Федеральное государственное автономное образовательное учреждение
высшего образования
«Московский физико-технический институт
(национальный исследовательский университет)»
Физтех-школа физики и исследований им. Ландау
Кафедра проблем теоретической физики (теоргруппа Горькова)

**Направление подготовки / специальность:** 03.04.01 Прикладные математика и физика
**Направленность (профиль) подготовки:** Общая и прикладная физика

# АНОМАЛИЯ ТУННЕЛЬНОЙ ПЛОТНОСТИ СОСТОЯНИЙ В НЕУПОРЯДОЧЕННОЙ ЭЛЕКТРОННОЙ СИСТЕМЕ, РЕАЛИЗУЮЩЕЙ СПИНОВЫЙ КВАНТОВЫЙ ЭФФЕКТ ХОЛЛА

(магистерская диссертация)

**Студент:**
Коковин Артём Дмитриевич

__________________________
*(подпись студента)*

**Научный руководитель:**
Бурмистров Игорь Сергеевич,
д-р физ.-мат. наук

__________________________
*(подпись научного руководителя)*

**Консультант** *(при наличии)***:**
__________________________
__________________________
*(подпись консультанта)*

Москва 2025

Abstract

We study the zero-bias anomaly in disordered superconductors of symmetry class C, where time-reversal symmetry is broken while spin-rotational invariance is preserved. Using the Finkel'stein nonlinear sigma model, we compute the disorder-averaged local density of states up to two-loop order in the presence of statically screened interaction. The resulting correction is derived analytically using dimensional regularization and evaluated numerically in the limit of strong interaction. We identify a parametric regime where the two-loop correction dominates over the leading one-loop result, leading to a breakdown of the previously established double-log-squared behavior at low energies. Renormalization group equations are obtained within the minimal subtraction scheme. We also propose a possible extension of this analysis to symmetry class A, where similar logarithmic structures are expected to emerge. Our findings refine the perturbative understanding of ZBA in class C systems.

# Contents

# 1 Introduction

## 1.1 Class C

In our work, we study low energy quasiparticles in a disordered superconductor with broken time-reversal symmetry (TRS) and with preserved spin-rotational symmetry. The Hamiltonian of such a system, denoted by $H$, is of the Bogoliubov-de Gennes type and must satisfy certain symmetry constraints. First, particle-hole symmetry implies:

$$\pi_x H^\top \pi_x = -H, \tag{1.1.1}$$

where $\pi_x = \sigma_x$ is a Pauli matrix acting in the particle-hole space with basis $\psi = (\psi_\uparrow, \psi_\downarrow, \psi_\uparrow^\dagger, \psi_\downarrow^\dagger)$. Spin-rotational symmetry imposes:

$$\sigma_i H \sigma_i = H. \tag{1.1.2}$$

Finally, broken TRS is expressed as:

$$U_T^{-1} H^*(\boldsymbol{k}) U_T \neq H(-\boldsymbol{k}), \quad T = U_T K, \tag{1.1.3}$$

where $T$ is a time-reversal operator, $U_T$ is a unitary operator that reverses spin, and $K$ denotes complex conjugation. Here, $H$ is taken to be diagonal in the momentum $k$ basis. The Hamiltonian obeying these symmetries, according to the classification by Altland and Zirnbauer [1], belongs to symmetry class C.

This class has attracted significant attention due to the possibility of realizing the spin quantum Hall effect (sQHE)—a topological phase characterized by the presence of gapless edge modes that carry spin current. It has been extensively studied in the literature [2, 3, 4, 5, 6]. A prototypical system exhibiting class C behavior is a spin-singlet $d$-wave superconductor with a complex order parameter of the form:

$$\Delta(\boldsymbol{k}) = i\sigma_y \big(d_{x^2-y^2}(\boldsymbol{k}) + i d_{xy}(\boldsymbol{k})\big), \quad \Delta^*(\boldsymbol{k}) \neq \Delta(-\boldsymbol{k}). \tag{1.1.4}$$

The material with this order parameter was proposed by Can et al. [7]. The authors suggest coupling two superconducting monolayers with $d$-wave order parameter (e.g., $CuO_2$ layers), twisted relative to one another, as illustrated in Fig. 1. For twist angles above a certain critical value $\theta_c$, the complex order parameter (1.1.4) minimizes the free energy. The emergence of the edge mode was demonstrated numerically in the same work.

This setup was recently investigated [8, 9, 10] for the Josephson diode effect—a hallmark of broken TRS (for theoretical exploration see [11]). These developments make twisted $Bi_2Sr_2CaCu_2O_{8+x}$ bilayers promising candidates for the experimental realization of the spin quantum Hall effect.

In the regime of weak disorder, class C superconductors can be described by a nonlinear sigma model (NL$\sigma$M). A key assumption in deriving this model is that the disorder-averaged order parameter vanishes, which is consistent with the known suppression of $d$-wave supercon-

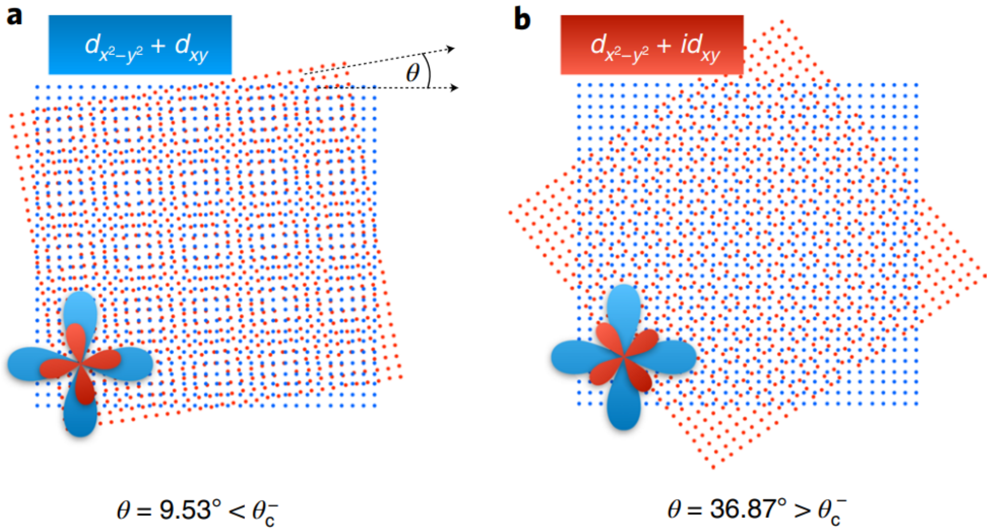


Figure 1: Adopted from Ref. [7]

ductivity by disorder [12]. Nevertheless, local superconducting fluctuations are allowed. The resulting system may be interpreted as a normal metal interspersed with randomly distributed superconducting regions. Another important assumption is that the average phase acquired by an electron undergoing Andreev reflection at a normal–superconductor (NS) interface is zero [1]; this condition is essential for the existence of gapless quasiparticle excitations.

## 1.2 Zero-Bias Anomaly

The interplay between disorder and Coulomb interaction in electronic systems gives rise to the phenomenon known as the zero-bias anomaly (ZBA). It manifests as a pronounced dip in the tunneling conductivity near the Fermi level, and has been observed in numerous experiments [13, 14, 15, 16]. The tunneling conductivity measured in the experiment, conducted by McMillan and Mochel [17], is shown in Fig. 2. They studied tunneling conductivity in gold-doped germanium alloys as a function of bias voltage. The level of doping is denoted by $x$: higher values of $x$ correspond to higher concentrations of gold. Gold introduces states within the band gap of germanium, thereby enabling conduction. For $x > 0.12$, close to the metal–insulator transition, a clear square-root-type dependence emerges. That is qualitatively consistent with the theoretical prediction by Aaronov and Altshuler in [18], which we discuss below. The curve for $x = 0.08$ corresponds to the semiconducting state, in which Coulomb interaction is known to produce a Coulomb gap [19].

The relation between the correction to the tunneling conductivity and the density of states (DoS) of electrons could be obtained using the tunneling Hamiltonian formalism. At zero temperature, it takes the form:

$$\frac{\delta G(V)}{G_0} = \frac{\delta\rho(eV)}{\rho_0}, \tag{1.2.1}$$

where $\delta G(V)$ is the interaction-induced correction to the tunneling conductivity $G_0$ at bias

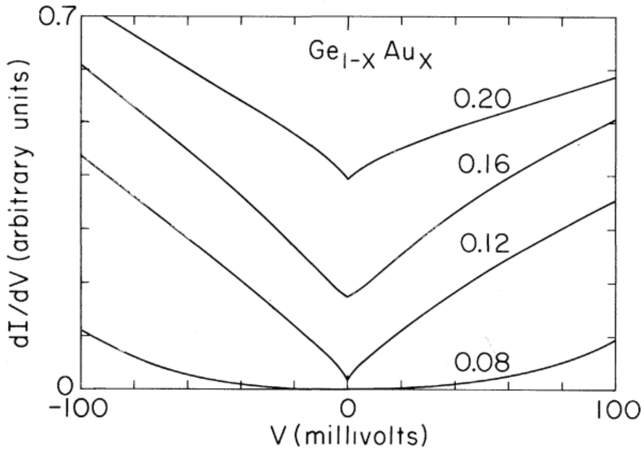


Figure 2: Differential conductance $dI/dV$ of the tunnel junction as a function of voltage for $Ge_{1-x}Au_x$ samples with $x = 0.08$, 0.12, 0.16, and 0.20. Conductance is normalized to unity at +0.3V. Adopted from [17].

voltage $V$, and $\delta\rho(eV)$ is the corresponding correction to the DoS $\rho_0$ at the energy $eV$, where $e$ is an elementary charge. This relation enables a direct interpretation of tunneling data in terms of DoS modifications.

Altshuler and Aronov showed in [18] that, in three dimensions, weak disorder and short-range, energy-independent electron interactions give rise to a square-root correction:

$$\delta\rho(E) \propto \sqrt{E}. \tag{1.2.2}$$

The same calculation can be carried out for the Coulomb interaction in two dimensions, although assumptions should be modified. Indeed, Coulomb interaction is not short-ranged, and acquires energy dependence due to Debye screening in the presence of disorder. In a subsequent paper by Altsuler, Aronov and Lee [20], the following expression was obtained for the screened-Coulomb-induced correction in 2D:

$$\frac{\delta\rho(E)}{\rho_0} = -\frac{1}{2\pi E_F \tau} \ln(|E|\tau) \ln\left|\frac{E}{D\kappa^2}\right|, \tag{1.2.3}$$

where $E_F$ - Fermi energy, $D$ - diffusion coefficient, $\tau$ is mean free time and $\kappa$ - Debye screening length in 2D.

This result is valid when the correction to the DoS is small relative to the bare DoS. That works only in a diffusive regime, when the disorder is small. Moreover, it diverges as $E \to 0$. Another approach to this problem was introduced by Levitov and Shytov in 1997 [21]. They completely avoided the weak disorder problem, considering a system at thermal equilibrium with known conductivity $\sigma$. They explore action in imaginary time and find an instanton that corresponds to the tunneling. Within the saddle-point approximation, they derived the following expression for the tunneling conductivity:

$$\frac{G(V)}{G_0} = \exp\left(-\frac{1}{\hbar}S(V)\right), \quad S(V) = \frac{e^2}{8\pi^2\sigma} \ln\left(\frac{e}{4\pi^2\sigma V\tau}\right) \ln\left(\frac{e\tau\sigma\left(\nu e^2\right)^2}{4\pi^2 V}\right), \tag{1.2.4}$$

where $\nu$ - compressibility and could be expressed using Einstein relation $\nu = \sigma/(e^2 D)$. This result is self-consistent when $eV < e^2/\sigma\tau$ and in the range of applicability of the perturbative result coincides with (1.2.3).

The same result was previously obtained by Finkel'stein [22], Nazarov [23], and was later obtained by Kamenev and Andreev [24] using the Keldysh NL$\sigma$M and $\mathcal{K}$-gauge.

In symmetry class C, ZBA can in principle also be observed. This is because the superconducting gap is, on average, suppressed by disorder, leaving quasiparticle states at the Fermi level that are susceptible to interaction effects. A key difference, however, is that interactions in class C occur in the triplet channel, whereas traditional ZBA calculations typically involve the singlet channel. Nevertheless, this distinction does not prevent meaningful comparison: class C and class A (the symmetry class of disordered metals with broken TRS) are connected via a continuous crossover, during which the triplet channel effectively transforms into the singlet channel [25]. Our perturbative analysis permits extension of the results to class A, enabling a direct comparison of the corresponding corrections to the LDoS.

