theory with respect to α (staying in the regime of validity of the KL boundary conditions) but we do not take into account higher-order terms with respect to t from the Nazarov boundary conditions. This is legitimate since $\alpha \gg t$ [see Eq. (1.27)] due to the diffusive limit condition $\xi \gg l$.

For example, the next-order term from the Nazarov boundary conditions would lead to contributions of the order of αt in the right-hand sides of Eqs. (1.19) and (1.20) (and in the solutions). At the same time, the proximity effect treated within the KL boundary conditions leads to corrections of the order of α^2. Since $\alpha^2 \gg \alpha t$, the main effect is captured by the self-consistent theory based on the KL boundary conditions.

2 Perturbation theory with respect to the interface conductance: Josephson current

In this section, we carry out perturbation theory in the small parameter α and calculate the order parameter and the current.

2.1 First order of the perturbation theory

The starting point of our perturbation theory is the solution of the Usadel equations with the KL boundary conditions at $\alpha = 0$.

From the boundary conditions (1.25)–(1.26) one can see that $d\theta(\pm 0)/dz = d\chi(\pm 0)/dz = 0$. Therefore, we can try to find the solution in the form:

$$\theta(z) \equiv \theta_0 = \text{const}, \tag{2.1}$$

$$\chi(z) \equiv \chi_0(z) = \frac{\delta\varphi}{2}\,\text{sgn}\, z = \varphi_0(z). \tag{2.2}$$

The form of $\chi(z)$ from the oddness of the function Eq. (1.17) and the value of the constant follows from the fact that in the bulk of the superconductor $\chi(z \to \infty, \omega) = \varphi(z \to \infty)$. We consider the order-parameter phase jump at the interface:

$$\delta\varphi = \varphi(+0) - \varphi(-0), \tag{2.3}$$

as the parameter that defines the current-carrying state of the Josephson junction. This parameter enters the full self-consistent set of equations and determines, in particular, the strength of the proximity effect between the superconducting banks and the current at any point of the junction.

To find θ_0 one must substitute the solution Eq. (2.1) in the Usadel equation Eq. (1.1):

$$|\Delta| \cos\theta_0 - \omega \sin\theta_0 = 0. \tag{2.4}$$

From this equation one can see, that $|\Delta| \equiv \Delta_0 = \text{const}$. Therefore, the solution for θ_0 and Δ_0 has the form:

$$\tan\theta_0 = \frac{\Delta_0}{\omega}, \tag{2.5}$$

$$\Delta_0 = \pi\lambda T \sum_{|\omega|<\omega_D} \sin\theta_0 = \pi\lambda T \sum_{|\omega|<\omega_D} \frac{\Delta_0}{\sqrt{\omega^2 + \Delta_0^2}}, \tag{2.6}$$

where the last equation is the standard BCS self-consistency equation. It is known that the solution of this equation cannot be expressed in a compact form, thus we suppose that we know Δ_0.

The next step is to consider $\alpha \neq 0$. Expanding θ, χ, Δ, and φ in powers of α, we get:

$$\theta(z,\omega) = \theta_0(\omega) + \alpha\theta_1(z,\omega) + \alpha^2\theta_2(z,\omega) \tag{2.7}$$

$$\Delta(z) = \Delta_0 + \alpha\Delta_1(z) + \alpha^2\Delta_2(z) \tag{2.8}$$

$$\chi(z,\omega) = \frac{\delta\varphi}{2}\,\mathrm{sgn}\,z + \alpha\chi_1(z,\omega) + \alpha^2\chi_2(z,\omega) \tag{2.9}$$

$$\varphi(z) = \frac{\delta\varphi}{2}\,\mathrm{sgn}\,z + \alpha\varphi_1(z,\omega) + \alpha^2\varphi_2(z,\omega) \tag{2.10}$$

$$\delta\chi(\omega) = \delta\varphi + \alpha\cdot\delta\chi_1(\omega) + \alpha^2\cdot\delta\chi_2(\omega) \tag{2.11}$$

In case of non-zero current, $J \neq 0$, the phases $\chi(z,\omega)$ and $\varphi(z)$ grow linearly in the bulk of the superconductor, because in the bulk, far from the barrier (I), θ and $d\chi/dz$ become constant. Corrections $\chi_{1(2)}$ and $\varphi_{1(2)}$ therefore become large which may seem to create a problem for our perturbation theory. However, this problem is purely formal because the quantities that actually enter our perturbation theory are not χ and φ themselves but their derivatives $d\chi/dz$ and $d\varphi/dz$ as well as their difference $\chi - \varphi$; all those quantities are finite in the bulk.

Our goal is to find the answer for J up to the α^2 order. The current given by Eq. (1.4) contains $d\chi/dz \sim \alpha$ [i.e. $d\chi_0/dz = 0$]; so, in order to obtain the answer up to α^2, it is sufficient to find θ_1, Δ_1, and χ_2:

$$J = \frac{2\pi\nu_0 DTSe}{\xi}\sum_{|\omega|<\omega_D}\left(\alpha\sin^2\theta_0\frac{d\chi_1}{dz} + \alpha^2\sin^2\theta_0\frac{d\chi_2}{dz} + 2\alpha^2\sin\theta_0\cos\theta_0\theta_1\frac{d\chi_1}{dz}\right). \tag{2.12}$$

We start with calculating θ_1 and Δ_1. In the first order of the perturbation theory, equations for θ_1 and Δ_1 separate from equations for χ_1 and φ_1, the pair-breaking term $(d\chi/dx)^2$ in Eq. (1.1) should be dropped out, and $\cos(\chi-\varphi)$ should be substituted by 1.

The Usadel equation (1.1) and the boundary condition (1.25) up to the first power in α have the form:

$$\frac{d^2\theta_1}{dz^2} + \frac{\Delta_1(z)}{\Delta_0}\cos\theta_0 - \frac{\theta_1}{\sin\theta_0} = 0, \tag{2.13}$$

$$\frac{d\theta_1(\pm 0)}{dz} = \pm\frac{1}{2}\left(1-\cos\delta\varphi\right)\sin 2\theta_0. \tag{2.14}$$

The boundary condition (2.14) can be included into Eq. (2.13) by emploing the Dirac delta function:

$$\frac{d^2\theta_1}{dz^2} + \frac{\Delta_1(z)}{\Delta_0}\cos\theta_0 - \frac{\theta_1}{\sin\theta_0} = \sin 2\theta_0\left(1-\cos\delta\varphi\right)\delta(z). \tag{2.15}$$

We can solve this linear system with the help of the Fourier transformation (with respect to z):

$$f(k) \equiv \int_{-\infty}^{\infty} f(z) e^{-ikz} dz; \quad f(z) = \int_{-\infty}^{\infty} f(k) e^{ikz} \frac{dk}{2\pi}. \tag{2.16}$$

In the Fourier space we find

$$\boxed{\theta_1(k) = \frac{\sin\theta_0 \cos\theta_0}{k^2 \sin\theta_0 + 1} \left[\frac{\Delta_1(k)}{\Delta_0} - 2\left(1 - \cos\delta\varphi\right) \sin\theta_0 \right]} \tag{2.17}$$

In the first order of the perturbation theory, the real part of the self-consistency equation (1.3) yields:

$$\Delta_1(k) = \pi\lambda T \sum_{|\omega|<\omega_D} \theta_1(k,\omega) \cos\theta_0(\omega). \tag{2.18}$$

The answer for Δ_1 can be written in terms of Δ_0 without any explicit information on ω_D and λ. The bulk self-consistency equation (2.6) can be written as

$$\frac{1}{\lambda} = \frac{\pi T}{\Delta_0} \sum_{|\omega|<\omega_D} \sin\theta_0. \tag{2.19}$$

Substituting this expression for λ into Eq. (2.18), we can rewrite the latter equation in the form

$$\pi T \sum_{|\omega|<\omega_D} \left(\frac{\Delta_1}{\Delta_0} \sin\theta_0 - \theta_1 \cos\theta_0 \right) = 0. \tag{2.20}$$

Substituting Eq. (2.17) into Eq. (2.20) we obtain the correction to the order parameter $\Delta_1(k)$:

$$\frac{\Delta_1(k)}{\Delta_0} \sum_{\omega} \frac{k^2 \sin^2\theta_0 + \sin^3\theta_0}{k^2 \sin\theta_0 + 1} = -2\left(1 - \cos\delta\varphi\right) \sum_{\omega} \frac{\sin^2\theta_0 \cos^2\theta_0}{k^2 \sin\theta_0 + 1}. \tag{2.21}$$

Since all the sums in this equation converge, we can extend the limits of summation to infinity, formally putting $\omega_D = \infty$. We introduce the following notation for the relevant class of sums:

$$L_n(k,T) \equiv \frac{2\pi T}{\Delta_0} \sum_{\omega>0} \frac{\sin^n\theta_0}{k^2 \sin\theta_0 + 1}. \tag{2.22}$$

Using the definition of Eq. (2.22) one can easily rewrite the answer for $\Delta_1(k)$ in the following form:

$$\boxed{\frac{\Delta_1(k,T)}{\Delta_0(T)} = -2\left(1 - \cos\delta\varphi\right) \frac{L_2(k,T) - L_4(k,T)}{k^2 L_2(k,T) + L_3(k,T)}} \tag{2.23}$$

The next step is to find χ_1 and φ_1. This can be done by using the linearized form of the continuity equation (1.2), the imaginary part of the self-consistency equation (1.3), (A.42), and the boundary condition (1.26) for the velocity of the Cooper pairs:

$$\frac{d^2\chi_1}{dz^2}\sin^2\theta_0 = (\chi_1 - \varphi_1)\sin\theta_0, \tag{2.24}$$

$$\sum_{|\omega|<\omega_D}(\chi_1 - \varphi_1)\sin\theta_0 = 0, \tag{2.25}$$

$$\frac{d\chi_1(\pm 0)}{dz} = \sin\delta\varphi. \tag{2.26}$$

This system has trivial solution of the form:

$$\boxed{\chi_1(z,\omega) = \varphi_1(z) = z\cdot\sin\delta\varphi}, \tag{2.27}$$

which follows from the homogeneity of the system of equations.

This formula tells us that in the main order with respect to the interface conductance, the Josephson relation have the standard form $J \propto \sin\delta\varphi$:

$$J = 2\pi\nu_0 DTSe\sum_\omega \sin^2\theta_0\cdot\alpha\sin\delta\varphi \propto \sin\delta\varphi. \tag{2.28}$$

Moreover, $\chi_1(z,\omega) = \varphi_1(z)$ are continuous functions at $z = 0$, unlike $\chi_0(z,\omega) = \varphi_0(z) = (\delta\varphi/2)\,\mathrm{sgn}\,z$. Therefore, $\chi_1(+0,\omega) - \chi_1(-0,\omega) = \delta\chi_1(\omega) = 0$.