## 1.3 Anderson Localization

The phenomenon of Anderson localization has been known since the middle of the previous century [26]. In his seminal work, P.W. Anderson studied non-interacting electrons on a disordered lattice, where the on-site energies were randomly distributed. He discovered that, beyond a critical strength of disorder, the electronic wavefunctions become localized. The resulting disorder-driven metal–insulator transition occurs at zero temperature and serves as a prototypical example of a quantum phase transition.

A notable feature of this transition is the anomalous scaling of the disorder-averaged moments of the local density of states (LDoS) with system size. Near the critical point, one finds:

$$\langle \rho^q(E,x) \rangle \sim L^{-x_q}, \quad x_q = d(q-1) + \Delta_q. \tag{1.3.1}$$

$x_q$ are called multifractal exponents and $\Delta_q$ represent the corresponding anomalous dimensions. In the delocalized (metallic) phase, these anomalous dimensions vanish. Such multifractal behavior is not restricted to the LDoS: many other operators exhibit similar scaling properties [27].

In class C systems without interactions, an infinite set of these anomalous dimensions can be obtained analytically through a mapping to the percolation problem [4, 28, 29, 30, 31, 32]. In the presence of interactions, a two-loop perturbative analysis using the Finkel'stein NL$\sigma$M was performed by Babkin and Burmistrov [33]. Although anomalous dimensions for several operators were computed, the two-loop result for the LDoS was not derived.

The goal of the present work is to compute the two-loop correction to the LDoS operator in class C, thereby completing this aspect of the analysis. Furthermore, the presence of the ZBA is expected to modify the pure power-law scaling. This deviation should manifest itself in the structure of the two-loop correction.

### 1.4 Outline of the Paper

In Section 2, we introduce the Finkelstein NL$\sigma$M, a field-theoretical approach for describing disordered superconductors. We explain its symmetries and discuss their physical origins. In Section 3, we describe the parametrization of matrix fields used for perturbative calculations in the weak-disorder regime and discuss the diagrammatic correspondence between propagators in the NL$\sigma$M and Green's function formalism.

In Section 4, we identify the disorder-averaged operator corresponding to the LDoS and develop its expansion up to the two-loop approximation. In Section 5, we reproduce the one-loop result and discuss how the known double-log-squared law is recovered in the limit of Coulomb interaction in class A.

Section 6 presents our detailed two-loop calculations, including analytical expressions for each contribution and the evaluation of results in the limit of zero temperature and zero energy. Section 7 is dedicated to deriving the renormalization group (RG) equations.

In Sections 8, 9, and 12, we present analytical expressions for the RG coefficients, analyze them analytically and numerically in the $\gamma \to -1$ limit, and discuss the resulting implications for the ZBA. Finally, we address the possibility of extending our results to symmetry class A.

## 2 Finkel'stein NL$\sigma$M with Interaction

We use Finkel'stein NL$\sigma$M to study the behavior of diffusive modes in the presence of the interaction between quasiparticles. In this section, we outline the symmetries of this field theory. The partition function takes the form:

$$Z = \int D\big[Q\big] \exp(S_0 + S_{\text{int}}), \tag{2.1}$$

where $S_0$ - non-interacting part of the action, and $S_{\text{int}}$ is attributed to the quasiparticle interaction:

$$\begin{aligned} S_0 &= -\frac{g}{16}\int_{\boldsymbol{x}} \text{Tr}\big(\nabla Q\big)^2 + Z_\omega \int_{\boldsymbol{x}} \text{Tr}\big(\hat{\varepsilon} Q\big); \\ S_{\text{int}} &= -\frac{\pi T \Gamma_t}{4}\sum_{\alpha,n}\int_{\boldsymbol{x}} \text{Tr}\big(I^\alpha_n \mathrm{s} Q\big)\, \text{Tr}\big(I^\alpha_{-n} \mathrm{s} Q\big), \end{aligned} \tag{2.2}$$

where Tr denotes the trace over all degrees of freedom, excluding spatial integration. We briefly summarize the parameters appearing above. Conductance is denoted as $g$, which will serve as a parameter of the perturbation theory. $Z_\omega$ is a counterterm and controls the renormalization of the corresponding term in the action. $T$ is the temperature, and $\Gamma_t$ is an interaction amplitude, which is taken to be statically screened to ensure its short-range nature. We use the notation $\int_{\boldsymbol{x}} = \int d^d\boldsymbol{x}$.

$Q$ is a matrix field over which we integrate, and we shall discuss its degrees of freedom. It is a matrix in a replica space of dimension $N_r$, and in a 2-dimensional Nambu space. It also acts in a $2N_m$-dimensional Matsubara space ($N_m$ for positive and $N_m$ for negative frequencies). The latter is necessary for the discussion of the interacting system. Strictly speaking, the Matsubara space is infinite-dimensional, and one should take the limit $N_m \to \infty$ with care. Nonetheless, this subtlety is irrelevant for our purposes. Another limit we need to take is $N_r \to 0$, as this corresponds to physical observables. In (2) there are operators:

$$\begin{aligned} \hat{\varepsilon}^{\alpha\beta}_{nm} = \varepsilon_n \delta_{n,m}\delta^{\alpha\beta}\mathrm{s}_0, \quad \varepsilon_n = 2\pi T\big(n + \frac{1}{2}\big), \\ \mathrm{s} = (\mathrm{s}_1, \mathrm{s}_2, \mathrm{s}_3), \\ \big(I^\alpha_n\big)^{\beta\gamma}_{km} = \delta_{k-m,n}\delta^{\beta\alpha}\delta^{\alpha\gamma} s_0. \end{aligned} \tag{2.3}$$

We use superscripts for replica space and subscripts for Matsubara frequencies, $\mathrm{s}_i$ acts on Nambu space as a Pauli matrix $\sigma_i$ with the convention, that $\sigma_0 = \hat{1}$.

$Q$ matrices are not entirely arbitrary: $Q$ is an Hermitian matrix $Q^\dagger = Q$ and satisfies a nonlinear constraint $Q^2(x) = \hat{1}$. These constraints are general for any NL$\sigma$M, the specific symmetries of class C superconductors, such as spin rotational and Bogolyubov-de Gennes symmetries (time-reversal symmetry is broken), further restrict the structure of $Q$:

$$Q = -\bar{Q}, \quad \bar{Q} = \mathrm{s}_2 L_0 Q^\top L_0 \mathrm{s}_2, \quad (L_0)^{\alpha\beta}_{nm} = \delta_{\varepsilon_n,-\varepsilon_m}\delta^{\alpha\beta}\mathrm{s}_0, \\ (L_0)^{\alpha\beta}_{nm} = \delta_{\varepsilon_n,-\varepsilon_m}\delta^{\alpha\beta}\mathrm{s}_0. \tag{2.4}$$

These constraints could be satisfied with unitary transformations T of a saddle point configuration $\Lambda$:

$$Q = \mathrm{T}^{-1}\Lambda\mathrm{T}, \quad \Lambda^{\alpha\beta}_{nm} = \mathrm{sign}(\epsilon_n)\delta_{nm}\delta^{\alpha\beta}\mathrm{s}_0. \tag{2.5}$$

Unitary transformation matrices, according to (2), satisfy the relation:

$$\left(\mathrm{T}^{-1}\right)^\top L_0\mathrm{s}_2 = \mathrm{s}_2 L_0 \mathrm{T}. \tag{2.6}$$

These matrices $T$ form a target manifold of the class C NL$\sigma$M - $\mathrm{Sp}(4N_rN_m)/U(2N_rN_m)$. The fluctuations around the saddle point $\Lambda$ will be small with respect to $g^{-1}$, as we will discuss in the next section. The only interaction term here corresponds to the triplet particle-hole channel. Other interactions are prohibited in this symmetry class: Cooper channel is suppressed due to the broken time-reversal symmetry, and the singlet particle-hole channel vanishes due to the symmetry (2):

$$\mathrm{Tr}\, I^\alpha_n \mathrm{s}_0 Q = 2\sum_k Q^{\alpha\alpha}_{k,k+n} = -2\sum_k Q^{\alpha\alpha}_{k,k+n} = 0. \tag{2.7}$$

## 3 Perturbation Theory

In this section, we discuss the main method of the work - perturbative expansion. We assume that the corrections to the saddle-point approximation are small with respect to the parameter $g^{-1}$. We start by parametrizing $Q$ matrix field with matrix field $W$:

$$Q = W + \Lambda\sqrt{1 - W^2} \tag{3.1}$$

In order for that parametrization to be consistent with nonlinear constraints, we need to impose another condition:

$$\Lambda W + W\Lambda = \{\Lambda, W\} = 0. \tag{3.2}$$

Condition (2) and hermiticity are also imposed on $W$:

$$W = W^\dagger, \quad W = -\bar{W}. \tag{3.3}$$

These conditions allow us to write $W$ as a block matrix:

$$W = \begin{pmatrix} 0 & w \\ w^\dagger & 0 \end{pmatrix}, \quad w^{\alpha\beta}_{nm} \neq 0 \quad \text{if} \quad \varepsilon_n > 0 \quad \text{and} \quad \varepsilon_m < 0,\ w^{\alpha\beta}_{nm} = 0 \quad \text{otherwise}. \tag{3.4}$$

$W$ is off-diagonal in Matsubara space. Since $w$ is Hermitian in Nambu space, we could decompose it using Pauli matrices: $w = \sum_{i=0}^{3} w_i \mathrm{s}_i$. Finally, we can rewrite (2) in terms of $w$:

$$(w_i)^{\alpha\beta}_{n,m} = \mathrm{v}_i (w_i)^{\beta\alpha}_{-m,-n}, \quad \mathrm{v} = (-1,1,1,1). \tag{3.5}$$

This relation is incredibly useful for calculations. Now, we can expand the action in terms of $w$ matrices and obtain gaussian part:

$$S^{(2)} = \sum_{n_i,\alpha,\beta,\mu,\nu,i} \int_{\boldsymbol{q}} (w_{i,q})^{\alpha\beta}_{n_1,n_2} (w^\dagger_{i,-q})^{\mu\nu}_{n_4,n_3} \Bigg[ \Bigg( -\frac{g}{4} q^2 - \frac{Z_\omega}{2} (\varepsilon_{n_1} - \varepsilon_{n_2}) \Bigg) \delta_{n_1,n_3} \delta_{n_2,n_4} \delta^{\alpha\nu} \delta^{\beta\mu} + \tag{3.6}$$

$$+ (-2\pi T \Gamma_t)(1 - \delta_{i,0}) \delta_{n_1-n_2,n_3-n_4} \delta^{\alpha\nu} \delta^{\beta\mu} \delta^{\alpha\beta} \Bigg]. \tag{3.7}$$

Here we used another notation $\int_{\boldsymbol{q}} = \int d^d q (2\pi)^{-d}$ for integration in momentum space. One can obtain correlation functions of the field $w$ that respect symmetry (2):

$$\langle (w_{i,q})^{\alpha\beta}_{n_1,n_2} (w^\dagger_{i,-q})^{\mu\nu}_{n_4,n_3} \rangle = \frac{2}{g} \delta_{n_1-n_2,n_3-n_4} \Bigg[ \delta^{\alpha\nu} \delta^{\beta\mu} \delta_{n_1 n_3} + \mathrm{v}_i \delta^{\alpha\mu} \delta^{\beta\nu} \delta_{n_1,-n_4} - \\ - \frac{4\pi T \gamma}{D} (1 - \delta_{i,0}) \delta^{\alpha\nu} \delta^{\beta\mu} \delta^{\alpha\beta} \mathcal{D}^t_q (i\omega_{n_1-n_2}) \Bigg] \mathcal{D}_q (i\omega_{n_1-n_2}), \tag{3.8}$$

where $i\omega_{n_1-n_2} = i\varepsilon_{n_1} - i\varepsilon_{n_2}$. Here we introduced $D = g/4Z_\omega$ - diffusion coefficient, $\gamma = \gamma_t / Z_\omega$ - interaction parameter and diffusive propagators:

$$\mathcal{D}_q(i\omega) = \frac{1}{q^2 + \frac{\omega}{D}}; \\ \mathcal{D}^t_q(i\omega) = \frac{1}{q^2 + (1+\gamma)\frac{\omega}{D}}. \tag{3.9}$$

These show the diffusive nature of fluctuations. We note that due to (3.8) $g^{-1}$ is indeed a small parameter of our perturbation theory. This expansion can be understood in terms of the disorder-averaged diagrammatic perturbation theory. The propagators in Eq.(3.9) correspond to diffusons, while the combination $\gamma \mathcal{D}\mathcal{D}^t$ captures the dynamically screened interaction mediated by impurity scattering, as illustrated in Fig.3. This formulation recasts the perturbation theory in a more convenient language of interacting diffusive modes.