Expanding Eq. (1.26) up to α^2, we obtain

$$\frac{d\chi_2(\pm 0)}{dz} = 0, \quad \forall\omega. \tag{2.29}$$

This boundary condition implies that in order to calculate J (which can be done at $z = 0$), we do not actually need to calculate $\chi_2(z)$. To find the current up to the α^2 order, we thus only need Δ_1 and θ_1 [see Eq. (2.12)]. We know θ_1 and Δ_1 from Eqs. (2.17),(2.23), thus we are able to find the current:

$$J = \frac{2\pi\nu_0 DTSe}{\xi}\sum_{|\omega|<\omega_D}\left(\alpha\sin^2\theta_0\frac{d\chi_1}{dz} + \alpha^2\theta_1\frac{d\chi_1}{dz}\sin 2\theta_0\right). \tag{2.30}$$

First of all, due to current conservation (1.4), we can calculate the current at any point. It is convinient to do that at the interface $z = 0$. Secondly, here we can use well-known relations:

$$2\nu_0 De^2 = \sigma, \tag{2.31}$$

$$\frac{2\pi T}{\Delta_0}\sum_{\omega>0}\sin^2\theta_0 = \frac{2\pi T}{\Delta_0}\sum_{\omega>0}\frac{\Delta_0^2}{\Delta_0^2+\omega^2} = \frac{\pi}{2}\tanh\left(\frac{\Delta_0}{2T}\right). \tag{2.32}$$

Finally, one can see that the sums in the Eq. (2.30) converge, which allows us to extend the limits of summation to infinity, formally putting $\omega_D = \infty$. Therefore, the answer for the current can be written in the following form:

$$J = \frac{\pi\Delta_0}{2eR_N}\tanh\left(\frac{\Delta_0}{2T}\right)\sin\delta\varphi\left(1+\alpha\frac{\sum_{\omega>0}\theta_1(z=0)\sin 2\theta_0}{\sum_{\omega>0}\sin^2\theta_0}\right), \tag{2.33}$$

where R_N is the resistance of the junction $1/G_N = 1/(g_N S)$. To find the under the bracket one has to use Eq. (2.17):

$$\frac{2\pi T}{\Delta_0}\sum_{\omega>0}\theta_1(k)\sin 2\theta_0 = -4\left(1-\cos\delta\varphi\right)\frac{2\pi T}{\Delta_0}\sum_{\omega>0}\frac{\sin^2\theta_0\cos^2\theta_0}{k^2\sin\theta_0+1}\left[\frac{L_2(k,T)-L_4(k,T)}{k^2L_2(k,T)+L_3(k,T)}+\sin\theta_0\right] =$$
$$= -4\left(1-\cos\delta\varphi\right)\left[\frac{\left(L_2(k,T)-L_4(k,T)\right)^2}{k^2L_2(k,T)+L_3(k,T)}+L_3(k,T)-L_5(k,T)\right]. \tag{2.34}$$

Thus, the answer for the current can be expressed like:

$$\boxed{J = J_0\sin\delta\varphi\left[1-4\alpha\left(1-\cos\delta\varphi\right)V(T)\right]}, \tag{2.35}$$
$$J_0 = \frac{\pi\Delta_0}{2eR_N}\tanh\left(\frac{\Delta_0}{2T}\right), \tag{2.36}$$

where $V(T)$ is the positive constant, which depends on the temperature T, of the form:

$$\boxed{V(T)\equiv\coth\left(\frac{\Delta_0}{2T}\right)\int\limits_{-\infty}^{\infty}\frac{dk}{\pi^2}\left[\frac{\left(L_2(k,T)-L_4(k,T)\right)^2}{k^2L_2(k,T)+L_3(k,T)}+L_3(k,T)-L_5(k,T)\right]}. \tag{2.37}$$

Indeed, to obtain the sum in the current Eq. (2.33) one should apply the inverse Fourier transfomation on Eq. (2.34). The positiveness of the constant follows from the fact, that $L_3(k,T)-L_5(k,T)\geq 0,\ \forall k$.

The answer for the current given by Eq. (2.35) contains not only the standard part of the Josephson relation, $J_0\sin\delta\varphi$, but also the second harmonic ($\sin 2\delta\varphi$ with positive coefficient) and a negative correction to the first harmonic:

$$J = J_0\left[\sin\delta\varphi\left(1-4\alpha V(T)\right)+2\alpha V(T)\sin 2\delta\varphi\right] \tag{2.38}$$

Below we present results in the limiting cases of $T=0$ and $T\to T_c$.

2.2 Solvable temperature cases

2.2.1 $T \to 0$

In the $T \to 0$ limit, the sums of the form Eq. (2.22) can be replaced by the integrals

$$L_n(k,0) = \int_0^\infty \frac{1}{(w^2+1)^{\frac{n-1}{2}}} \frac{1}{k^2+\sqrt{w^2+1}} dw \tag{2.39}$$

There is a relationship between the sum of L_n and the sums with lower indices:

$$L_{n+1}(k,T) = \frac{2\pi T}{\Delta_0} \sum_{\omega>0} \frac{\sin^{n+1}\theta_0}{k^2 \sin\theta_0 + 1} = \frac{2\pi T}{\Delta_0} \sum_{\omega>0} \frac{(k^2\sin\theta_0 + 1 - 1)\sin^n\theta_0}{k^2(k^2\sin\theta_0+1)} = \frac{1}{k^2}\left[L_n(0,T) - L_n(k,T)\right] \tag{2.40}$$

Therefore, to calculate the sums at an arbitrary n, we use the following identity:

$$L_{n+1}(k,T) = \sum_{j=0}^{n-2} \frac{(-1)^j}{k^{2j+2}} L_{n-j}(0,T) - \frac{(-1)^n}{k^{2n-2}} L_2(k,T). \tag{2.41}$$

In the $T \to 0$ limit the integrals of the form Eq. (2.39) for $k = 0$ can be calculated:

$$L_n(0,0) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{n}{2}\right)}. \tag{2.42}$$

The sum $L_2(k,0)$ can also be evaluated and has the form:

$$L_2(k,0) = \begin{cases} \dfrac{1}{\sqrt{1-k^4}}\left[\dfrac{\pi}{2} - \arctan\left(\dfrac{k^2}{\sqrt{1-k^4}}\right)\right], & |k| \le 1 \\ \dfrac{1}{2\sqrt{k^4-1}} \ln\left|\dfrac{k^2+\sqrt{k^4-1}}{k^2-\sqrt{k^4-1}}\right|, & |k| \ge 1 \end{cases}. \tag{2.43}$$

Then we obtain:

$$\begin{aligned} L_3(k,0) =& \frac{\pi}{2k^2} - \frac{L_2(k,0)}{k^2}, \\ L_4(k,0) =& \frac{1}{k^2} - \frac{\pi}{2k^4} + \frac{L_2(k,0)}{k^4}, \\ L_5(k,0) =& \frac{\pi}{4k^2} - \frac{1}{k^4} + \frac{\pi}{2k^6} - \frac{L_2(k,0)}{k^6}. \end{aligned} \tag{2.44}$$

Plugging the obtained expressions into Eq. (2.23), we find $\Delta_1(k)$ and $\theta_1(k)$ [see Fig. 2.1]. We are unable to find the inverse Fourier transformation of $\Delta_1(k)$ analitically. In this work we find $\Delta_1(z)$ numerically. Figure 2.2 illustrates the correction to the order parameter $\Delta_1(z)/\Delta_0$ in the coordinate space at $T = 0$. Since Δ_1 is proportional to $2(1-\cos\delta\varphi)$, the plot is shown without this factor. As one can see, the result of calculations is in line with the expectations shown schematically in Fig. 1.1.

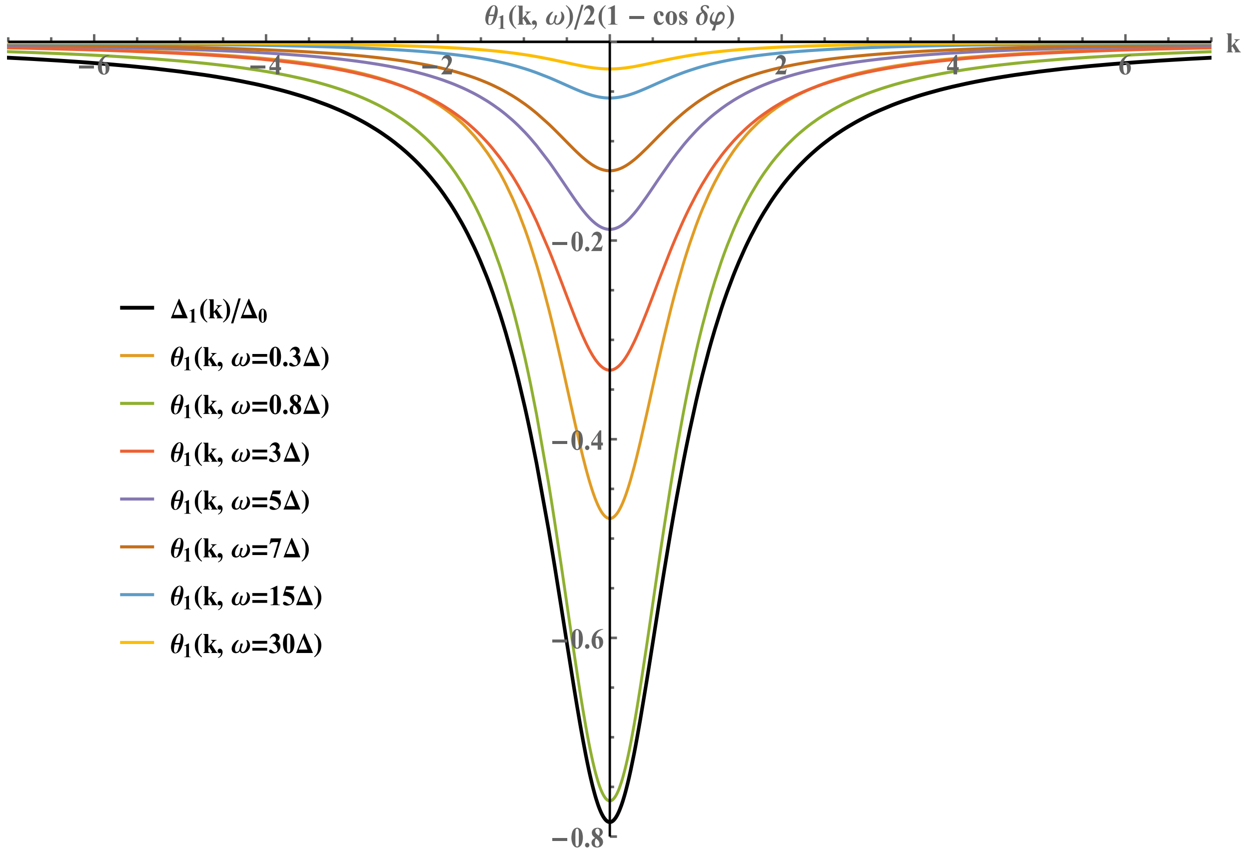

Figure 2.1: The plot of $\theta_1(k, \omega)$ and $\Delta_1(k)$ in the Fourier space at $T = 0$ without the factor $2(1 - \cos\delta\varphi)$.