We now discuss the physical interpretation of these propagators. The emergent diffusive poles are typically associated with conservation laws. Castellani et al. [34] demonstrated that, in a disordered interacting electron gas (class AI), the Ward identity relates the spin susceptibility to the renormalized triplet propagator $\mathcal{D}^t$ and its pole structure.

The NL$\sigma$M describes dynamics in the hydrodynamic regime, i.e., at small momenta $k \ll p_F$ (with $p_F$ the Fermi momentum) and low energies $E \ll \tau^{-1}$. Corrections outside this regime correspond to Landau Fermi-liquid effects and result in renormalizations of the theory's parameters, such as the single-particle density of states.

In Landau Fermi-liquid theory, the spin susceptibility is related to the triplet interaction

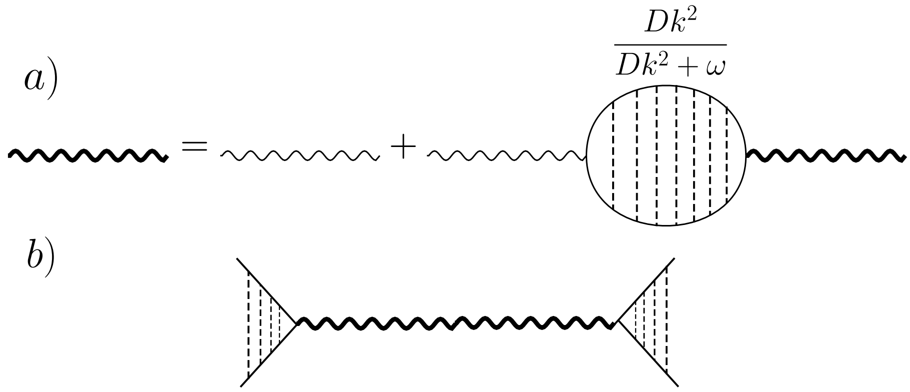


Figure 3: a) Statically screened interaction (wiggly line) is dressed by the diffusive ladder (dashed lines); b) Vertices of the interaction are also dressed by the diffusive ladder.

amplitude via the standard expression [35]:

$$\gamma = -\frac{F_t}{1+F_t}, \tag{3.10}$$

where $F_t$ is the $s$-wave Fermi-liquid parameter in the triplet channel. For repulsive interactions, $F_t < 0$, so $\gamma$ is positive. Moreover, since $\gamma$ receives no one-loop corrections in the sigma model [33], this sign persists at large momentum scales.

We also need to add a term to our action:

$$S_h = \frac{gh^2}{8}\int_{\boldsymbol{x}} \mathrm{Tr}\big(\Lambda Q\big). \tag{3.11}$$

In terms of expansion around the saddle-point, this term changes diffusive propagators :

$$\begin{aligned} \mathcal{D}_q(i\omega) &\to \frac{1}{q^2+h^2+\frac{\omega}{D}}; \\ \mathcal{D}^t_q(i\omega) &\to \frac{1}{q^2+h^2+(1+\gamma)\frac{\omega}{D}}. \end{aligned} \tag{3.12}$$

It is very useful to us, since it regularizes infrared divergencies without affecting the UV and, hence, the renormalizability. In our calculations it will be more convenient not to keep track of Matsubara frequencies, but rather express corrections in terms of $h$. The possibility to restore the energy and temperature dependence from this limit is discussed in [36]. In the next section, we derive a perturbative expression for the LDoS correction.

# 4 Corrections to LDoS

The expansion of action in terms of $w$ matrices produces vertices of the perturbation theory. There are three terms in action, which will be relevant for the consideration of two-loop corrections to the LDoS:

$$S_{\text{int}}^{(3)} = \frac{\pi T\Gamma_t}{4}\sum_{\alpha,n}\int_x \operatorname{Tr} I_n^\alpha \mathrm{s}W \operatorname{Tr} I_{-n}^\alpha \mathrm{s}\Lambda W^2$$

$$S_{\text{int}}^{(4)} = -\frac{\pi T\Gamma_t}{16}\sum_{\alpha n}\int_x \operatorname{Tr} I_n^\alpha \mathrm{s}\Lambda W^2 \operatorname{Tr} I_{-n}^\alpha \mathrm{s}\Lambda W^2$$

$$S_0^{(4)} = \frac{g}{64}\prod_{i=1}^{4}\int_{q_i}\sum_{\alpha_i,n_i}\delta\left(\sum_i q_i\right)\times\left(q_{12}q_{34}+q_{14}q_{23}-2h^2-\frac{\omega_{n_{12}+n_{34}}}{D}\right)$$
$$\times \operatorname{tr}\left[(w_{q_1})^{\alpha_1\alpha_2}_{n_1n_2}\left(w^\dagger_{q_2}\right)^{\alpha_2\alpha_3}_{n_2n_3}(w_{q_3})^{\alpha_3\alpha_4}_{n_3n_4}\left(w^\dagger_{q_4}\right)^{\alpha_4\alpha_1}_{n_4n_1}\right], \tag{4.1}$$

where $q_{ij} = q_i + q_j$, and $\omega_{n_{12}} = \epsilon_{n_1} - \epsilon_{n_2}$, and tr denotes the trace over Nambu degrees of freedom. The first and second lines originate from the expansion of the interaction term in action, whereas the last term comes from the non-interacting part. The LDoS corresponds to the single $Q$ matrix operator:

$$\rho(i\epsilon_n) = \frac{\rho_0}{2}\operatorname{tr}\left(Q^{\alpha\alpha}_{nn}\right). \tag{4.2}$$

Note that the replica and Matsubara indices are fixed. In order to obtain corrections to the LDoS, one should use the expanded action, as well as the expansion of $Q$ matrix up to two loops:

$$\operatorname{tr}\langle Q^{\alpha\alpha}_{nn}\rangle = 2-\frac{1}{2}\operatorname{tr}\left\langle(\Lambda W^2)^{\alpha\alpha}_{nn}\right\rangle-\frac{1}{8}\operatorname{tr}\left\langle(\Lambda W^4)^{\alpha\alpha}_{nn}\right\rangle-\frac{1}{2}\operatorname{tr}\left\langle(\Lambda W^2)^{\alpha\alpha}_{nn}[S_0^{(4)}+S_{\text{int}}^{(4)}+(S_{\text{int}}^{(3)})^2/2]\right\rangle+\dots \tag{4.3}$$

Here $\langle\rangle$ denotes averaging using Wick's theorem and the correlation functions (3.8). For convenience, we assume $\epsilon_n > 0$. The main goal of this work is to compute these contributions explicitly. Once obtained, they will allow us to derive the RG equations and ultimately access the correction to the ZBA.

# 5 One-Loop Correction

This section is dedicated to reproducing the known one-loop result for the LDoS. In doing so, we will apply our assumptions and introduce notations that will be used in two-loop analysis as well. The one-loop correction arises from the second term on the right-hand side of (4.3). Rewriting it in terms of propagators (3.12), we obtain:

$$\operatorname{tr}\langle Q^{\alpha\alpha}_{nn}\rangle = 2-\sum_{i,k,\gamma}\langle(w_i)^{\alpha\gamma}_{nk}(x)(w_i^\dagger)^{\gamma\alpha}_{kn}(x)\rangle = 2-\frac{4}{g}\int_p \mathcal{D}_p(i\omega_{2n})+\frac{2}{g}\sum_{k:\epsilon_k<0}\frac{12\pi T\gamma}{D}\mathcal{D}_p\mathcal{D}_p^t(i\omega_{n-k}). \tag{5.1}$$

From this point on, we adopt a shorthand notation for products of multiple propagators evaluated at the same frequency argument, as exemplified above:

$$\mathcal{D}_p^n\left(\mathcal{D}_p^t\right)^m(i\omega) = \mathcal{D}_p^n(i\omega)\left(\mathcal{D}_p^t\right)^m(i\omega). \tag{5.2}$$

Note that there are only two contributions, not three. That is because one of the terms in (3.8) vanishes in the replica limit $N_r \to 0$. From now on, we will only consider contractions in this limit. We will also consider a zero-temperature limit, in which the summation over Matsubara frequencies is replaced by integration:

$$\sum_n f(i\omega_n) = \int \frac{d\omega}{2\pi T} f(i\omega). \tag{5.3}$$

The integrals over momenta will be computed in the $\epsilon$-regularization, assuming, that $d = 2 + \epsilon$. Moreover, we retain only the pole parts in $\epsilon$, as these are the only terms relevant for the minimal subtraction scheme of renormalization [37]. Changing the variables of integration from $Q$-matrices to $W$ introduces a Jacobian, which usually regularizes the UV divergences. Since we use $\epsilon$-regularization, it can be disregarded [37].

We can now obtain the result for the one-loop correction:

$$\mathrm{tr}\langle Q^{\alpha\alpha}_{nn}\rangle = 2 - \frac{4}{g} A(\epsilon)\big[1 - 3\ln(1+\gamma)\big]. \tag{5.4}$$

Here we introduce another quantity:

$$A(\epsilon) = -\frac{S_d}{(2\pi)^d}\frac{\Gamma(1+\epsilon/2)\Gamma(1-\epsilon/2)}{\epsilon}, \qquad \int_p \mathcal{D}_p(i\omega) = A(\epsilon)\big(\frac{\omega}{D} + h^2\big), \tag{5.5}$$

where $S_d$ is the surface area of a sphere, embedded in a $d$-dimensional space (volume of a $d-1$-dimensional manifold). This factor emerges frequently in our calculations, and we will use it to redefine RG-charge in Section 7. It is clear, that for $\gamma = -1$, which corresponds to the Coulomb interaction in class A, theory becomes nonrenormalizable - instead of a first order-pole, the correction has a second-order pole in $\epsilon$. That is consistent with the ZBA, where the correction to the LDoS in one-loop approximation is proportional to the squared logarithm of energy. We will show, that in two loops the LDoS is also renormalizable, with a similar divergence of the exponent as $\gamma \to -1$.

Under these same assumptions, we now proceed to identify the diagrams contributing to the second-order correction.

# 6 Two-Loop Corrections

The two-loop corrections emerge from the third and the fourth contribution on the right-hand side of the (4.3). These are the only contributions that are quadratic in $g^{-1}$, as will become evident when the expressions are rewritten in terms of propagators. We assume $\epsilon_n > 0$ (moreover, for some calculations it is convenient to assume $n > 0$, which we adopt where appropriate) and proceed with the derivation of the two-loop contributions.