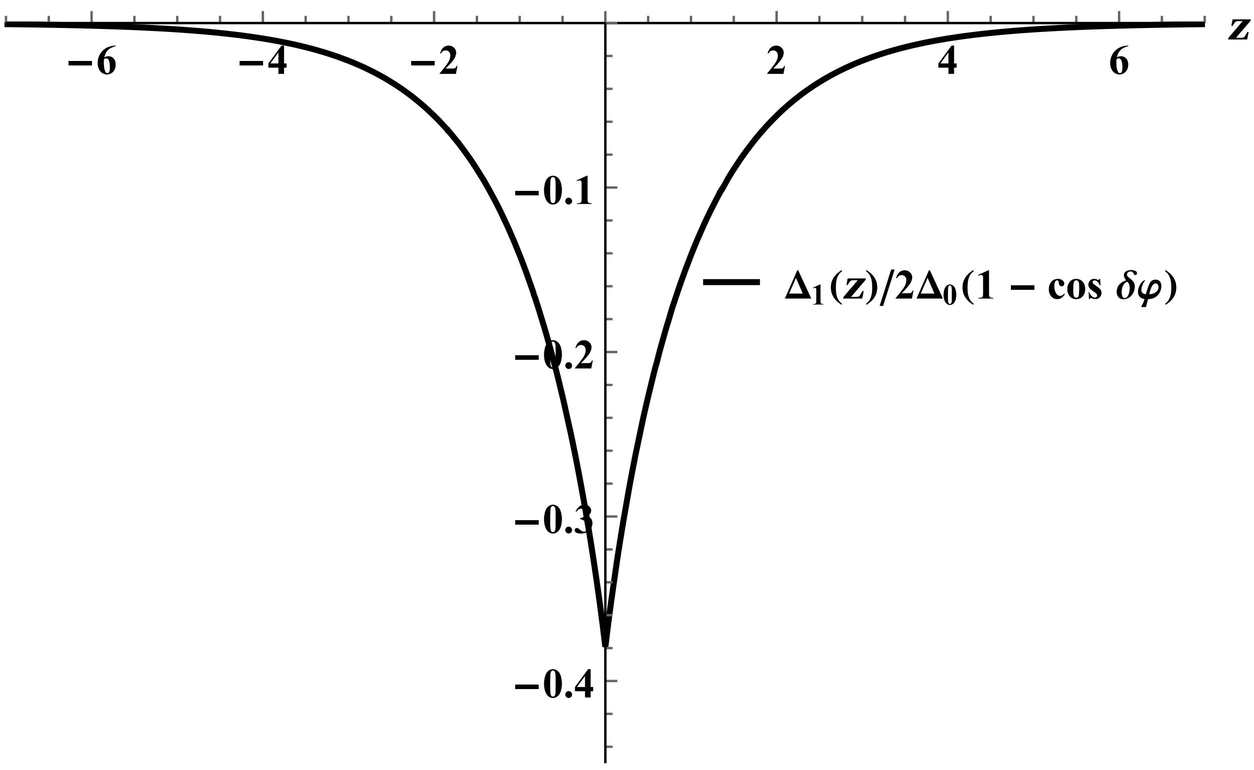

Figure 2.2: $\Delta_1(z)/\Delta_0$ plot at $T = 0$ without factor $2(1-\cos\delta\varphi)$. The correction to the order parameter is negative and the order parameter is most strongly suppressed in the vicinity of the interface.

At the same time, the current (2.35) in determined by $V(T)$ containing an integral with the L_n sums, see Eq. (2.37). Although we are not able to calculate the integral in Eq. (2.37) at $T = 0$ analytically, we can do it numerically obtaining:

$$\boxed{V(0) \approx 0.272}.$$

In the $T \to 0$ limit, the characteristic length scale becomes $\xi(0)$. At the same time, since $\xi(0) \sim \xi_{\rm GL}(0)$, we can write the answer with the help of the same definition for $G_D(T)$ as:

$$\boxed{J = \frac{\pi\Delta_0(0)}{2eR_N}\left\{[1 - 0.93\gamma(0)]\sin\delta\varphi + 0.46\gamma(0)\sin 2\delta\varphi\right\}}, \tag{2.45}$$

$$\gamma(T) \equiv \frac{G_N}{G_D(T)}, \tag{2.46}$$

where $G_D(T) = \sigma S/\xi_{\rm GL}(T)$ is the diffusive conductance of the superconductor on the length $\xi_{\rm GL}(T)$.

2.2.2 $T \to T_c$

In this limit, Δ_0 becomes small and has the form [23] (for more details see Appendix B):

$$\Delta_0 = \sqrt{\frac{8\pi^2 T_c(T_c - T)}{7\zeta(3)}}. \tag{2.47}$$

Therefore, we have to keep only the leading orders in Δ_0 in Eq. (2.23), Eq.(2.37) [i.e. one has to neglect higher orders/powers of $\sin\theta_0 \approx \Delta_0/\omega$ in the sums Eq. (2.22)]:

$$\frac{\Delta_1(k)}{\Delta_0} = -2\,(1 - \cos\delta\varphi)\,\frac{1}{k^2 + L_3(k,T)/L_2(k,T)}, \tag{2.48}$$

$$V(T) = \frac{2T_c}{\pi^2\Delta_0}\int\limits_{-\infty}^{\infty} dk \frac{L_2(k,T)}{k^2 + L_3(k,T)/L_2(k,T)}. \tag{2.49}$$

In these formulas, we may neglect the k dependence in the L_n sums putting $k = 0$. Indeed, in the $T \to T_c$ limit, we have $\omega \sim T_c$ and $\Delta_0 \ll T_c$, hence $\sin\theta_0 \approx \Delta_0/\omega \ll 1$ and $L_3/L_2 \sim \Delta_0/T_c \ll 1$. From Eq. (2.48) we see that $\Delta_1(k)$ varies on the scale of $k \sim \sqrt{\Delta_0/T_c}$. At the same time, the L_n sums, Eq. (2.22), vary on the scale of $k \sim \sqrt{T_c/\Delta_0} \gg \sqrt{\Delta_0/T_c}$.

We thus obtain:

$$\frac{\Delta_1(k)}{\Delta_0} = -\frac{2\,(1-\cos\delta\varphi)}{k^2+7\zeta(3)\Delta_0/\pi^3 T_c} \Rightarrow \frac{\Delta_1\,(z)}{\Delta_0} = -\pi\,(1-\cos\delta\varphi)\cdot\frac{\exp\left\{-\frac{[56\zeta(3)(1-T/T_c)]^{1/4}}{\pi}\,|z|\right\}}{[56\zeta(3)\,(1-T/T_c)]^{1/4}}, \tag{2.50}$$

$$\boxed{V(T) = \sqrt{\frac{\pi^3 T_c}{28\zeta(3)\Delta_0}} = \frac{\pi}{2}\left[\frac{1}{56\zeta(3)\,(1-T/T_c)}\right]^{1/4}}. \tag{2.51}$$

Thus, the current Eq. (2.35) has the form:

$$\boxed{J = \frac{\pi\Delta_0^2(T)}{4eR_N T_c}\left\{\left[1-\sqrt{2}\gamma(T)\right]\sin\delta\varphi + \frac{\gamma(T)}{\sqrt{2}}\sin 2\delta\varphi\right\}}. \tag{2.52}$$

2.3 Discussion of the results

Equations (2.35),(2.45),(2.52) are the main results of this work. In the limit $T \to T_c$, Eq. (2.35) reduces to Eq. (2.52) and reproduces the result by Kupriyanov [1]. The answer (2.35) for arbitrary T and its $T \to 0$ limit, Eq. (2.45), are new results.

The Josephson relation with the second harmonic in the SIS tunnel junction at arbitrary T was previously derived by Golubov and Kupriyanov [2]. In their paper, the Usadel equations (1.1)–(1.3), (1.19), and (1.20) were solved in the coordinate space. The authors employed a conjecture for the form of the solution to the full self-consistent problem[2] . On the contrary, our perturbation theory allows systematic rigorous calculation of the solution. The results of Ref. [2] for the Josephson current turn out to be parametrically correct but with wrong numerical coefficients in front of γ corrections. Our theory provides exact values of the coefficients.

2.4 Role of self-consistency

While self-consistency for the order parameter is inherent in our calculations, it may be instructive to discuss its role, considering what changes if the self-consistency is neglected and we simply put $\Delta = \Delta_0$. Below, we discuss how this would change the results for the Josephson current J.

[2]In Ref. [2], the perturbation theory was developed in the coordinate space. Two issues indicate that the presented form of solution is not rigorous (we call it "conjectured"). (i) In Eq. (31) of Ref. [2], the order parameter and the quantity parameterizing the Green functions are expanded in the system of decaying exponents. The system does not form a full basis in the functional space, which means that actually only a certain class of functions is considered. (ii) Equation (34) in Ref. [2] is obtained from Eqs. (32) and (33) according to the procedure described below Eq. (33). This procedure leads to equality between two sums running over *different* quantities (ω and Ω). In order to obtain Eq. (34), one should equate term-by-term the elements of these *different* sums. This assumption also implies a certain conjecture about the form of solution.

Neglecting self-consistency implies putting $\Delta_1(k) = 0$ in Eq. (2.17). Following step-by-step the algorithm described above in subsection (2.1), one would then obtain

$$V(T) = \coth\left(\frac{\Delta_0}{2T}\right) \int_{-\infty}^{\infty} \frac{dk}{\pi^2} \left(L_3(k,T) - L_5(k,T)\right). \tag{2.53}$$

Neglecting self-consistency thus leads to dropping out the first term under the integral in Eq. (2.37).

In the limit of low temperatures, $T \to 0$, neglecting self-consistency makes the result for the numerical coefficient V, Eq. (2.53), valid only by the order of magnitude. Indeed, for the frequencies $\omega \sim \Delta_0$, we have $\sin\theta_0 \sim 1$, which means that the L_n sums defined in Eq. (2.22) are all of the same order and vary on the scale of $k \sim 1$. Therefore, both the terms under the integral in Eq. (2.37) are of the same order. Numerical calculations in the limit $T \to 0$ give for Eq.(2.53) the answer:

$$V(T) = 0.152. \tag{2.54}$$

In the case of approaching the critical temperature, $T \to T_c$, self-consistency begins to play a major role. Indeed, in this limit, we have $\sin\theta_0 \ll 1$, hence the L_n sums are of the order of $(\Delta_0/T_c)^{n-1}$ and vary on the scale of $k \sim \sqrt{T_c/\Delta_0} \gg 1$. Substituting this into the integral in Eq. (2.37), one finds that the first term (which is due to self-consistency) gives a contribution of the order of $\sqrt{T_c/\Delta_0}$ [see the answer for $V(T)$ for $T \to T_c$ limit, Eq. (2.51)], while the $(L_3 - L_5)$ term gives a contribution of the order of $\sqrt{\Delta_0/T_c} \ll \sqrt{T_c/\Delta_0}$. The major role of self-consistency in this case is expectable since in the $T \to T_c$ limit, the Usadel equations reduce to the GL equations (for more details see Appendix B or [24]) , so all information about spatial variations of superconducting characteristics inside the superconducting banks must be encoded in the $\Delta(x)$ function. Neglecting this spatial dependence would mean neglecting the corrections due to finite interface conductance, which is the main effect considered in this paper.

We thus conclude that taking into account self-consistency in our problem is necessary in order to obtain quantitatively and qualitatively correct results.

2.5 Applicability conditions of the perturbation theory

The condition of weak proximity effect, which we assumed when developing our perturbation theory, can be formulated according to Eq. (2.8) as

$$\alpha \frac{|\Delta_1(z=0)|}{\Delta_0} \ll 1. \tag{2.55}$$

In the Fourier representation, the result for Δ_1 is given by Eq. (2.23). At T not too close to T_c, this yields $|\Delta_1(z=0)| \sim \Delta_0$, so that the condition becomes $\alpha \ll 1$. At the same time, at $T \to T_c$, Eq. (2.50) demonstrates that $|\Delta_1(z=0)|/\Delta_0 \sim \sqrt{T_c/\Delta_0(T)} \sim (1 - T/T_c)^{-1/4}$.

Summarizing, at all temperatures, the condition of smallness of α can be written as

$$\boxed{\alpha \ll \left(1 - \frac{T}{T_c}\right)^{1/4}}. \tag{2.56}$$

Note that the α parameter itself depends on T in the vicinity of T_c as $\alpha \propto (1 - T/T_c)^{-1/4}$ [see the definition of $\xi(T)$, Eq. (1.23), and Eq. (2.47)].

Alternatively, condition (2.56) can be written as $\gamma(T) \ll 1$, where $\gamma(T)$ is defined by Eq. (2.46). The limiting results for the Josephson current, Eqs. (2.52) and (2.45), confirm that this is indeed the condition of smallness of the corrections to the Josephson relation.

3 Second-order perturbation theory for the phases

3.1 Arbitrary temperatures perturbation theory

As we have shown in Eq. (2.27) in the first order of the perturbation theory, the phases $\chi_1(z,\omega)$ and $\varphi_1(z)$ are the same at all frequencies. In this section, we show that the second-order perturbation theory yields $\chi_2(z,\omega) \neq \varphi_2(z)$.

We start the second-order perturbation theory by expanding Eqs. (1.2), (A.42), and (1.26) up to α^2. Thus, we obtain:

$$\sin^2\theta_0 \frac{d^2\chi_2}{dz^2} + 2\frac{d\theta_1}{dz}\cos\theta_0 \sin\delta\varphi = \chi_2 - \varphi_2, \tag{3.1}$$

$$\sum_{|\omega|<\omega_D} (\chi_2 - \varphi_2)\sin\theta_0 = 0, \tag{3.2}$$

$$\frac{d\chi_2(\pm 0)}{dz} = 0. \tag{3.3}$$

Before doing the basic calculations, we will calculate the helper functions $\chi_2^{(0)}(z,\omega)$ and $\varphi_2^{(0)}(z)$. By definition, these functions satisfy the Eqs. (3.1), (3.2), are continuous at any z and vanish in the bulk. The equations for these functions have the form:

$$\sin^2\theta_0 \frac{d^2\chi_2^{(0)}}{dz^2} + 2\frac{d\theta_1}{dz}\cos\theta_0 \sin\delta\varphi = \chi_2^{(0)} - \varphi_2^{(0)}, \tag{3.4}$$

$$\sum_{|\omega|<\omega_D} \left(\chi_2^{(0)} - \varphi_2^{(0)}\right)\sin\theta_0 = 0. \tag{3.5}$$

The system is linear, thus it can be solved via Fourier transformation. The Fourier transformation of Eq. (3.4) gives:

$$\chi_2^{(0)}(k) = \frac{1}{k^2\sin\theta_0 + 1}\left[\varphi_2^{(0)}(k) + 2ik\theta_1(k)\cos\theta_0\sin\delta\varphi\right]. \tag{3.6}$$

(we omit the ω argument of $\chi_2^{(0)}$, $\varphi_2^{(0)}$, θ_1, and θ_0 for brevity). Substituting Eq. (3.6) into Eq. (3.5), we find:

$$\varphi_2^{(0)}(k) = \frac{i}{kL_2(k)} \cdot \frac{2\pi T}{\Delta_0} \sum_{\omega>0} \frac{\theta_1(k)\sin 2\theta_0}{k^2\sin\theta_0 + 1}\sin\delta\varphi. \tag{3.7}$$

Since the sum in this equation converge, we can extend the limits of summation to infinity, formally putting $\omega_D = \infty$.

Our actual problem for finding χ_2 and φ_2, defined by Eqs. (3.1)–(3.3), is more complicated than the one for $\chi_2^{(0)}$ and $\varphi_2^{(0)}$ due to two circumstances. First, current conservation, Eq. (1.4), leads to nonzero correction to the velocity of the Cooper pairs in the bulk, i.e.,

$d\varphi_2/dz \neq 0$ at $z \to \infty$, which leads to delta-functional singularity in the Fourier transformation of χ_2 and φ_2. Since $\chi = \varphi$ in the bulk, it is possible to solve the system of equations for

$$\phi_2(z,\omega) \equiv \chi_2(z,\omega) - \varphi_2(z), \tag{3.8}$$

from which the singularity drops out. Second, χ_2 can be discontinuous at $z = 0$, with a (yet unknown) phase jump $\delta\chi_2(\omega)$, see Eq. (2.11), which leads to a singularity in Eq. (3.1):

$$\sin\theta_0 \frac{d^2\phi_2}{dz^2} + \sin\theta_0 \frac{d^2\varphi_2}{dz^2} + 2\frac{d\theta_1}{dz}\cos\theta_0 \sin\delta\varphi = \phi_2 + \delta\chi_2 \delta'(z)\sin\theta_0. \tag{3.9}$$

The Fourier transformation of Eq. (3.9) gives:

$$\phi_2(k) = \frac{ik}{k^2\sin\theta_0 + 1}\left[(ik\varphi_2(k) - \delta\chi_2)\sin\theta_0 + 2\theta_1\cos\theta_0\sin\delta\varphi\right]. \tag{3.10}$$

The self-consistency equation (3.2) in the Fourier space have the form:

$$\sum_{|\omega|<\omega_D} \phi_2(k)\sin\theta_0 = 0. \tag{3.11}$$

Finally, due to discontinuity at $z = 0$, the derivative of χ_2 contains the delta-functional contribution $\delta\chi_2\delta(z)$. The boundary condition (3.3) contains only one-sided limits at $z = 0$, so in order to write Eq. (3.3) in the Fourier space, we have to subtract from the Fourier transform the singularity due to the phase jump $\delta\chi_2\delta(z)$:

$$\lim_{z\to 0}\left(\int_{-\infty}^{\infty} \frac{dk}{2\pi} ik\phi_2(k)e^{ikz} - \delta\chi_2\delta(z)\right) = 0. \tag{3.12}$$

The system of Eqs. (3.10), (3.11), and (3.12) determines $\phi_2(z,\omega)$ and $\delta\chi_2(\omega)$. In order to find these functions, we employ an algorithm similar to the one used in the case of θ and Δ. First, we substitute Eq. (3.10) into Eq. (3.11) and then find $ik\varphi_2(k)$ and $\phi_2(k)$. We still do not know $\delta\chi_2(\omega)$ but we can find it from Eq. (3.12).

From the self-consistency equation (3.11) we find the connection between $\varphi_2(k)$ and $\delta\chi_2$:

$$k^2\varphi_2(k) = \frac{ik}{L_2(k)} \cdot \frac{2\pi T}{\Delta_0} \sum_{\omega>0} \frac{\theta_1 \sin 2\theta_0}{k^2\sin\theta_0 + 1} \sin\delta\varphi - \frac{ik}{L_2(k)} \cdot \frac{2\pi T}{\Delta_0} \sum_{\omega>0} \frac{\delta\chi_2 \sin^2\theta_0}{k^2\sin\theta_0 + 1}. \tag{3.13}$$

Here we define *the phase functional* $F[\delta\chi_2](k)$:

$$\boxed{F\left[\delta\chi_2\right](k) \equiv \frac{2\pi T}{\Delta_0} \sum_{\omega>0} \frac{\delta\chi_2 \sin^2\theta_0}{k^2\sin\theta_0 + 1}}. \tag{3.14}$$

Using Eqs. (3.10), (3.6), (3.7), and the definition of the phase functional, Eq. (3.14), we obtain:

$$k^2\varphi_2(k) = k^2\varphi_2^{(0)}(k) - \frac{ikF(k)}{L_2(k)}, \tag{3.15}$$

$$\phi_2(k) = \chi_2^{(0)}(k) - \varphi_2^{(0)}(k) + \frac{ik\sin\theta_0}{k^2\sin\theta_0+1}\frac{F(k)}{L_2(k)} - \frac{ik\delta\chi_2\sin\theta_0}{k^2\sin\theta_0+1}. \tag{3.16}$$

The next step is to use the boundary condition (3.12). Substituting there Eq. (3.16), we obtain:

$$\frac{\delta\chi_2}{2\sqrt{\sin\theta_0}} = \int\limits_{-\infty}^{\infty}\frac{dk}{2\pi}\left[ik\left(\chi_2^{(0)} - \varphi_2^{(0)}\right) - \frac{k^2\sin\theta_0}{k^2\sin\theta_0+1}\frac{F(k)}{L_2(k)}\right]. \tag{3.17}$$

This can be transformed as:

$$\frac{\delta\chi_2}{2\sqrt{\sin\theta_0}} = \frac{d\chi_2^{(0)}(z=0)}{dz} - \frac{d\varphi_2^{(0)}(z=0)}{dz} - \int\limits_{-\infty}^{\infty}\frac{F(k)}{L_2(k)}\frac{dk}{2\pi} + \int\limits_{-\infty}^{\infty}\frac{1}{k^2\sin\theta_0+1}\frac{F(k)}{L_2(k)}\frac{dk}{2\pi}. \tag{3.18}$$

Below for brevity we denote $d\chi_2^{(0)}(z=0)/dz$ by $\chi_2^{\prime(0)}(0)$, and similar notation is used for $\varphi_2^{(0)}$.