## 6.1 $\Lambda W^2 S_0^{(4)}$

We begin with this contribution, even though it is not the simplest. However, it serves as a convenient example for illustrating some details of the derivation. We first express it in terms of the $w$ -matrices in momentum space:

$$-\frac{1}{2}\langle \mathrm{Tr}\left(\Lambda W^2\right)^{\alpha\alpha}_{nn} S_0^{(4)}\rangle = -\frac{g}{64}\int_{\boldsymbol{q},\boldsymbol{q'}} e^{i(\boldsymbol{q}+\boldsymbol{q'})\boldsymbol{x}} \sum_{k,m,\gamma,i} \langle (w_{i,q})^{\alpha\gamma}_{nk} \left(w^{\dagger}_{i,q'}\right)^{\gamma\alpha}_{kn} \prod_{j=1}^{4}\int_{q_i}\sum_{\alpha_j,n_j}\delta(\sum_j q_j)\times$$
$$\times\left(q_{12}q_{34}+q_{14}q_{23}-2h^2-\frac{\omega_{n_{12}+n_{34}}}{D}\right)\mathrm{tr}\left[(w_{q_1})^{\alpha_1\alpha_2}_{n_1n_2}\left(w^{\dagger}_{q_2}\right)^{\alpha_2\alpha_3}_{n_2n_3}(w_{q_3})^{\alpha_3\alpha_4}_{n_3n_4}\left(w^{\dagger}_{q_4}\right)^{\alpha_4\alpha_1}_{n_4n_1}\right]\rangle. \quad (6.1.1)$$

Details of the subsequent derivation can be found in the section 10. The result of the contraction is:

$$-\frac{1}{2}\langle \mathrm{Tr}\left(\Lambda W^2\right)^{\alpha\alpha}_{nn} S_0^{(4)}\rangle = +\frac{1}{2g^2}\int_{\boldsymbol{q_1},\boldsymbol{q_3}}\Big[16\mathcal{D}_{q_3}(i\omega_{2n})\mathcal{D}^2_{q_1}(i\omega_{2n})\left(\mathcal{D}^{-1}_{q_3}(i\omega_{2n})+\mathcal{D}^{-1}_{q_1}(i\omega_{2n})\right)-\frac{96\pi T\gamma}{D}\times$$
$$\times\sum_{n_1>0}\mathcal{D}^t_{q_1}(i\omega_{n_1+n})\mathcal{D}^2_{q_1}(i\omega_{n_1+n})\mathcal{D}_{q_3}(i\omega_{2n_1})\left(\mathcal{D}^{-1}_{q_3}(i\omega_{2n_1})+\mathcal{D}^{-1}_{q_1}(i\omega_{n_1+n})\right)-\frac{96\pi T\gamma}{D}\times$$
$$\times\sum_{n_1>0}\mathcal{D}^t_{q_1}(i\omega_{n_1+n})\mathcal{D}^2_{q_1}(i\omega_{n_1+n})\mathcal{D}_{q_3}(i\omega_{2n})\left(\mathcal{D}^{-1}_{q_3}(i\omega_{2n})+\mathcal{D}^{-1}_{q_1}(i\omega_{n_1+n})\right)+\frac{192\pi^2T^2\gamma^2}{D^2}\times$$
$$\times\sum_{\substack{n_3>0\\ n_2<0}}\theta(n_3-n_2-n)\left(\mathcal{D}^t_{q_1}\right)^2(i\omega_{n_3-n_2})\mathcal{D}^2_{q_1}(i\omega_{n_3-n_2})\mathcal{D}_{q_3}(i\omega_{-2n_2})\left(\mathcal{D}^{-1}_{q_3}(i\omega_{-2n_2})+\mathcal{D}^{-1}_{q_1}(i\omega_{n_3-n_2})\right)-$$
$$-\frac{96\pi T\gamma}{D}\sum_{n_1>0}\mathcal{D}_{q_3}(i\omega_{n+n_1})\mathcal{D}^t_{q_3}(i\omega_{n+n_1})\mathcal{D}^2_{q_1}(i\omega_{2n})\left(\mathcal{D}^{-1}_{q_3}(i\omega_{n+n_1})+\mathcal{D}^{-1}_{q_1}(i\omega_{2n})\right)+$$
$$+\frac{576\pi^2T^2\gamma^2}{D^2}\sum_{\substack{n_1>0\\ n_2<0}}\mathcal{D}^t_{q_3}(i\omega_{n_1-n_2})\mathcal{D}_{q_3}(i\omega_{n_1-n_2})\mathcal{D}^t_{q_1}(i\omega_{n-n_2})\mathcal{D}^2_{q_1}(i\omega_{n-n_2})\times$$
$$\times\left(\mathcal{D}^{-1}_{q_3}(i\omega_{n_1-n_2})+\mathcal{D}^{-1}_{q_1}(i\omega_{n-n_2})\right)+\frac{576\pi^2T^2\gamma^2}{D^2}\sum_{\substack{n_1>0\\ n_2<0}}\mathcal{D}^t_{q_3}(i\omega_{n_1+n})\mathcal{D}_{q_3}(i\omega_{n_1+n})\mathcal{D}^t_{q_1}(i\omega_{n-n_2})\mathcal{D}^2_{q_1}(i\omega_{n-n_2})\times$$
$$\times\left(\mathcal{D}^{-1}_{q_3}(i\omega_{n_1+n})+\mathcal{D}^{-1}_{q_1}(i\omega_{n-n_2})\right)-\frac{576\pi^3T^3\gamma^3}{D^3}\sum_{\substack{n_3\geq 0\\ n_2<0}}\theta(n_3-n_2-n)\left(\mathcal{D}^t_{q_1}\right)^2(i\omega_{n_3-n_2})\mathcal{D}^2_{q_1}(i\omega_{n_3-n_2})\times$$
$$\times\Bigg(\sum_{v\geq -n_2}\left(\mathcal{D}^{-1}_{q_3}(i\omega_v)+\mathcal{D}^{-1}_{q_1}(i\omega_{n_3-n_2})\right)\mathcal{D}^t_{q_3}(i\omega_v)\mathcal{D}_{q_3}(i\omega_v)+$$
$$+\sum_{v>n_3}\left(\mathcal{D}^{-1}_{q_3}(i\omega_v)+\mathcal{D}^{-1}_{q_1}(i\omega_{n_3-n_2})\right)\mathcal{D}^t_{q_3}(i\omega_v)\mathcal{D}_{q_3}(i\omega_v)\Bigg)\Big]. \quad (6.1.2)$$

This result is valid for an arbitrary $n>0$, next step is to assume $T=0$ and calculate the result for $\epsilon_n \to 0$:

$$-\frac{1}{2}\operatorname{tr}\Big\langle (W^2)^{\alpha\alpha}_{nn} S_0^{(4)}\Big\rangle \to \frac{1}{2g^2}\int_{pq}\Big[16(D_p^{-1}(0)+\underset{①}{D_q^{-1}}(0))D_p(0)D_q^2(0)-$$

$$-48\gamma\int_0^\infty dz(D_p^{-1}(z)+\underset{②}{D_q^{-1}}(0))D_pD_p^t(z)D_q^2(0)-$$

$$-48\gamma\int_0^\infty dz(D_p^{-1}(0)+\underset{③}{D_q^{-1}}(z))D_p(0)D_q^2D_q^t(z)+144\gamma^2\int_0^\infty dydz(D_p^{-1}(y)+\underset{④}{D_q^{-1}}(z))DD_p^t(y)D_q^2D_q^t(z)$$

$$-48\gamma\int_0^\infty dz(D_p^{-1}(2z)+\underset{⑤}{D_q^{-1}}(z))D_p(2z)D_q^2D_q^t(z)+$$

$$+144\gamma^2\int_0^\infty dydz(D_p^{-1}(y+z)+\underset{⑥}{D_q^{-1}}(z))D_pD_p^t(y+z)D^2D_q^t(z)$$

$$+48\gamma^2\int_0^\infty dydz(D_p^{-1}(2y)+\underset{⑦}{D_q^{-1}}(y+z))D_p(2y)D_q^2D_q^{t2}(y+z)$$

$$-144\gamma^3\int_0^\infty dxdydz(D_p^{-1}(x+y)+\underset{⑧}{D_q^{-1}}(y+z))D_pD_p^t(x+y)D_q^2D_q^{t2}(y+z)\Big]. \tag{6.1.3}$$

The propagator used here is slightly different:

$$D_q^{(t)}(\omega/D)=\mathcal{D}^{(t)}(i\omega). \tag{6.1.4}$$

Extracting the poles in $\epsilon$, we obtain:

$$-\frac{1}{2}\operatorname{tr}\Big\langle (W^2)^{\alpha\alpha}_{nn} S_0^{(4)}\Big\rangle \to \frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\Bigg(\frac{8-2\gamma(2+9\gamma)-12(4+\gamma)\ln(1+\gamma)+72(1+\gamma)\ln^2(1+\gamma)}{1+\gamma}+$$
$$+\epsilon\frac{3\gamma(3\gamma-2(5+\ln 4))+6\ln(1+\gamma)\left(1-3(-2+\gamma)\gamma+3(2+\gamma)\ln(1+\gamma)\right)}{2(1+\gamma)}\Bigg). \tag{6.1.5}$$

Note that the prefactor $A^2(\epsilon)$ diverges as $\epsilon^{-2}$; therefore, the expansion in brackets must be carried out up to first order in $\epsilon$ in order for the result to contain poles.

## 6.2 $\Lambda W^2 S_{\text{int}}^{(4)}$

This contribution is of the same order in $g^{-1}$, as the previous one. Upon evaluating the traces over Nambu space, we obtain:

$$-\frac{1}{2}\langle \operatorname{Tr}\left(\Lambda W^2\right)^{\alpha\alpha}_{nn} S_{\text{int}}^{(4)}\rangle = -\frac{\pi T\Gamma_t}{16}\langle \sum_{\substack{m,n_1,n_4\\ m_1,m_4,\alpha_4\\ \alpha',\alpha_4'}}\int_{\boldsymbol{p_1},\boldsymbol{p_2}} e^{i(\boldsymbol{p_1}+\boldsymbol{p_2})\boldsymbol{x}}\left[[(w_{i,p_1})^{\alpha\alpha'}_{nk}(w^\dagger_{i,p_2})^{\alpha'\alpha}_{kn}\right]B_{\beta\delta,\lambda\rho}\times$$
$$\times\int_{\boldsymbol{q_i}}\delta\left(\sum_i \boldsymbol{q_i}\right)\operatorname{sign}(\varepsilon_{n_1-m})\left[(w_{\beta,q_1})^{\xi\alpha_4}_{n_1-m,n_4}(w^\dagger_{\delta,q_2})^{\alpha_4\xi}_{n_4n_1}+h.c.\right]\operatorname{sign}(\varepsilon_{m_1+m})$$
$$\left[(w_{\lambda,q_3})^{\xi\alpha_4'}_{m_1+m,m_4}(w^\dagger_{\rho,q_4})^{\alpha_4'\xi}_{m_4m_1}+h.c.\right]\rangle, \tag{6.2.1}$$

where $B_{\beta\delta,\lambda\rho} = \sum_{i>0} \mathrm{tr}(\sigma_i\sigma_\beta\sigma_\delta)\,\mathrm{tr}(\sigma_i\sigma_\lambda\sigma_\rho)$. $h.c.$ stands for hermitian conjugation. Summarizing all of the possible corrections one could obtain:

$$
\begin{aligned}
-\frac{1}{2}\langle \mathrm{Tr}\left(\Lambda W^2\right)^{\alpha\alpha}_{nn} S^{(4)}_{\mathrm{int}}\rangle = \frac{+\pi T\Gamma_t}{g^3} \times \int_{\boldsymbol{q_1},\boldsymbol{q_3}} \sum_{m>0} 192\mathcal{D}_{q_1}\left(i\omega_m\right)\mathcal{D}^2_{q_3}\left(0\right) + \\
+96\mathcal{D}_{q_1}\left(0\right)\mathcal{D}^2_{q_3}\left(i\omega_m\right) + 96\mathcal{D}_{q1}\left(i\omega_{2m}\right)\mathcal{D}^2_{q_3}\left(i\omega_m\right) - \\
-\frac{4\pi\gamma T}{D} \times 2\sum_{k>0}\left[192\theta(k-m)\mathcal{D}_{q_1}(i\omega_{k-m})\mathcal{D}^t_{q_3}\mathcal{D}^2_{q_3}(i\omega_k) + \left\{288+48\theta(k-m)\right\}\mathcal{D}_{q_1}(i\omega_{k+m})\mathcal{D}^t_{q_3}\mathcal{D}^2_{q_3}(i\omega_k)\right] - \\
-\frac{4\pi\gamma T}{D}\sum_{k>0}\left[144\theta(k-m)\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k-m})\mathcal{D}^2_{q_3}(i\omega_k) + 384\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k+m})\mathcal{D}^2_{q_3}(i\omega_k)\right] + \\
+\frac{16\pi^2\gamma^2T^2}{D^2}\sum_{k>0}\Big\{192(k-m)\theta(k-m)\mathcal{D}_{q_1}(i\omega_{k-m})\left(\mathcal{D}^t_{q_3}\right)^2\mathcal{D}^2_{q_3}(i\omega_k) + \\
+\left[144k+48(k-m)\theta(k-m)\right]\mathcal{D}_{q_1}(i\omega_{k+m})\left(\mathcal{D}^t_{q_3}\right)^2\mathcal{D}^2_{q_3}(i\omega_k)\Big\} + \\
+2\times\frac{16\pi^2\gamma^2T^2}{D^2}\sum_{k>0}96\left\{(k-m)\theta(k-m)\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k-m})\mathcal{D}^t_{q_3}\mathcal{D}^2_{q_3}(i\omega_k) + 2k\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k+m})\mathcal{D}^t_{q_3}\mathcal{D}^2_{q_3}(i\omega_k)\right\} - \\
-\frac{64\pi^3T^3\gamma^3}{D^3}\sum_{k>0}96\Big\{(k-m)^2\theta(k-m)\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k-m})\left(\mathcal{D}^t_{q_3}\right)^2\mathcal{D}^2_{q_3}(i\omega_k) + \\
+k^2\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k+m})\left(\mathcal{D}^t_{q_3}\right)^2\mathcal{D}^2_{q_3}(i\omega_k)\Big\}.
\end{aligned} \tag{6.2.2}
$$

In the limit $T \to 0, n = 0$, one derives:

$$
\begin{aligned}
-\frac{1}{2}\langle \mathrm{Tr}\left(\Lambda W^2\right)^{\alpha\alpha}_{nn} S^{(4)}_{\mathrm{int}}\rangle \to \frac{12\gamma}{g^2}\int_{pq}\int_0^\infty dz\Big[\underset{①}{2D_q(z)D^2_p(0)} + \underset{②}{D_q(0)D^2_p(z)} + \underset{③}{D^2_q(z)D_p(2z)}\Big] \\
-\frac{12\gamma^2}{g^2}\int_{pq}\int_0^\infty dydz\Big[\underset{④}{8D_p(y)D^2_qD^t_q(y+z)} + \underset{⑤}{12D_p(y+z)D^2_qD^t_q(y)} + \underset{⑥}{2D_p(y+2z)D^2_qD^t_q(y+z)} \\
\underset{⑦}{-8y\gamma D_p(y)D^2_qD^{t2}_q(y+z)} \underset{⑧}{-6y\gamma D_p(y+z)D^2_qD^{t2}_q(y)} \underset{⑨}{-2y\gamma D_p(y+2z)D^2_qD^{t2}_q(y+z)} + \underset{⑩}{3D_pD^t_p(y)D^2_q(y+z)} \\
\underset{⑪}{+8D_pD^t_p(y+z)D^2_q(y)} \underset{⑫}{-8y\gamma D_pD^t_p(y)D^2_qD^t_q(y+z)} \underset{⑬}{-16y\gamma D_pD^t_p(y+z)D^2_qD^t_q(y)} \\
\underset{⑭}{+8y^2\gamma^2D_pD^t_p(y)D^2_qD^{t2}_q(y+z)} + \underset{⑮}{8y^2\gamma^2D_pD^t_p(y+z)D^2_qD^{t2}_q(y)}\Big].
\end{aligned} \tag{6.2.3}
$$

One could evaluate this integrals in the limit $\epsilon \to 0$:

$$
\begin{aligned}
-\frac{1}{2}\langle \mathrm{Tr}\left(\Lambda W^2\right)^{\alpha\alpha}_{nn} S^{(4)}_{\mathrm{int}}\rangle \to \frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\Bigg[\frac{6\gamma\left(3+4\gamma-(11+3\gamma)\ln(1+\gamma)\right)}{1+\gamma} + \\
+\epsilon\frac{3\Big(\gamma\left(10+\gamma(7+\ln 4)+\ln 1024\right) - 2\left(1+4\gamma(3+4\gamma+\ln 2)+\ln 16\right)\ln(1+\gamma) + \gamma(27+19\gamma)\ln^2(1+\gamma)\Big)}{2(1+\gamma)}\Bigg].
\end{aligned} \tag{6.2.4}
$$

## 6.3 $\Lambda W^4$

This contribution is given by the expression:

$$-\frac{1}{8}\,\mathrm{tr}\langle\Big(\Lambda W^4\Big)^{\alpha\alpha}_{nn}\rangle = -\frac{1}{g^2}\int_{q,p}\Big(8\mathcal{D}_q(i\omega_{2n})\mathcal{D}_p(i\omega_{2n}) - 36\gamma\frac{2\pi T}{D}\sum_{m>0}\mathcal{D}^t_p\mathcal{D}_p(i\omega_{n+m})\mathcal{D}_q(i\omega_{2n}) -$$

$$-12\gamma\frac{2\pi T}{D}\sum_{m>0}\mathcal{D}^t_p\mathcal{D}_p(i\omega_{n+m})\mathcal{D}_q(i\omega_{2m}) + 36\gamma^2\left(\frac{2\pi T}{D}\right)^2\sum_{m,l>0}\mathcal{D}^t_p\mathcal{D}_p(i\omega_{n+m})\mathcal{D}^t_q\mathcal{D}_q(i\omega_{n+l}) +$$

$$+36\gamma^2\left(\frac{2\pi T}{D}\right)^2\sum_{m,l>0}\mathcal{D}^t_p\mathcal{D}_p(i\omega_{n+m})\mathcal{D}^t_q\mathcal{D}_q(i\omega_{m+l})\Big). \qquad (6.3.1)$$

For $n = 0$:

$$-\frac{1}{8}\,\mathrm{tr}\langle\Big(\Lambda W^4\Big)^{\alpha\alpha}_{nn}\rangle \to -\frac{1}{g^2}\int_{q,p}\Big(8D_q(0)D_p(0) - 36\gamma\int dx D^t_p D_p(x)D_q(0) - 12\gamma\int dx D^t_p D_p(x)D_q(2x) +$$

$$+36\gamma^2\int dxdy D^t_p D_p(x)D^t_q D_q(y) + 36\gamma^2\int dxdy D^t_p D_p(x)D^t_q D_q(x+y)\Big). \qquad (6.3.2)$$

Evaluating these integrals yields:

$$-\frac{1}{8}\,\mathrm{tr}\langle\Big(\Lambda W^4\Big)^{\alpha\alpha}_{nn}\rangle \to \frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\Big[-8+42\ln(1+\gamma)-54\ln(1+\gamma)^2+\epsilon\big(3\ln(2)\ln(1+\gamma)-\frac{3}{2}\ln^2(1+\gamma)\big)\Big]. \qquad (6.3.3)$$

## 6.4 $S^{(3)}_{\mathrm{int}}$

Now, we write the whole expression:

$$-\frac{1}{4}\langle\mathrm{tr}\left(\Lambda W^2\right)^{\alpha\alpha}_{nn}\left[S^{(3)}_{\mathrm{int}}\right]^2\rangle = -2\left(\frac{\pi T\Gamma_t}{4}\right)^2\sum_{\substack{n_i,m,m',k\\ i,p,r,s,t,j,l}}\int_{\boldsymbol{p_1},\boldsymbol{p_2}} e^{i(\boldsymbol{p_1}+\boldsymbol{p_2})\boldsymbol{x}}\left[(w_{i,p_1})^{\alpha\alpha'}_{nk}(w^\dagger_{i,p_2})^{\alpha'\alpha}_{kn}\right]C_{prs}C_{tjl}\times$$

$$\times\int_{\boldsymbol{q_i},\boldsymbol{q'_i}}\delta\left(\sum_i \boldsymbol{q_i}\right)\delta\left(\sum_i \boldsymbol{q'_i}\right)\left[(w_{p,q_1})^{\xi\xi}_{n_1-m,n_1} + (w^\dagger_{p,q_1})^{\xi\xi}_{n_1-m,n_1}\right]\times$$

$$\times\left[(w_{r,q_2})^{\xi\alpha_1}_{n_2+m,n_3}(w^\dagger_{s,q_3})^{\alpha_1\xi}_{n_3,n_2} - (w^\dagger_{r,q_2})^{\xi\alpha_1}_{n_2+m,n_3}(w_{s,q_3})^{\alpha_1\xi}_{n_3,n_2}\right]\times$$

$$\times\left[(w_{t,q'_1})^{\xi'\xi'}_{m_1-m',m_1} + (w^\dagger_{t,q'_1})^{\xi'\xi'}_{m_1-m',m_1}\right]\left[(w_{j,q'_2})^{\xi'\alpha'_1}_{m_2+m',m_3}(w^\dagger_{l,q'_3})^{\alpha'_1\xi'}_{m_3,m_2} - (w^\dagger_{j,q'_2})^{\xi'\alpha'_1}_{m_2+m',m_3}(w_{l,q'_3})^{\alpha'_1\xi'}_{m_3,m_2}\right], \qquad (6.4.1)$$

where $C_{tjk} = \mathrm{tr}(\sigma_t\sigma_j\sigma_k)$. Using Wick's theorem, one could obtain four different nonzero contributions. We present the expressions for them.

### 6.4.1 $S^{(3)}_{\text{int},1}$

First contraction and (6.2.2) are very similar:

$$
\begin{aligned}
S^{(3)}_{\text{int},1} = &-\frac{\pi T\Gamma_t}{g^3}\gamma \times \int_{\boldsymbol{q_1},\boldsymbol{q_3}} \sum_{m>0} \frac{\omega_m}{D}\mathcal{D}^t_{q_1-q_3}(i\omega_{|m|}) \times \int_{\boldsymbol{q_1},\boldsymbol{q_3}} 192\mathcal{D}_{q_1}\left(i\omega_m\right)\mathcal{D}^2_{q_3}\left(0\right) + \\
&+96\mathcal{D}_{q_1}\left(0\right)\mathcal{D}^2_{q_3}\left(i\omega_m\right) + 96\mathcal{D}_{q1}\left(i\omega_{2m}\right)\mathcal{D}^2_{q_3}\left(i\omega_m\right) - \\
&-\frac{4\pi\gamma T}{D} \times 2\sum_{k>0}\left[192\theta(k-m)\mathcal{D}_{q_1}(i\omega_{k-m})\mathcal{D}^t_{q_3}\mathcal{D}^2_{q_3}(i\omega_k) + \left\{288+48\theta(k-m)\right\}\mathcal{D}_{q_1}(i\omega_{k+m})\mathcal{D}^t_{q_3}\mathcal{D}^2_{q_3}(i\omega_k)\right] - \\
&-\frac{4\pi\gamma T}{D}\sum_{k>0}\left[144\theta(k-m)\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k-m})\mathcal{D}^2_{q_3}(i\omega_k) + 384\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k+m})\mathcal{D}^2_{q_3}(i\omega_k)\right] + \\
&+\frac{16\pi^2\gamma^2T^2}{D^2}\sum_{k>0}\Big\{192(k-m)\theta(k-m)\mathcal{D}_{q_1}(i\omega_{k-m})\left(\mathcal{D}^t_{q_3}\right)^2\mathcal{D}^2_{q_3}(i\omega_k) + \\
&+\left[144k+48(k-m)\theta(k-m)\right]\mathcal{D}_{q_1}(i\omega_{k+m})\left(\mathcal{D}^t_{q_3}\right)^2\mathcal{D}^2_{q_3}(i\omega_k)\Big\} + \\
&+2\times\frac{16\pi^2\gamma^2T^2}{D^2}\sum_{k>0}96\left\{(k-m)\theta(k-m)\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k-m})\mathcal{D}^t_{q_3}\mathcal{D}^2_{q_3}(i\omega_k) + 2k\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k+m})\mathcal{D}^t_{q_3}\mathcal{D}^2_{q_3}(i\omega_k)\right\} - \\
&-\frac{64\pi^3T^3\gamma^3}{D^3}\sum_{k>0}96\Big\{(k-m)^2\theta(k-m)\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k-m})\left(\mathcal{D}^t_{q_3}\right)^2\mathcal{D}^2_{q_3}(i\omega_k) + \\
&+k^2\mathcal{D}^t_{q_1}\mathcal{D}_{q_1}(i\omega_{k+m})\left(\mathcal{D}^t_{q_3}\right)^2\mathcal{D}^2_{q_3}(i\omega_k)\Big\}.
\end{aligned}
\tag{6.4.2}
$$

We now proceed by implementing the same limit:

$$
\begin{aligned}
S^{(3)}_{\text{int},1} \to &-\frac{12\gamma}{g^2}\int_{pq}\int_0^\infty dz\left[\gamma z D^t_{\boldsymbol{p+q}}(z)\right]\Big[\underset{①}{2D_q(z)D^2_p(0)} + \underset{②}{D_q(0)D^2_p(z)} + \underset{③}{D^2_q(z)D_p(2z)}\Big] + \\
&+\frac{12\gamma^2}{g^2}\int_{pq}\int_0^\infty dydz\left[\gamma z D^t_{\boldsymbol{p+q}}(z)\right]\Big[\underset{④}{8D_p(y)D^2_qD^t_q(y+z)} + \underset{⑤}{12D_p(y+z)D^2_qD^t_q(y)} + \underset{⑥}{2D_p(y+2z)D^2_qD^t_q(y+z)} \\
&\underset{⑦}{-8y\gamma D_p(y)D^2_qD^{t2}_q(y+z)} \underset{⑧}{-6y\gamma D_p(y+z)D^2_qD^{t2}_q(y)} \underset{⑨}{-2y\gamma D_p(y+2z)D^2_qD^{t2}_q(y+z)} + \underset{⑩}{3D_pD^t_p(y)D^2_q(y+z)} \\
&\underset{⑪}{+8D_pD^t_p(y+z)D^2_q(y)} \underset{⑫}{-8y\gamma D_pD^t_p(y)D^2_qD^t_q(y+z)} \underset{⑬}{-16y\gamma D_pD^t_p(y+z)D^2_qD^t_q(y)} \\
&\underset{⑭}{+8y^2\gamma^2D_pD^t_p(y)D^2_qD^{t2}_q(y+z)} + \underset{⑮}{8y^2\gamma^2D_pD^t_p(y+z)D^2_qD^{t2}_q(y)}\Big]
\end{aligned}
\tag{6.4.3}
$$