Now, we multiply Eq. (3.18) by $2\pi T\sin^2\theta_0/\Delta_0$ and sum over $\omega > 0$. Then by definition of L_2, see Eq. (2.22), we have

$$\frac{\pi T}{\Delta_0}\sum_{\omega>0}\delta\chi_2\sin^{3/2}\theta_0 - \int\limits_{-\infty}^{\infty}F(k)\frac{dk}{2\pi} = \frac{2\pi T}{\Delta_0}\sum_{\omega>0}\sin^2\theta_0\left(\chi_2^{\prime(0)}(0) - \varphi_2^{\prime(0)}(0) - \int\limits_{-\infty}^{\infty}\frac{F(k)}{L_2(k)}\frac{dk}{2\pi}\right). \tag{3.19}$$

The left-hand side turns to zero after integration of the phase functional, Eq. (3.14), over k. At the same time, in the right-hand side we have a contribution:

$$\boxed{\int\limits_{-\infty}^{\infty}\frac{F(k)}{L_2(k)}\frac{dk}{2\pi} = \frac{\sum_{\omega>0}\chi_2^{\prime(0)}(0)\sin^2\theta_0}{\sum_{\omega>0}\sin^2\theta_0} - \varphi_2^{\prime(0)}(0)}. \tag{3.20}$$

We denote:

$$V_0 \equiv \frac{\sum_{\omega>0}\chi_2^{\prime(0)}(0)\sin^2\theta_0}{\sum_{\omega>0}\sin^2\theta_0}. \tag{3.21}$$

Substituting this result into Eq. (3.18), we obtain:

$$\boxed{\frac{\delta\chi_2}{2\sqrt{\sin\theta_0}} = \chi_2^{\prime(0)}(0) - V_0 + \int\limits_{-\infty}^{\infty}\frac{1}{k^2\sin\theta_0+1}\frac{F(k)}{L_2(k)}\frac{dk}{2\pi}}. \tag{3.22}$$

In order to calculate V_0, we consider the Fourier transform of $\varphi_2'(z)$ and employing Eq. (3.15) we get:

$$ik\varphi_2(k) = ik\varphi_2^{(0)}(k) + \frac{F(k)}{L_2(k)} + \beta\delta(k), \tag{3.23}$$

where β is an unknown coefficient. Since $\varphi_2(z)$ is a continuous function at $z = 0$ and $\varphi_2'(0) = 0$ due to the boundary condition (3.3), we obtain:

$$0 = \varphi_2'^{(0)}(0) + \int\limits_{-\infty}^{\infty} \frac{F(k)}{L_2(k)} \frac{dk}{2\pi} + \frac{\beta}{2\pi}. \tag{3.24}$$

From equation (3.20) we find:

$$\beta = -2\pi V_0. \tag{3.25}$$

Finally, due to the current conservation, we can consider the current in the bulk where $\theta_1 = 0$. Employing Eq. (2.35), we obtain [also see the expansion for the current up to α^2, Eq. (2.12)]:

$$J_0 \sin\delta\varphi \left[1 - 4\alpha\left(1 - \cos\delta\varphi\right) V(T)\right] = J_0 \left(\sin\delta\varphi + \alpha\frac{d\varphi_2(z\to\infty)}{dz}\right). \tag{3.26}$$

Expressing $d\varphi_2(z = \infty)/dz$ with the help of Eq. (3.23), we obtain:

$$\boxed{V_0 = -\frac{d\varphi_2(z\to\infty)}{dz} = 4V\left(1 - \cos\delta\varphi\right)\sin\delta\varphi}. \tag{3.27}$$

The answer for the phases χ_2 and φ_2 thus reads [see Eq. (3.10) for the connection between $ik\chi_2(k)$ and $ik\varphi_2(k)$]:

$$\boxed{ik\chi_2(k) = ik\chi_2^{(0)}(k) + \frac{1}{k^2\sin\theta_0 + 1}\frac{F(k)}{L_2(k)} + \frac{\delta\chi_2\cdot k^2\sin\theta_0}{k^2\sin\theta_0 + 1} - 2\pi V_0\delta(k)}, \tag{3.28}$$

$$\boxed{ik\varphi_2(k) = ik\varphi_2^{(0)}(k) + \frac{F(k)}{L_2(k)} - 2\pi V_0\delta(k)}. \tag{3.29}$$

The inverse Fourier transformation gives the derivatives $d\chi_2/dz$ and $d\varphi_2/dz$, from which we can find $\chi_2(z,\omega)$ and $\varphi_2(z)$, respectively. Even without explicit implementation of this algorithm, we can make sure that $\chi_2 \neq \varphi_2$. Indeed, if we assume that $\chi_2 = \varphi_2$, then Eq. (3.22) immediately simplifies to the form:

$$\frac{d\chi_2^{(0)}(z=0,\omega)}{dz} = V_0, \tag{3.30}$$

which cannot be satisfied since both $\chi_2^{(0)}(z,\omega)$ and its derivative at $z = 0$ have nontrivial dependence on ω (as witnessed, for example, by numerical calculations). This proves that

$\chi_2 \neq \varphi_2$. Moreover, this result is a consequence of Eq. (3.1), which contains $d\theta_1/dz$ that plays the role of the nonzero source in this equation.

3.1.1 Bulk behavior

The final step is to find out, how does $\chi_2(z, \omega)$ behave in the bulk, where we have supposed it to be equal to $\varphi_2(z)$. From Eqs. (3.28) and (3.29), one can see that in the bulk, the phases χ_2 and φ_2 are equal and vary linearly as:

$$\chi_2(z \to \infty, \omega) = a \,\mathrm{sgn}\, z + bz, \tag{3.31}$$

with constant coefficients a and b. Our goal now is to find them.

Since $\chi_2(k)$ is an odd function, we can write:

$$\chi_2(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} \chi_2(k) \int_{-\infty}^{\infty} \frac{dk}{2\pi} (\cos kz + i \sin kz)\, \chi_2(k) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\sin kz}{k} ik\chi_2(k). \tag{3.32}$$

Substituting here the relation:

$$\frac{\sin kz}{\pi k} \underset{z\to\infty}{\to} \mathrm{sgn}(z)\delta(k) \tag{3.33}$$

and employing Eq. (3.28), we find:

$$a = \frac{1}{2}\left(\lim_{k\to 0} ik\chi_2^{(0)}(k) + \frac{F(0)}{L_2(0)}\right), \quad b = -V_0. \tag{3.34}$$

The constant a can be found with the use of Eq. (3.6), and in terms of the sums defined in Eq. (2.22), it acquires the form:

$$a = 2\,(1 - \cos\delta\varphi)\sin\delta\varphi \left[\frac{(L_2 - L_4)^2 + L_3\,(L_3 - L_5)}{L_2 L_3} + \frac{F}{L_2}\right]_{k=0}. \tag{3.35}$$

Both constants a and b do not depend on ω, which shows that $\chi_2 = \varphi_2$ in the bulk.

3.2 Numerical results for $T = 0$

Equation (3.22) can be solved numerically, and we present the results of this procedure in the case of $T = 0$ in Figs. 3.1 and 3.2. Both the figures confirm that $\chi \neq \varphi$. From Fig. 3.1, we see that $\delta\chi_2(\omega)$ is an alternating function, which could be inferred from Eq. (3.2) at

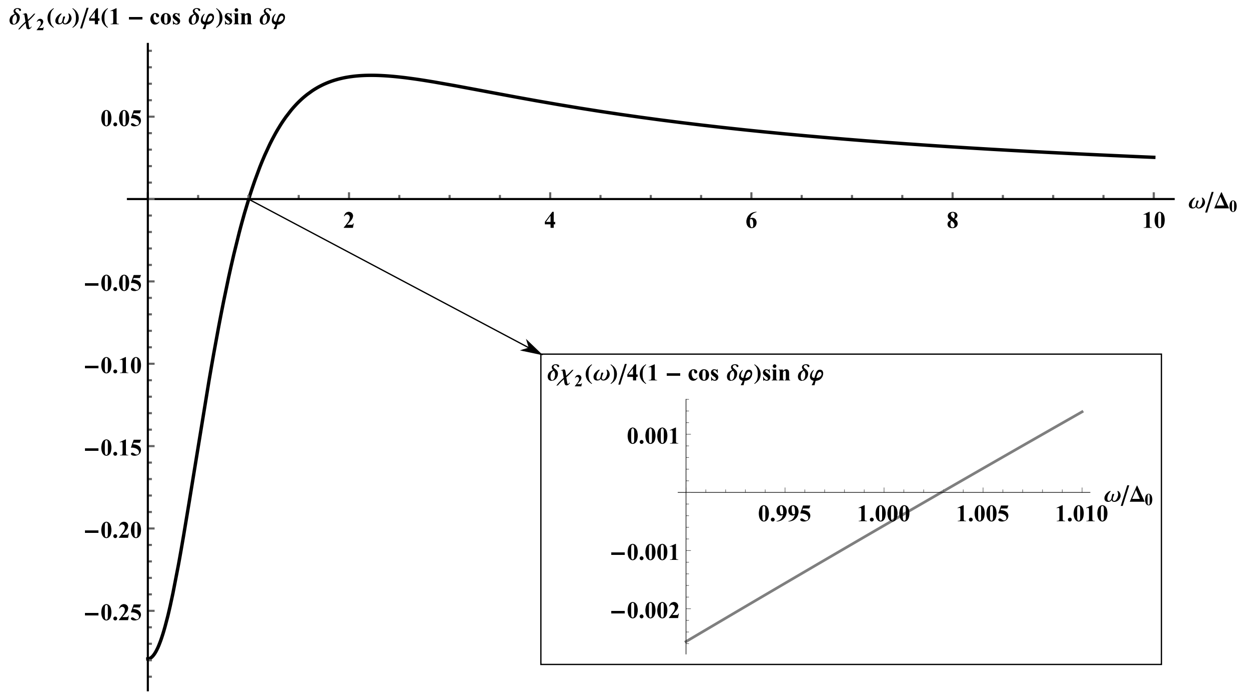

Figure 3.1: $\delta\chi_2(\omega)$ plot at $T = 0$ without factor $4(1-\cos\delta\varphi)\sin\delta\varphi$. Interestingly, the curve crosses the abscissa very close to the $\omega = \Delta_0$ point (see the inset, the difference of the point of intersection from 1 may be due to the error of the numerical method). While this may be a hint to an exact property, we do not have a proof for that.

$z = \pm 0$. Indeed, due to the continuity of corrections $\varphi_1(z)$ and $\varphi_2(z)$, we obtain:

$$\sum_{\omega} \delta\chi_2(\omega)\sin\theta_0 = 0. \tag{3.36}$$

The sum can turn to zero only if $\delta\chi_2(\omega)$ changes its sign.

From the plot Fig. 3.2 we see how the phases $\chi_2(z,\omega)$ and $\varphi_2(z)$ depend on z in a nonlinear manner (such nonlinear dependence was discussed in Ref. [13] in the case of SNS junction). $\chi_2(z,\omega)$ and $\varphi_2(z)$ become equal in the bulk and change linear with the slope $-V_0(T)$. The overall nonlinear spatial dependence of the phases corresponds to increased velocity of the superconducting condensate in the vicinity of the interface. This compensates for the interface suppression of the order parameter (see Fig. 2.2) and, hence, of the condensate density (due to the proximity effect between the superconducting banks) in order to provide position-independent Josephson current throughout the system.