Details of the integral evaluations are provided in section 11. The result in the form of a series is given by the expression:

$$S^{(3)}_{\text{int},1} \to \frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\times$$
$$\times\Bigg[\frac{-6\gamma(4+\gamma(9+\gamma(20+7\gamma)))+6(1+\gamma)\ln(1+\gamma)(4+\gamma(19+\gamma(32+3\gamma))-3(1+\gamma)(4+3\gamma)\ln(1+\gamma))}{\gamma(1+\gamma)^2}+$$
$$+\epsilon\frac{3}{2\gamma(1+\gamma)^2}\Big[-8(-3+\gamma)\gamma^3+(1+\gamma)\ln(1+\gamma)(16+2\gamma(14+19\gamma(1+\gamma))+\ln(1+\gamma)(-44-\gamma(77+\gamma(25+6\gamma))-$$
$$-6(1+\gamma)(2+\gamma)\ln(1+\gamma)))+2(1+\gamma)(8+2\gamma(7+17\gamma)-3(1+\gamma)(8+5\gamma)\ln(1+\gamma))\,\mathrm{Li}_2(-\gamma)-$$
$$-6\gamma(1+\gamma)^2\ln(1+\gamma)(\mathrm{Li}_2(-\gamma)+\frac{1}{2}\ln^2(1+\gamma))\Big]+$$
$$+\epsilon\Big(R_5(\gamma)+E_5(\gamma)+T_2(\gamma)+R_6(\gamma)+T_3(\gamma)+E_6(\gamma)+E_7(\gamma)+R_7(\gamma)\Big)\Bigg]. \tag{6.4.4}$$

Contributions, denoted as capital letters, are analytic expressions in the form of converging integrals; some of them are irrelevant, namely $E$ contributions, since they cancel out exactly. The expressions for the relevant contributions will be written down below:

$$R_5(\gamma) = -3\gamma\int_1^{1+\gamma}dz\int[]\frac{u_2}{(u_iu_j)(u_1(1+\gamma)+u_2+2u_3)^2}$$
$$R_6(\gamma) = -6\gamma\int_1^{1+\gamma}dz(1+\gamma-z)\int[]\frac{u_2^2}{(u_iu_j)(u_1(1+\gamma)+zu_2+2u_3)^2(u_3+u_2z)}$$
$$R_7(\gamma) = 6\gamma\int_1^{1+\gamma}dz(z-1)(1+\gamma-z)\int[]\frac{u_2^3}{(u_iu_j)(u_1(1+\gamma)+zu_2+2u_3)^2(u_3+u_2z)^2}$$
$$T_2(\gamma) = -18\gamma^2\int_1^{1+\gamma}dz\int[]\frac{u_3^2}{(u_iu_j)z^2(u_3+u_2z)(u_1(1+\gamma)+u_3)^2}$$
$$T_3(\gamma) = 9\gamma^2\int_1^{1+\gamma}dz\int[]\frac{u_3^2}{(u_iu_j)z(u_3+u_2z)(u_1(1+\gamma)+u_3)^2}. \tag{6.4.5}$$

Here we used the notation $\int[] = \left(\prod_i\int_0^1 du_i\right)\delta(1-\sum_i u_i)$ for integration over Feynman parameterization variables and $u_iu_j = u_1u_2+u_2u_3+u_1u_3$.

These expressions look huge, but each contribution corresponds to a diagram in Green's function formalism. Let us consider, for example, one of the contributions above. First, we notice that it is possible to add (6.4.3) and (6.2.3), and using the relation

$$1-\gamma x D^t(x) = D^{-1}D^t(x), \tag{6.4.6}$$

we obtain a contribution that is expressed in terms of propagators as:

$$\int_{q_1,q_3}\int_0^\infty dxdy D^{-1}_{q_1+q_3}D^t_{q_1+q_3}(y)D_{q_1}(x+y)D^t_{q_3}D^2_{q_3}(x). \tag{6.4.7}$$

This contribution is a sum of the fifth contributions in (6.4.3) and (6.2.3). Here, the powers of $\gamma$ and other prefactors were dropped for convenience. Combination $D^{-1}D^t$ corresponds to the screened interaction without the dressed vertices: that is due to the fact, that two incoming retarded Green's functions don't get dressed by the diffusive ladder. The diagram, corresponding to that expression, is presented in Fig. 4.

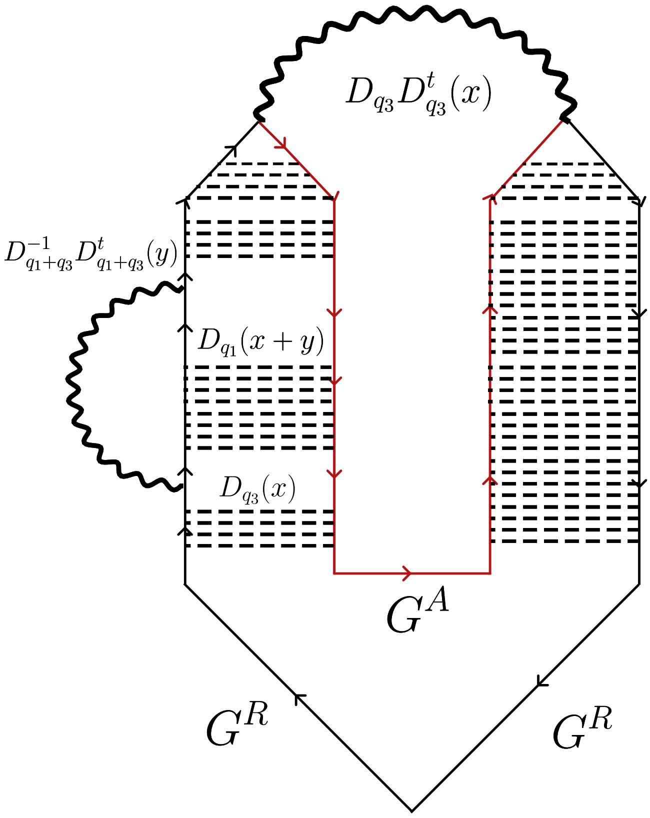


Figure 4: Fifth contributions from (6.4.3) and (6.2.3), depicted as a diagram in Green's function formalism. Black lines denote retarded Green's functions, red lines - advanced.

In principle, not every contribution we obtain could be mapped back to the corresponding one in Green's function formalism, but the sum of all contributions is inevitably equivalent to the one obtained with Green's functions. That could be traced to the fact that the expansion of action and $Q$ matrix depends on the parametrization we chose, while the result should be independent of it. Therefore, it is not a coincidence that in order to draw a graph with Green's functions, we need to consider a sum of expressions in terms of propagators, as we did here, when we added up (6.4.3) and (6.2.3).

### 6.4.2 $S^{(3)}_{\text{int},2}$

The second contraction in this series is given by:

$$S^{(3)}_{\text{int},2} = -8\frac{16}{g^4}\left(\frac{\pi T\Gamma_t}{4}\right)^2 \int_{\boldsymbol{q_1},\boldsymbol{q_2}} \sum_{m,k<0} \left\{2 - \frac{4\pi T\gamma}{D}(n-k)\mathcal{D}^t_{q_1+q_2}(i\omega_{n-k})\right\} \mathcal{D}_{q_1+q_2}(i\omega_{n-k})\times$$
$$\times \left\{2 - \frac{4\pi T\gamma}{D} m\theta(m)\mathcal{D}^t_{q_1}(i\omega_m)\right\} \mathcal{D}_{q_1}(i\omega_m)\mathcal{D}_{q_1+q_2}(i\omega_{n-k})\times$$
$$\times \Bigg[(-48)\left\{\theta(m+k)+\theta(m-n) - \frac{4\pi T\gamma}{D}\mathcal{D}^t_{q_1+q_2}(i\omega_{n-k})\sum_{n_2>0}\theta(n-k-n_2)\theta(m-(n-k-n_2))\right\}+$$
$$96\frac{4\pi T\gamma}{D} m\theta(m)\mathcal{D}^t_{q_2}(i\omega_{n-k+m})\left\{\theta(n+m)+\theta(m-k) - \frac{4\pi T\gamma}{D}(n-k)\mathcal{D}^t_{q_1+q_2}(i\omega_{n-k})\right\}\Bigg]\mathcal{D}_{q_2}(i\omega_{n-k+m}). \quad (6.4.8)$$

In the limit $n = 0, T \to 0$ one obtains:

$$S^{(3)}_{\text{int},2} \to\to \frac{24\gamma^2}{g^2}\int_{pq}\int_0^\infty dydz\Big[\underset{①}{D_q D^t_q(y) D^t_p(z) D_{\boldsymbol{q+p}}(y+z)} + \underset{②}{D_q D^t_q(y) D^t_p(y+z) D_{\boldsymbol{q+p}}(z+2y)} -$$
$$-\underset{③}{2\gamma z D_q D^{t2}_q(y+z) D^t_p(z) D_{\boldsymbol{q+p}}(y+2z)} - \underset{④}{2\gamma z D_q D^{t2}_q(z) D^t_p(y+z) D_{\boldsymbol{q+p}}(y+2z)} -$$
$$-\underset{⑤}{8\gamma z D^{t2}_q(y) D^t_p(z) D_{\boldsymbol{q+p}} D^t_{\boldsymbol{q+p}}(y+z)}\Big]. \quad (6.4.9)$$

One could obtain poles in $\epsilon$:

$$S^{(3)}_{\text{int},2} \to \frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\Bigg[\frac{24\gamma\left(4\gamma - 3(1+\gamma)\ln(1+\gamma)\right)}{(1+\gamma)^2} - \epsilon\frac{6\left(\gamma\left(16\gamma + 3(1+\gamma)\ln^2(1+\gamma) + 12(1+\gamma)\,\text{Li}_2(-\gamma)\right)\right)}{(1+\gamma)^2}+$$
$$+\epsilon\big(T_1(\gamma) + R_3(\gamma) + R_4(\gamma) + R_8(\gamma) + E_4(\gamma)\big)\Bigg], \quad (6.4.10)$$

where relevant contributions are given by expressions:

$$T_1(\gamma) = 6\gamma\int_1^{1+\gamma} dz \int [] \frac{u_3}{(u_i u_j) z (u_3 + u_2 z)(u_1(1+\gamma) + u_3)}$$
$$R_3(\gamma) = -6\gamma\int_1^{1+\gamma} dz \int [] \frac{(2u_3 + u_1(1+\gamma))}{(u_i u_j)(u_1(1+\gamma) + z u_2 + 2u_3)(u_3 + u_1(1+\gamma)))}$$
$$R_4(\gamma) = 12\gamma\int_1^{1+\gamma} dz \int [] \frac{(2u_3 + u_1(1+\gamma))}{(u_i u_j)(1+\gamma)(u_1(1+\gamma) + (1+\gamma)u_2 + 2u_3)(u_3 + u_1(1+\gamma))}$$
$$R_8(\gamma) = 12\gamma\int_1^{1+\gamma} dz(z-1) \int [] \frac{u_2^2}{(u_i u_j)(u_3 + u_2 z)(u_1(1+\gamma) + u_2 z + 2u_3)^2}. \quad (6.4.11)$$