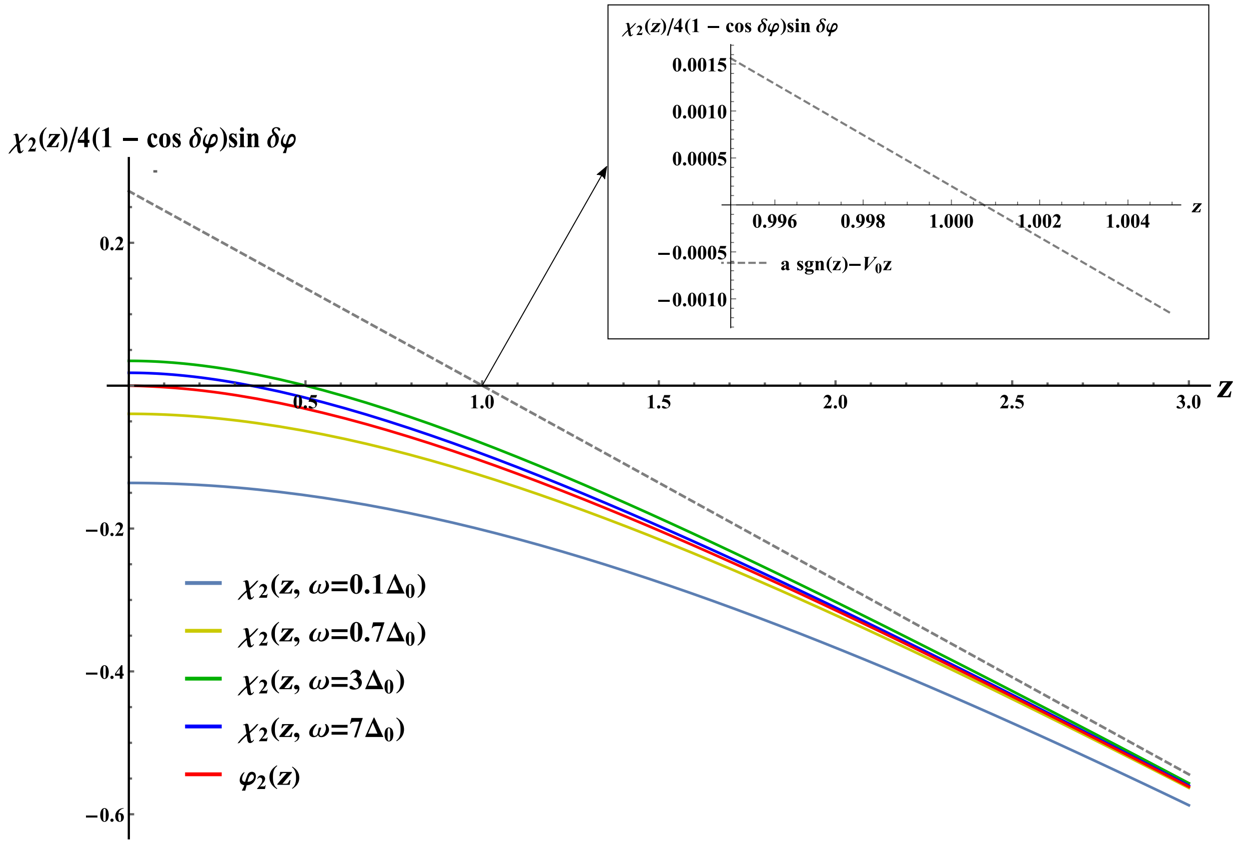

Figure 3.2: $\chi_2(z)$ plot at different ω without factor $4(1-\cos\delta\varphi)\sin\delta\varphi$. Since χ_2 is an odd function, nonzero values $\chi_2(z=0)\neq 0$ signify that this function is discontinuous at $z=0$. At $z\to\infty$, all the curves become linear. Interestingly, this linear dependence of the form $a\,\mathrm{sgn}\,z+bz$ crosses the z axis very close to $z=1$ (see the inset, the difference of the point of intersection from 1 may be due to the error of the numerical method). While this may be a hint to an exact property, we do not have a proof for that. The figure demonstrates that $\delta\chi_2$ changes nonmonotonically as the function of ω: the curve for $\omega=7\Delta_0$ is lower than for $\omega=3\Delta_0$, but higher than for $\omega=0.7\Delta_0$ (this correlates with the result of Fig. 3.1). In the $\omega\to\infty$ limit, the $\chi_2(z,\omega)$ curves converge to $\varphi_2(z)$.

4 Conclusions and results

We have considered the Josephson effect in a planar diffusive SIS-type junction at arbitrary temperature T and constructed fully self-consistent perturbation theory with respect to the dimensionless conductance parameter $\alpha \ll 1$, which is the ratio of the interface conductance to the conductance of the superconducting material on the coherence length. We have presented analytical analysis of two orders of the perturbation theory.

The first order of the perturbation theory provides correction Δ_1 to the absolute value of the order parameter, see Eq. (2.23) and Fig. 2.2. In the coordinate space, $\Delta(z)$ is suppressed in the vicinity of the interface. Knowledge of Δ_1 makes it possible to find θ_1. In its turn, θ_1 provides the answer for the Josephson current up to the α^2 order, which contains not only the standard part $J(\delta\varphi) \propto \sin\delta\varphi$ but also a (negative) correction to the first harmonic and the second harmonic $\sin 2\delta\varphi$ (with a positive amplitude). We further analyze the general answer given by Eq. (2.35), in two limiting cases, see Sec. 2.2.1 and 2.2.2. In the $T \to T_c$ limit, we reproduce the result by Kupriyanov [1], while our results in the $T \to 0$ limit (as well as in the case of arbitrary temperature) are new. Although the same problem at arbitrary temperature has been considered before in Ref. [2], the corrections to the Josephson relation obtained there were only parametrically correct due to a conjectured form of solution. Our theory provides rigorous solution, which results in exact numerical coefficients.

Our perturbation theory also provides solutions for the superconducting phases of the anomalous Green functions and of the order parameter, χ and φ, respectively. In the zeroth order, the phases are equal constants corresponding to the standard main-order solution for the Josephson effect in tunnel junctions. In the first order, the phases are still equal but acquire the linear part which describes finite velocity of the superconducting condensate at each point of the superconductors. Finally, in the second order, we find that $\chi \neq \varphi$. We present the plot of $\chi_2(z,\omega)$ at different Matsubara frequencies and of $\varphi_2(z)$ at $T = 0$ in Fig. 3.2. We also illustrate the frequency dependence of the phase jumps $\delta\chi_2(\omega)$ at $T = 0$ in Fig. 3.1 (note that the phase jumps $\delta\varphi_2$ are absent by definition).

The overall spatial dependence of the phases is nonlinear, corresponding to increased velocity of the superconducting condensate in the vicinity of the interface. This compensates for the interface suppression of the order parameter and, hence, of the condensate density (due to the proximity effect between the superconducting banks) in order to provide position-independent Josephson current throughout the system, see Fig. 1.1.

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A The Usadel equations

In this appendix, we give a brief description of the equations describing superconductivity, as well as some important definitions and properties of the Usadel equations.

One of the key characteristics describing superconductivity in solids are the Green's functions, which are defined as follows:

$$G\left(1;2\right) \equiv -\left\langle T_\tau \psi_\uparrow(1)\psi_\uparrow^\dagger(2)\right\rangle,\ i = (\mathbf{r}_i, \tau_i),\ i = \overline{1,2}, \tag{A.1}$$

$$\tilde{G}\left(1;2\right) \equiv \left\langle T_\tau \psi_\downarrow^\dagger(1)\psi_\downarrow(2)\right\rangle, \tag{A.2}$$

$$F\left(1;2\right) \equiv \left\langle T_\tau \psi_\uparrow(1)\psi_\downarrow(2)\right\rangle, \tag{A.3}$$

$$\tilde{F}\left(1;2\right) \equiv -\left\langle T_\tau \psi_\downarrow^\dagger(1)\psi_\uparrow^\dagger(2)\right\rangle, \tag{A.4}$$

where ψ, $\psi^\dagger$ are the annihilation/creation operators of the electron respectively, $\mathbf{r}$ is the coordinate of the electron in the Cooper pair, τ is the time, and T_τ is the time ordering operator .

In order to obtain these Green functions one must write Gorkov equations [25]:

$$\left[i\omega\hat{\sigma}_z + \frac{1}{2m}\nabla^2_{\mathbf{r}_1} + \mu + i\hat{\Delta}\left(\mathbf{r}_1\right)\right]\hat{G}_\omega = \delta\left(\mathbf{r}_1 - \mathbf{r}_2\right), \tag{A.5}$$

$$\left[i\omega\hat{\sigma}_z + \frac{1}{2m}\nabla^2_{\mathbf{r}_2} + \mu + i\hat{\Delta}^{\mathrm{T}}\left(\mathbf{r}_2\right)\right]\hat{G}^{\mathrm{T}}_\omega = \delta\left(\mathbf{r}_1 - \mathbf{r}_2\right), \tag{A.6}$$

$$\Delta\left(\mathbf{r}\right) = igT\sum_{|\omega|<\omega_D} F_\omega\left(\mathbf{r},\mathbf{r}\right), \tag{A.7}$$

$$\hat{G} \equiv \begin{pmatrix} G & F \\ \tilde{F} & \tilde{G}\end{pmatrix}, \quad \hat{\Delta} \equiv \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0\end{pmatrix}, \tag{A.8}$$

where μ is the chemical potential, m is the mass of the electron, $\omega = \pi T(2n+1)$, $n \in \mathbb{Z}$ are Matsbara frequencies, g – BCS coupling constant, Δ is the orer parameter, and T is the temperature.

Analyzing SIS contact using this set of equations is a daunting task. Therefore, it is necessary to make a number of approximations.

It is convinient to change variables with the use of Wigner transformation:

$$\mathbf{R} \equiv \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \quad \mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2, \tag{A.9}$$

$$f\left(\mathbf{R},\mathbf{p}\right) \equiv \int_{-\infty}^{\infty} f\left(\mathbf{R},\mathbf{r}\right)e^{-i\mathbf{pr}}d\mathbf{r}, \tag{A.10}$$

$$\mathbf{p} \approx \mathbf{n}\left(p_F + \frac{X}{v_F}\right), \tag{A.11}$$

where $\mathbf{R}$ is the coordinate of the center of mass of the Cooper pair, $\mathbf{p}$ is the relative momentum of the electron in Cooper pair, p_F, v_F are Fermi momentum and velocity respectively. The first approximation lies in the fact that the Fermi wave length λ_F is much less that the coherence length ξ – the scale of $\mathbf{R}$ at which the Green functions changes. This means that the relative motion of electrons in a pair is much faster than the movement of the pair itself as a whole. Therefore, one should consider derivatives on $\mathbf{R}$ in the Gorkov Eqs. (A.5)–(A.6) to be small. This approximation allows one to derive semiclassical superconductivity equations, called the Eilenberger equations [26]:

$$v_F \mathbf{n} \nabla_{\mathbf{R}} \hat{g}\left(\mathbf{R}, \mathbf{n}\right) + \left[\hat{\omega}, \hat{g}\left(\mathbf{R}, \mathbf{n}\right)\right] = 0, \tag{A.12}$$

$$\hat{\omega} \equiv \omega \hat{\sigma}_z + \hat{\Delta}\left(\mathbf{R}\right) + \frac{1}{2\tau} \int \frac{d\mathbf{n}}{4\pi} \hat{g}\left(\mathbf{R}, \mathbf{n}\right), \tag{A.13}$$

$$\hat{g}\left(\mathbf{R}, \mathbf{n}\right) \equiv 2i \int\limits_{-\infty}^{\infty} \frac{dX}{2\pi} \hat{G}\left(\mathbf{R}, \mathbf{n}, X\right), \tag{A.14}$$

$$\Delta\left(\mathbf{R}\right) = \pi \underbrace{\nu_0 g}_{\lambda} T \sum_{|\omega| < \omega_D} \int \frac{d\mathbf{n}}{4\pi} f_\omega\left(\mathbf{R}, \mathbf{n}\right), \tag{A.15}$$

$$\hat{g}^2 = 1, \tag{A.16}$$

$$\tilde{g} = -g, \tag{A.17}$$

where ν_0 is the density of states at the Fermi level in the normal state, τ is the mean free time (we consider a superconductor with impurities). The combination $\nu_0 g = \lambda$ is called BCS coupling constant.

In the diffusive, or so-called dirty, limit, when the superconducting coherence length ξ is much larger than the mean free path l, superconductors can be described by the Usadel equations [20]. The Usadel equations is the approximation of Eilenberger Eqs. (A.12)–(A.17) that uses the fact in very dirty superconductors all directions of the momentum $\mathbf{p}$ are equal, which means that the Green function Eq. (A.14) is close to isotropized one Eq. (A.19):

$$\hat{g}\left(\mathbf{R}, \mathbf{n}\right) = \hat{g}_0\left(\mathbf{R}\right) + \mathbf{n} \hat{\mathbf{g}}_1\left(\mathbf{R}, \mathbf{n}\right), \quad |\mathbf{g}_1| \ll g_0 \tag{A.18}$$

$$\hat{g}_0\left(\mathbf{R}\right) \equiv \int \frac{d\mathbf{n}}{4\pi} \hat{g}\left(\mathbf{R}, \mathbf{n}\right). \tag{A.19}$$

The Usadel equations have the following form:

$$D\left(-i\nabla\right)\left[\hat{g}_0\left(-i\nabla\right)\hat{g}_0\right]+\left[\hat{\Omega},\hat{g}_0\right]=0, \tag{A.20}$$

$$\Delta\left(\mathbf{r}\right)=\pi\lambda T\sum_{|\omega|<\omega_D}f_0\left(\mathbf{r},\omega\right), \tag{A.21}$$

$$\hat{\Omega}\equiv\begin{pmatrix}\omega & \Delta\\ \Delta^* & -\omega\end{pmatrix} \tag{A.22}$$

$$\hat{g}_0^2=1, \tag{A.23}$$

where $D=v_F l/3$ is a diffusion constant and l is the mean free path.

In order to include a magnetic field with the vector potential $\mathbf{A}$ one should replace $-i\nabla$ by $\hat{P}\equiv-i\nabla-e\mathbf{A}\left[\hat{\sigma}_z,\ldots\right]$, where e is the charge of the electron.

A componentwise representation of the Usadel equations gives:

$$D\nabla\left[g\nabla g+f\nabla\tilde{f}+2ie\mathbf{A}f\tilde{f}\right]-\tilde{f}\Delta+f\Delta^*=0, \tag{A.24}$$

$$\frac{D}{2}\left[g\left(\nabla-2ie\mathbf{A}\right)^2f-f\nabla^2g\right]+g\Delta-\omega f=0, \tag{A.25}$$

$$\frac{D}{2}\left[g\left(\nabla+2ie\mathbf{A}\right)^2\tilde{f}-\tilde{f}\nabla^2g\right]+g\Delta^*-\omega\tilde{f}=0, \tag{A.26}$$

$$D\nabla\left[g\nabla g+\tilde{f}\nabla f-2ie\mathbf{A}f\tilde{f}\right]-f\Delta^*+\tilde{f}\Delta=0, \tag{A.27}$$

$$g^2+f\tilde{f}=1, \tag{A.28}$$

$$\Delta\left(\mathbf{r}\right)=\pi\lambda T\sum_{|\omega|<\omega_D}f\left(\mathbf{r},\omega\right). \tag{A.29}$$

Here we omit 0 index for brevity, supposing that f, $\tilde{f}$, g are isotropized.

The system of Eqs. (A.24)–(A.29) is overflowing. Indeed, one can easily see that from so-called normalisation condition Eq. (A.28) it follows that Eqs. (A.24) and (A.27) coincide up to sign. Less trivial fact is that Eq. (A.26) follows from Eqs. (A.24)–(A.25). Indeed, one can multiply Eq. (A.25) by $\tilde{f}$ and subtract from it multiplied by f Eq. (A.26) and obtain (A.24). We can also use the symmetries of the Green functions with respect to the reversal of the frequency ω sign:

$$-g^*_{-\omega}=g_\omega;\quad f^*_{-\omega}=\tilde{f}_\omega. \tag{A.30}$$

Thus, the Usadel equations have the form:

$$\frac{D}{2}\left[g\left(\nabla-2ie\mathbf{A}\right)^2f-f\nabla^2g\right]+g\Delta-\omega f=0, \tag{A.31}$$

$$g^2+|f|^2=1, \tag{A.32}$$

$$\Delta\left(\mathbf{r}\right)=\pi\lambda T\sum_{|\omega|<\omega_D}f\left(\mathbf{r},\omega\right). \tag{A.33}$$

For a complete description of superconductivity, it is also necessary to add a relation for the current and Maxwell equation:

$$\mathbf{j} = \pi i \nu_0 D T e \sum_{|\omega|<\omega_D} \left(f \left(\nabla + 2ie\mathbf{A} \right) f^* - f^* \left(\nabla - 2ie\mathbf{A} \right) f \right), \tag{A.34}$$

$$\nabla \times \nabla \times \mathbf{A} = 4\pi \mathbf{j}. \tag{A.35}$$

The Usadel equations can be rewritten in more convinient form with the use of so-called angular parameterisation [6, 7]:

$$g = \cos\theta, \quad f = e^{i\chi} \sin\theta, \tag{A.36}$$

$$\Delta = |\Delta| e^{i\varphi}, \tag{A.37}$$

where χ is the anomalous Green function f phase, θ – angular variable, and φ is the order parameter phase. In this parameterisation the Eqs. (A.31), (A.33), (A.34) have the form:

$$\frac{D}{2} \nabla^2 \theta + |\Delta| \cos\left(\chi - \varphi\right) \cos\theta - \omega \sin\theta - \frac{D}{2} \sin\theta \cos\theta \left(\nabla\chi - 2e\mathbf{A} \right)^2 = 0, \tag{A.38}$$

$$\frac{D}{2} \nabla \left(\left(\nabla\chi - 2e\mathbf{A} \right) \sin^2\theta \right) = |\Delta| \sin\left(\chi - \varphi\right) \sin\theta, \tag{A.39}$$

$$|\Delta| = \pi \lambda T \sum_{|\omega|<\omega_D} e^{i(\chi-\varphi)} \sin\theta, \tag{A.40}$$

$$\mathbf{j} = 2\pi \nu_0 D T e \sum_{|\omega|<\omega_D} \sin^2\theta \left(\nabla\chi - 2e\mathbf{A} \right). \tag{A.41}$$

One of the consequences of the Usadel equations is the continuity equation for the current [18, 24, 27, 28]. Indeed, due to $|\Delta| \in \mathbb{R}_+$, therefore from Eq. (A.40):

$$0 = \sum_{|\omega|<\omega_D} \sin\left(\chi - \varphi\right) \sin\theta. \tag{A.42}$$

The next step is to sum Eq. (A.39), which does play the role of the continuity equation with the source for spectral component of the current $\left(\nabla\chi - 2e\mathbf{A}\right) \sin^2\theta$, over ω. From current relation (A.41) we obtain:

$$\nabla \mathbf{j} = 0 \tag{A.43}$$

B The limit of temperatures close to critical. The Ginzburg-Landau equation.

In the limit $T \to T_c$, the order parameter Δ becomes small in comparison with T_c, which makes it possible to simplify the Usadel equations. This appendix is a short paragraph that contains the main ideas of this derivation. For more details see [24].

Before obtaining the equations describing the behavior of the system in this limit, let us find the characteristic scales at which the physical quantities in this system change.

We start from the Usadel equations (A.38), (A.39) and the self-consistency equation (A.40), which can be rewritten in the following form [24]:

$$|\Delta| \ln\left(\frac{T_c}{T}\right) = \pi T \sum_{\omega} \left(\frac{|\Delta|}{|\omega|} - \sin\theta e^{i(\chi-\varphi)}\right), \tag{B.1}$$

where the summation limits are already infinite.

Consider a homogeneous solution of this system of equations in a case of non-zero current J. The velocity of the Cooper pairs is:

$$\mathbf{p} = \nabla\varphi - 2e\mathbf{A}. \tag{B.2}$$

In $T \to T_c$ limit we have to keep only the leading orders in Δ in Eq. (A.38), from which one can obtain:

$$|\Delta| - \omega\theta - \frac{D}{2}p^2\theta \overset{\omega \geq 0}{=} 0 \Rightarrow \theta = \frac{|\Delta|}{\omega + \frac{D}{2}p^2}. \tag{B.3}$$

The direct substitution of this result in the self-consistency equation (B.1) gives us :

$$\left(1 - \frac{T}{T_c}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{n+\frac{1}{2}} - \frac{1}{n+\frac{1}{2}+\frac{D}{4\pi T_c}p^2}\right) = \psi\left(\frac{1}{2} + \frac{D}{4\pi T_c}p^2\right) - \psi\left(\frac{1}{2}\right), \tag{B.4}$$

where ψ is the digamma function.

Due to $T_c - T \ll T_c$, therefore expanding digamma function in Eq. (B.4) with respect to Dp^2/T_c we obtain the order of the scale ξ_c at which the physical quantities in this system change:

$$\frac{Dp^2}{T_c} \sim 1 - \frac{T}{T_c} \Rightarrow \xi_c = \frac{1}{p} \sim \sqrt{\frac{D}{T_c - T}} \sim \xi_{\mathrm{GL}}(T). \tag{B.5}$$

Here we have shown, that in the limit $T \to T_c$ angular variable θ and the order parameter Δ changes on the scale $\xi_{\mathrm{GL}}(T)$. Based on this, we can now directly derive the Ginzburg-Landau equation from the Usadel equations. Without loss of generality, we will consider a system with zero vector potential $\mathbf{A} = \mathbf{0}$. In order to derive Ginzburg-Landau equation it is convinient to use The Usadel equation in the form Eq. (A.31):

$$\frac{D}{2}\left[g\nabla^2 f - f\nabla^2 g\right] + g\Delta - \omega f = 0, \tag{B.6}$$

$$g^2 + |f|^2 = 1. \tag{B.7}$$

In $T \to T_c$ limit $\Delta \ll T_c$, which allows us to expand Green functions f, g with respect to Δ up to 3rd order:

$$f = f_0 + f_1 + f_2 + f_3 + o\left(|\Delta|^3\right), \tag{B.8}$$

$$g = g_0 + g_1 + g_2 + o\left(|\Delta|^2\right). \tag{B.9}$$

The expansion is carried out up to those orders when the gradient terms begin to influence, which will make it possible to monitor the contribution of various inhomogeneities of the system to the order parameter. With the use of Eq. (B.5) one can see that $|\nabla\theta| \sim \theta/\xi_{\mathrm{GL}}$. With the help of this fact after a direct substitution of expansions Eqs. (B.8)–(B.9) we obtain:

$$f = \frac{\Delta}{|\omega|} + \frac{D}{2\omega^2}\nabla^2\Delta - \frac{\Delta|\Delta|^2}{2\omega^2|\omega|}, \tag{B.10}$$

$$g = \frac{|\omega|}{\omega} - \frac{|\Delta|^2}{2\omega|\omega|}. \tag{B.11}$$

The next step is to substitute the obtained Green functions f, g in the self-consistency equation, which has the form:

$$\Delta \ln\left(\frac{T_c}{T}\right) = \pi T \sum_{|\omega|<\omega_D}\left(\frac{\Delta}{|\omega|} - f\right). \tag{B.12}$$

Substituting f, Eq. (B.10), into Eq. (B.12), we obtain:

$$\Delta\left(1 - \frac{T}{T_c}\right) = \frac{7\zeta(3)}{8\pi^2 T_c^2}\Delta|\Delta|^2 - \frac{\pi D}{8T_c}\nabla^2\Delta. \tag{B.13}$$

This is the Ginzburg-Landau equation. It can be rewritten in more compact form (here we will again bring the vector potential back into consideration):

$$\xi_{\mathrm{GL}}^2\left(\nabla - 2ie\mathbf{A}\right)^2\Delta + \Delta - \Delta\frac{|\Delta|^2}{\Delta_0^2} = 0, \tag{B.14}$$

$$\xi_{\mathrm{GL}}^2(T) \equiv \frac{\pi D}{8\left(T_c - T\right)}, \tag{B.15}$$

$$|\Delta_0|^2 \equiv \frac{8\pi^2 T_c^2}{7\zeta(3)}\left(1 - \frac{T}{T_c}\right), \tag{B.16}$$

where ξ_{GL} is the Ginzburg-Landau correlation length and Δ_0 is the bulk order parameter.