### 6.4.3 $S^{(3)}_{\text{int},3}$

The third contraction takes the form:

$$S^{(3)}_{\text{int},3}{=}{-}8\frac{16}{g^4}\left(\frac{\pi T\Gamma_t}{4}\right)^2\int_{\boldsymbol{q_1},\boldsymbol{q_2}}\sum_m\theta(m{-}n)\left[2{-}\frac{4\pi T\gamma}{D}m\theta(m)\mathcal{D}^t_{q_1+q_2}(i\omega_m)\right]^2\mathcal{D}^2_{q_1+q_2}(i\omega_m)$$

$$\sum_{n_1>0}\Bigg\{48\mathcal{D}_{q_1}(i\omega_{2n_1})\mathcal{D}_{q_2}(i\omega_{2n_1+m})+48\mathcal{D}_{q_1}(i\omega_{2n_1+m})\mathcal{D}_{q_2}(i\omega_{2n_1+2m})-\frac{4\pi T\gamma}{D}\sum_{n_2<0}\Big[72+24\theta(-m-n_2)-$$

$$-\frac{4\pi T\gamma}{D}(n_1-n_2)\,\mathcal{D}^t_{q_2}(i\omega_{n_1-n_2+m})\Big]\mathcal{D}_{q_2}(i\omega_{n_1-n_2+m})\mathcal{D}_{q_1}\mathcal{D}^t_{q_1}(i\omega_{n_1-n_2})-$$

$$-\frac{4\pi T\gamma}{D}\sum_{n_2<0}96\mathcal{D}_{q_1}(i\omega_{n_1-n_2})\mathcal{D}_{q_2}\mathcal{D}^t_{q_2}(i\omega_{n_1-n_2+m})\Bigg\}\tag{6.4.12}$$

In the limit $n=0, T\to 0$ one obtains:

$$S^{(3)}_{\text{int},3}\to-\frac{12\gamma^2}{g^2}\int_{qp}\int_0^\infty dydz\Big[\underset{①}{D^{t2}_p(y)D_q(z)D_{\boldsymbol{q+p}}(y+z)}+\underset{②}{D^{t2}_p(y)D_q(y+z)D_{\boldsymbol{q+p}}(2y+z)}$$

$$\underset{③}{-6\gamma zD^{t2}_p(y)D_qD^t_q(z)D_{\boldsymbol{q+p}}(y+z)}-\underset{④}{2\gamma zD^{t2}_p(y)D_qD^t_q(y+z)D_{\boldsymbol{q+p}}(2y+z)}-$$

$$\underset{⑤}{-8\gamma zD^{t2}_p(y)D^t_q(z)D_{\boldsymbol{q+p}}D^t_{\boldsymbol{q+p}}(y+z)}\tag{6.4.13}$$

One could obtain poles in $\epsilon$:

$$S^{(3)}_{\text{int},3}\to\frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\Bigg[\frac{12\gamma^2(-1{+}3\gamma)}{(1{+}\gamma)^2}+$$

$$+\epsilon\frac{\gamma^2\left(\pi^2(-1{+}4\gamma){-}24(-1{+}\gamma{+}6\gamma^2){+}36(1{+}\gamma)\ln^2(1{+}\gamma){+}6(7{+}2\gamma)\,\mathrm{Li}_2(-\gamma)\right)}{2(1{+}\gamma)^2(1{+}2\gamma)}+$$

$$+\epsilon\Big(R_1(\gamma)+R_2(\gamma)+E_3(\gamma)\Big)\Bigg],\tag{6.4.14}$$

where relevant contributions are given by expressions:

$$R_1(\gamma)=-3\gamma\int_1^{1+\gamma}dz\int[]\frac{u_2+2u_3}{(u_iu_j)(1+\gamma)(u_1(1+\gamma)+u_2+2u_3)(u_2+u_3)}$$

$$R_2(\gamma)=6\gamma^2\int_1^{1+\gamma}dz\int[]\frac{u_2(2u_3+u_2z)}{(u_iu_j)(1+\gamma)(u_1(1+\gamma)+zu_2+2u_3)(u_3+u_2z)^2}.\tag{6.4.15}$$

#### 6.4.4 $S^{(3)}_{\text{int},4}$

The fourth contraction in this series can be written as:

$$S^{(3)}_{\text{int},4}{=}{-}8\frac{16}{g^4}\left(\frac{\pi T\Gamma_t}{4}\right)^2\int_{\boldsymbol{q_1},\boldsymbol{q_2}}\sum_{m,m'}\left[2{-}\frac{4\pi T\gamma}{D}m'\theta(m')D^t_{q_1}(i\omega_{m'})\right]D_{q_1}(i\omega_{m'})$$
$$\left[2{-}\frac{4\pi T\gamma}{D}m\theta(m)D^t_{q_2}(i\omega_m)\right]D_{q_2}(i\omega_m)D^2_{q_1+q_2}(i\omega_{m+m'})\theta(m+m'-n)\times$$
$$\times\Bigg[-24\left\{\theta(n-m)\theta(n-m')+\theta(m-n)\theta(m'-n)\right\}+48\frac{4\pi T\gamma}{D}D^t_{q_1+q_2}(i\omega_{m+m'})\times$$
$$\times\Big\{m'\theta(m')\theta(n{-}m'){+}m'\theta(m')\theta(m{-}n){+}m\theta(m)\theta(n{-}m){+}m\theta(m)\theta(m'{-}n){-}$$
$$-\frac{4\pi T\gamma}{D}mm'\theta(m)\theta(m')D^t_{q_1+q_2}(i\omega_{m+m'})\Big\}\Bigg] \tag{6.4.16}$$

In the limit $n=0, T\to 0$ one obtains:

$$S^{(3)}_{\text{int},4}\to\frac{12\gamma^2}{g^2}\int_{pq}\int_0^\infty dydzD^t_q(y)D^t_p(z)D^2_{\boldsymbol{q+p}}(y+z)\Big[\underset{①}{1}-\underset{②}{4y\gamma D^t_{\boldsymbol{q+p}}(y+z)}-\underset{③}{4z\gamma D^t_{\boldsymbol{q+p}}(y+z)}+$$
$$+\underset{④}{8yz\gamma^2D^{t2}_{\boldsymbol{q+p}}(y+z)}\Big)\Big] \tag{6.4.17}$$

One could obtain poles in $\epsilon$:

$$S^{(3)}_{\text{int},4}\to\frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\epsilon\left[\frac{\gamma^2\left(-\pi^2{+}6\ln^2(1{+}\gamma){+}12\,\text{Li}_2(-\gamma)\right)}{2{+}4\gamma}+2E_1(\gamma)+E_2(\gamma)\right]. \tag{6.4.18}$$

# 7 Renormalization

At dimensionality $d=2$, the LDoS operator is renormalizable, which is shown in section 7. Therefore, once the two-loop correction to the LDoS is known, one can derive the corresponding renormalization-group (RG) equations. We adopt the minimal subtraction scheme [37], which enables the derivation of RG equations using only the divergent parts proportional to $\epsilon=d-2$. We start by writing down the expression for the field-renormalization constant, defined as follows:

$$\rho=\rho_0 Z. \tag{7.1}$$

Here $\rho_0$ denotes bare Fermi-level LDoS, and the so-called $Z$ factor is a field-renormalization constant. In $\epsilon$-regularization, this factor could be obtained as an asymptotic series in the small parameter $t\propto g^{-1}$:

$$Z=1+\frac{th^\epsilon}{\epsilon}A_1(\gamma)+\frac{t^2h^{2\epsilon}}{\epsilon^2}\big(B(\gamma)+C(\gamma)\epsilon\big). \tag{7.2}$$

There are also one-loop corrections to the conductivity and our IR cutoff $h^2$, which effectively plays the role of an inverse system size $L^{-1}$ or, at non-zero temperature, the inverse temperature scale $(\sqrt{D/T})^{-1}$. We are interested only in one-loop corrections to these quantities since only they will contribute to the two-loop result we are pursuing. We therefore write down the renormalized conductance $g'$ and the infrared scale $h'$:

$$g' = g\big[1 + a_1(\gamma)\frac{th^\epsilon}{\epsilon}\big], \quad h'^2 = h^2\big[1 - b(\gamma)\frac{th^\epsilon}{\epsilon}\big]. \tag{7.3}$$

It is convenient to define $t$ as

$$t = \frac{2^{-\epsilon}}{g\pi^{1+\epsilon/2}}\Gamma(1-\epsilon/2). \tag{7.4}$$

The correction to the infrared scale is intimately related to field-renormalization and conductance corrections. That is due to the fact that $h^2$ multiplies the elementary field, and not a composite operator. There are also one-loop corrections to $Z_\omega$ and $\Gamma_t$ [33]:

$$\frac{\delta Z_\omega}{Z_\omega} = \frac{\delta\Gamma_t}{\Gamma_t} = (1-3\gamma)\frac{th^\epsilon}{\epsilon}. \tag{7.5}$$

These corrections cancel each other out in the interaction strength correction:

$$\delta\gamma = \frac{\delta\Gamma_t}{Z_\omega} - \frac{\delta Z_\omega \Gamma_t}{Z_\omega^2} = \frac{\Gamma_t}{Z_\omega}\big(\frac{\delta\Gamma_t}{\Gamma_t} - \frac{\delta Z_\omega}{Z_\omega}\big) = 0. \tag{7.6}$$

Therefore, $\gamma$ is not renormalized in one-loop approximation. We now define a renormalized $t'$:

$$t'^{-1} = t^{-1}\big[1 + a_1(\gamma)\frac{th^\epsilon}{\epsilon}\big]. \tag{7.7}$$

I would like to get into some details of the derivation of the RG equations using the minimal subtraction scheme. First, the UV divergencies are treated with $\epsilon$-regularization - there is no need to introduce an additional cutoff. Since the only scale we are left with is $h'$, it is convenient to use dimensionless parameters:

$$\bar{t} = t'h'^\epsilon. \tag{7.8}$$

Expressing the $Z$ factor in terms of $\bar{t}$, we obtain:

$$Z = 1 + \frac{\bar{t}}{\epsilon}A_1(\gamma) + \frac{\bar{t}^2}{\epsilon^2}\big(B(\gamma) + A_1(\gamma)a_1(\gamma) + \big[C(\gamma) + A(\gamma)b(\gamma)\big]\epsilon\big). \tag{7.9}$$

For observables to be UV finite, we need to redefine bare parameters of the theory. Dimensional analysis allows us to rewrite $t$ in a form

$$t = \bar{t}h'^{-\epsilon}Z_t(\bar{t}). \tag{7.10}$$

$Z_t$ could be obtained from (7.7), it contains poles as a function of $\epsilon$. Furthermore, it is natural to assume the independence of $t$ from $h'$, since a bare parameter in a theory should be independent of the varying IR scale. Hence, we can evaluate $\beta$-function:

$$\left(\frac{\partial\bar{t}}{\partial\log(1/h')}\right)_t \equiv \beta(\bar{t}) = -\frac{\epsilon\bar{t}}{1 + (\partial\ln Z_t(\bar{t})/\ln(\bar{t}))_{h'}} = -\epsilon\bar{t} + a_1\bar{t}^2 + O(\bar{t}^3). \tag{7.11}$$

This result is useful to find an equation for the renormalization flow of the LDoS. We write

$$\left(\frac{\partial \ln Z(\bar{t})}{\partial \log(1/h')}\right)_t = \left(\frac{\partial \ln Z(\bar{t})}{\partial \ln(\bar{t})}\right)_{h'} \beta(\bar{t}) = -A_1(\gamma)\bar{t} - \big(2A_1(\gamma)b(\gamma) + 2C(\gamma)\big)\bar{t}^2 + \\ + \frac{\bar{t}^2}{\epsilon}\big(A_1^2(\gamma) - a_1(\gamma)A_1(\gamma) - 2B(\gamma)\big). \tag{7.12}$$

The requirement that the RG flow be finite implies a consistency condition on the coefficients:

$$B(\gamma) = \frac{1}{2}A_1(\gamma)\big(A_1(\gamma) - a(\gamma)\big). \tag{7.13}$$

# 8 Results

The result of this work is the analytic expression for $B(\gamma)$ and $C(\gamma)$. Substituting $\bar{t}$ as defined in (7.8) and (7.4), we obtain analytic expression for $B(\gamma)$:

$$B(\gamma) = \frac{1}{2}\big(-6 + 21\ln(1+\gamma) + 6\frac{\ln(1+\gamma)}{\gamma} - 9\ln^2(1+\gamma) - 18\frac{\ln^2(1+\gamma)}{\gamma}\big). \tag{8.1}$$

In order to verify whether this operator is renormalizable, we must check the validity of the identity (7.13). To this end, we use the expression for $a_1(\gamma)$ obtained in [33]:

$$a_1(\gamma) = 1 + 6\left(1 - \frac{(1+\gamma)\ln(1+\gamma)}{\gamma}\right). \tag{8.2}$$

According to (5.4), $A_1(\gamma)$ is given by the following expression:

$$A_1(\gamma) = 1 - 3\ln(1+\gamma). \tag{8.3}$$

By substituting these functions into the expression (7.13), we see that the equality holds.