C Numerical analysis of the second order perturbation theory for phases

In this appendix, our goal is to tell about the main idea of the numerical method for solving the integral equation Eq. (3.22) and, therefore, obtaining phases from it as a function of z.

The idea of the method is based on several factors. First, with the help of numerical integration, we managed to understand how the quantity $\chi_2'^{(0)}(0,\omega)$ is distributed depending on ω at $T=0$. It turns out that the following equalities hold (due to $\chi_2 \propto 4\sin\delta\varphi(1-\cos\delta\varphi)$ we omit this factor in all formulas containing F, χ_2, φ_2, $\chi_2^{(0)}$, $\varphi_2^{(0)}$ for brevity)

$$\chi_2'^{(0)}(0,\omega=0) \approx 0.15, \tag{C.1}$$

$$\varphi_2'^{(0)}(0) \approx 0.319, \tag{C.2}$$

$$V_0 \approx 0.272. \tag{C.3}$$

Therefore, from Eq. (3.20) we see, that

$$\int_{-\infty}^{\infty} \frac{F(k)}{L_2(k)} \frac{dk}{2\pi} \approx 0.05 \lesssim \chi_2'^{(0)}(0,\omega=0), V_0. \tag{C.4}$$

This inequality allows us to look at the phase functional F in the equation Eq. (3.22) as a small correction and solve it iteratively (i.e., first put $F=0$, find the phase jumps $\delta\chi_2$, and then find F, etc.).

Second, if the phase jumps $\delta\chi_2$ are calculated incorrectly, this will significantly affect the rate of decrease of the phase functional F. Indeed, using the equation Eq. (3.36) we understand that the phase functional, Eq. (3.14), must decay at infinity faster than $1/k^2$ for $k \gg 1$. If this is not the case, then the integral in the equation Eq. (3.22) will converge, but will give parametrically incorrect values, which will worsen the numerical scheme. Therefore, it is very important to ensure that the equality Eq. (3.36) was carried out. For this, it is necessary to consider at the iteration step the constant V_0 in the integral equation also changing. If the circuit converges, then the V_0 value obtained like this will simply approach its theoretical value.

iteration	value
$V_0^{[1]}$	0.297115
$V_0^{[2]}$	0.268929
$V_0^{[3]}$	0.272768
$V_0^{[4]}$	0.272190
$V_0^{[\text{th}]}$	0.272193

Table C.1: The table of $V_0(T=0)$ for different iterations. As we can see, the numerical scheme very quickly approaches the theoretical value of V_0, which is one of the indirect proofs of its convergence.

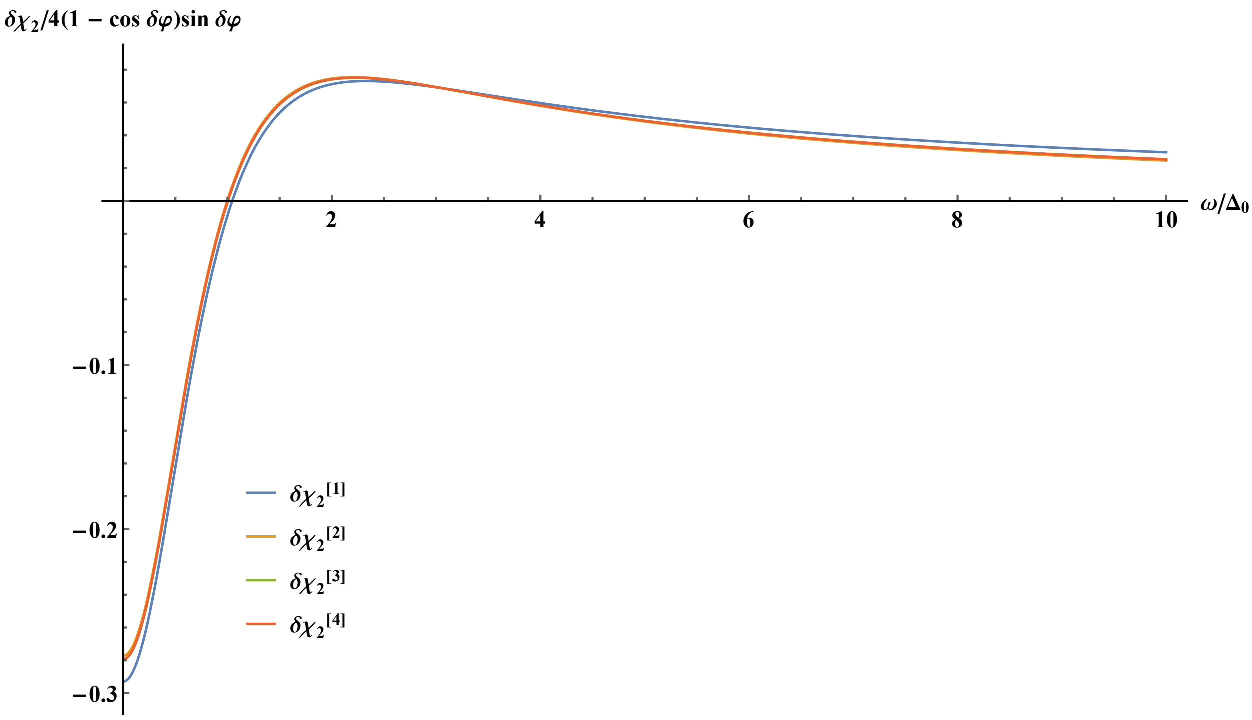

Figure C.1: $\delta\chi_2(\omega)$ plot for $T=0$ for different iterations of the numerical scheme. As can be seen from the graph, the numerical scheme converges very quickly at low frequencies.

Thus, the numerical scheme has the form

$$\frac{\delta\chi_2^{[i+1]}}{2\sqrt{\sin\theta_0}} = \chi_2'^{(0)}(0) - V_0^{[i+1]} + \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{1}{1+k^2\sin\theta_0} \frac{F^{[i]}(k)}{G(k)}, \tag{C.5}$$

$$V_0^{[i+1]} \int_0^\infty d\omega \sin^{3/2}\theta_0 = \int_0^\infty d\omega \sin^{3/2}\theta_0 \chi_2'^{(0)}(0) + \int_0^\infty d\omega \sin^{3/2}\theta_0 \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{1}{1+k^2\sin\theta_0} \frac{F^{[i]}(k)}{G(k)}, \tag{C.6}$$

$$F^{[0]}(k) \equiv 0. \tag{C.7}$$

This scheme gives for V_0 the following values for 4 iterations (see Tab. C.1). Below we also show the plots for $\delta\chi_2(\omega)$, Fig. C.1, and for $F(k)$, Fig. C.2, for different iterations, which shows a good convergence of the scheme.

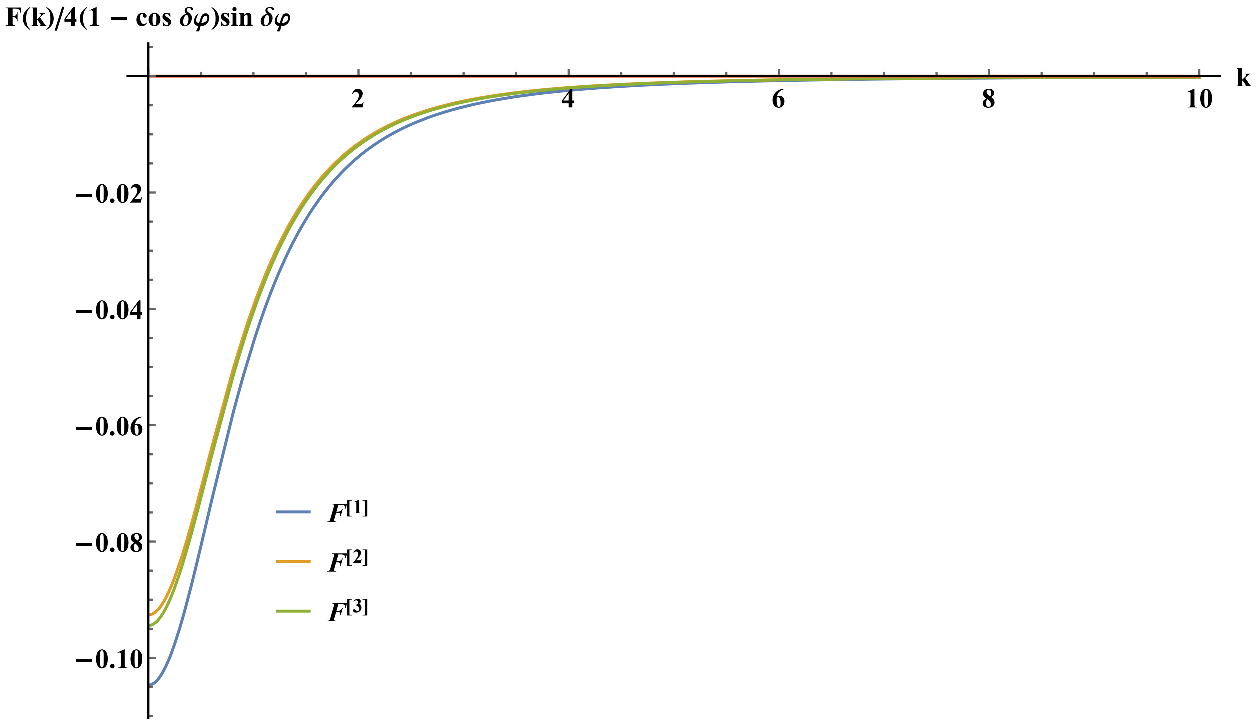

Figure C.2: $F(k)$ plot for $T = 0$ for different iterations of the numerical scheme. As can be seen from the graph, the numerical scheme converges very quickly.