As for $C(\gamma)$, the expression is a little more elaborate:

$$C(\gamma) = \frac{1}{16\gamma(1+\gamma)^2(1+2\gamma)}\Big(\gamma^2\big(-\gamma\left(\pi^2(1-4\gamma)+66(1+\gamma)(1+2\gamma)\right) + 6(1+\gamma)(3+\gamma)(1+2\gamma)\ln 2\big) \\ +3(1+\gamma)\ln(1+\gamma)\big(-2(1+2\gamma)\left(-8+\gamma(-14+\gamma(-13+\ln 8)) + \ln 8\right) \\ +\ln(1+\gamma)\left(-44+\gamma(-154+\gamma(-137+\gamma(15+26\gamma)))-3(1+\gamma)(1+2\gamma)(4+3\gamma)\ln(1+\gamma)\right)\big) \\ -6\left(-8-\gamma(38+\gamma(68+\gamma(65+22\gamma)))+6(1+\gamma)^2(1+2\gamma)(4+3\gamma)\ln(1+\gamma)\right)\mathrm{Li}_2(-\gamma)\Big) + \\ +\frac{1}{8}\big(R_1(\gamma) + R_2(\gamma) + R_3(\gamma) + R_4(\gamma) + R_5(\gamma) + R_6(\gamma) + R_7(\gamma) + R_8(\gamma) + T_1(\gamma) + T_2(\gamma) + T_3(\gamma)\big), \tag{8.4}$$

where $\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} x^k/k^2$ is dilogarithm.

In class C, $\gamma$ represents the interaction strength in the triplet channel. As mentioned in the introduction, under the crossover to class A, $\gamma$ becomes a dimensionless singlet channel amplitude, with the limit $\gamma \to -1$ corresponding to the Coulomb interaction. As seen in

Eq. (5.4), the divergence in this limit also appears in class C. It is therefore natural to study the behavior near $\gamma \to -1$ in the two-loop approximation, bearing in mind its relevance for the class A regime. The expression is:

$$C(\gamma) \xrightarrow{\gamma\to-1} -\frac{1}{8}(57.41 + \# - (33.96 + \#)\ln(1+\gamma) + (2.66 + \#)\ln(1+\gamma)^2 + \ln(1+\gamma)^3). \quad (8.5)$$

In the next section, we discuss the implications of this result.

# 9 Discussions

As we consider the limit $\gamma \to -1$ in class A, due to the static screening of Coulomb interaction introduces cutoff for the integrals over frequency, which is effectively proportional to the inverse system size $L^{-1}$, or $L_E^{-1} = (\sqrt{D/E})^{-1}$ for the energy dependence. Therefore, changing $1+\gamma$ with $(\kappa L)^{-1}$, where $\kappa$ is an inverse Debye screening length, will qualitatively allow us to check for the corrections to the ZBA.

First, the expression contains no terms of order $(1+\gamma)^{-1}$ or more singular, even though in the intermediate answers they were present. These divergencies cancel out exactly, which is shown analytically. Let us recall the answer in one-loop:

$$\rho \sim \exp\Big(-3t\ln(1+\gamma)\ln\big((L_E/l)^{-1}\big)\Big). \quad (9.1)$$

Here $l$ is a mean free path, and we assumed $\ln(1+\gamma) \gg 1$. It is suggested that this answer holds for arbitrarily small energies if we consider the renormalization of $g$. In the second order, integrating (7.12) we obtain:

$$\rho(E) \sim \exp\Big(-3\bar{t}\ln(1+\gamma)\ln\big((L_E/l)^{-1}\big) - \frac{\bar{t}^2}{4}\ln^3(1+\gamma)\ln\big((L_E/l)^{-1}\big)\Big). \quad (9.2)$$

As we mentioned above, to track the ZBA, one should substitute $1+\gamma$ with $(\kappa L_E)^{-1}$ before integrating the RG equation. Doing this, one could obtain that for energies that satisfy the relation:

$$\ln\big(L_E\kappa\big) \geq \sqrt{g} \gg 1 \quad \text{or} \quad \ln(D\kappa^2/E) \geq \sqrt{g} \gg 1, \quad (9.3)$$

the two-loop correction is larger than the first-order. In the same limit, renormalization of $g$ could still be small, since it is controlled by $-\ln\big(E\tau\big)/g$. Therefore, we predict the discrepancy from the double-squared-log behavior at small enough energies.

Nevertheless, there remains an uncertainty as to whether this divergence can be physically probed in superconducting systems, due to the lack of a known microscopic system that maps onto a class C NL$\sigma$M. If class C is viewed as a projection of class AI onto a gapless sector, then any repulsive interaction would correspond to $\gamma > 0$ in the triplet channel. Consequently, the divergence at $\gamma \to -1$ in class C lacks a clear physical realization.

# 10 Contractions

In this section we present a derivation of (6.1.2). First we consider the structure of the summation in Nambu space:

$$\sum_{k<0,\gamma,i}\langle (w_{i,q})^{\alpha\gamma}_{nk}\left(w^\dagger_{i,q'}\right)^{\gamma\alpha}_{kn}\sum_{\alpha_i,n_i}\mathrm{tr}\left[(w_{q_1})^{\alpha_1\alpha_2}_{n_1n_2}\left(w^\dagger_{q_2}\right)^{\alpha_2\alpha_3}_{n_2n_3}(w_{q_3})^{\alpha_3\alpha_4}_{n_3n_4}\left(w^\dagger_{q_4}\right)^{\alpha_4\alpha_1}_{n_4n_1}\right]\rangle =$$

$$\sum_{k<0,\gamma,i}(w_{i,q})^{\alpha\gamma}_{nk}\left(w^\dagger_{i,q'}\right)^{\gamma\alpha}_{kn}\sum_{\substack{\alpha_i,n_i\\p,r,s,t}}A_{prst}\left[(w_{q_1,p})^{\alpha_1\alpha_2}_{n_1n_2}\left(w^\dagger_{q_2,r}\right)^{\alpha_2\alpha_3}_{n_2n_3}(w_{q_3,s})^{\alpha_3\alpha_4}_{n_3n_4}\left(w^\dagger_{q_4,t}\right)^{\alpha_4\alpha_1}_{n_4n_1}\right]\rangle, \tag{10.1}$$

Where $A$ is a matrix, its components are:

$$A_{\alpha\beta\gamma\mu} = 2\left(\delta_{\alpha\beta}\delta_{\gamma\mu}-\delta_{\alpha\gamma}\delta_{\beta\mu}+\delta_{\alpha\mu}\delta_{\beta\gamma}\right)+4\left(\delta_{\alpha\gamma}\delta_{0\beta}\delta_{0\mu}+\delta_{\beta\mu}\delta_{0\alpha}\delta_{0\gamma}\right)-8\delta_{0\alpha}\delta_{0\beta}\delta_{0\gamma}\delta_{0\mu}+2i\sum_{(\alpha\beta\gamma\mu)}\varepsilon_{0\alpha\beta\gamma}\delta_{0\mu}. \tag{10.2}$$

Now we can look at different contractions. Let us first consider this contraction:

$$\sum_{k<0,\gamma,i}\delta\Big(\sum_i q_i\Big)\langle\underline{(w_{i,q})^{\alpha\gamma}_{nk}}\,\underline{\left(w^\dagger_{i,q'}\right)^{\gamma\alpha}_{kn}}\sum_{\substack{\alpha_i,n_i\\p,r,s,t}}A_{prst}\left[\underline{(w_{p,q_1})^{\alpha_1\alpha_2}_{n_1n_2}}\,\underline{\left(w^\dagger_{r,q_2}\right)^{\alpha_2\alpha_3}_{n_2n_3}}\,\underline{(w_{s,q_3})^{\alpha_3\alpha_4}_{n_3n_4}\left(w^\dagger_{t,q_4}\right)^{\alpha_4\alpha_1}_{n_4n_1}}\right]\rangle =$$

$$=\delta\Big(\sum_i q_i\Big)\sum_{\substack{\alpha_i,n_i,k,\gamma\\p,r,s,t,i}}A_{prst}\langle(w_{i,q})^{\alpha\gamma}_{nk}(w^\dagger_{r,q_2})^{\alpha_2\alpha_3}_{n_2n_3}\rangle\langle(w_{p,q_1})^{\alpha_1\alpha_2}_{n_1n_2}(w^\dagger_{q',i})^{\gamma\alpha}_{kn}\rangle\langle(w_{s,q_3})^{\alpha_3\alpha_4}_{n_3n_4}(w^\dagger_{t,q_4})^{\alpha_4\alpha_1}_{n_4n_1}\rangle = 2(2\pi)^6\delta(q+q_2)\times$$

$$\times\delta(q_1+q')\delta(q_3+q_4)\delta(q_1+q_2)\sum_{\substack{\alpha_i,n_i,k,\gamma\\p,r,s,t,i}}\langle(w_{r,-q_2})^{\alpha\gamma}_{nk}(w^\dagger_{r,q_2})^{\alpha_2\alpha_3}_{n_2n_3}\rangle\langle(w_{r,-q_2})^{\alpha_1\alpha_2}_{n_1n_2}(w^\dagger_{r,q_2})^{\gamma\alpha}_{kn}\rangle\langle(w_{s,q_3})^{\alpha_3\alpha_4}_{n_3n_4}(w^\dagger_{-q_s})^{\alpha_4\alpha_1}_{n_4n_1}\rangle, \tag{10.3}$$

where we used the equality $A_{rrss}=2$. Now, using (3.8), we rewrite the contractions (leaving delta-functions and other prefactors aside):

$$\langle(w_{r,-q_2})^{\alpha\gamma}_{nk}(w^\dagger_{r,q_2})^{\alpha_2\alpha_3}_{n_2n_3}\rangle\langle(w_{r,-q_2})^{\alpha_1\alpha_2}_{n_1n_2}(w^\dagger_{r,q_2})^{\gamma\alpha}_{kn}\rangle\langle(w_{s,q_3})^{\alpha_3\alpha_4}_{n_3n_4}(w^\dagger_{-q_s})^{\alpha_4\alpha_1}_{n_4n_1}\rangle =$$

$$=\frac{8}{g^3}\left[\delta^{\alpha\alpha_3}\delta^{\gamma\alpha_2}\delta_{nn_3}\delta_{kn_2}+\mathrm{v}_r\delta^{\alpha\alpha_2}\delta^{\gamma\alpha_3}\delta_{n,-n_2}\delta_{k,-n_3}-\frac{4\pi T\gamma}{D}(1-\delta_{r,0})\delta^{\alpha\gamma}\delta^{\alpha_2\alpha_3}\delta^{\alpha\alpha_2}\delta_{n-k,n_3-n_2}D^t_{q_1}(i\omega_{n-k})\right]\times$$

$$\left[\delta^{\alpha_1\alpha}\delta^{\alpha_2\gamma}\delta_{n_1n}\delta_{kn_2}+\mathrm{v}_r\delta^{\alpha_1\gamma}\delta^{\alpha_2\alpha}\delta_{n_1,-k}\delta_{n_2,-n}-\frac{4\pi T\gamma}{D}(1-\delta_{r,0})\delta^{\alpha\gamma}\delta^{\alpha_1\alpha_2}\delta^{\alpha\alpha_2}\delta_{n-k,n_1-n_2}D^t_{q_1}(i\omega_{n_1-n_2})\right]\times$$

$$\left[\delta^{\alpha_3\alpha_1}\delta^{\alpha_4\alpha_4}\delta_{n_3n_1}\delta_{n_4n_4}+\mathrm{v}_s\delta^{\alpha_3\alpha_4}\delta^{\alpha_4\alpha_1}\delta_{n_3,-n_4}\delta_{n_4,-n_1}-\frac{4\pi T\gamma}{D}(1-\delta_{s,0})\delta^{\alpha_3\alpha_4}\delta^{\alpha_1\alpha_4}\delta^{\alpha_1\alpha_3}\delta_{n_3-n_4,n_1-n_4}D^t_{q_3}(i\omega_{n_3-n_4})\right]$$

$$\times D_{q_3}(i\omega_{n_3-n_4})D_{q_1}(i\omega_{n-k})D_{q_1}(i\omega_{n_1-n_2}). \tag{10.4}$$

We proceed by summing over the replica indices and $s,r$ since there is no external dependence on them. We start with the latter line:

$$\sum_{s,\alpha_4,n_4}\left[\delta^{\alpha_3\alpha_1}\delta^{\alpha_4\alpha_4}\delta_{n_3n_1}\delta_{n_4n_4}+\mathrm{v}_s\delta^{\alpha_3\alpha_4}\delta^{\alpha_4\alpha_1}\delta_{n_3,-n_4}\delta_{n_4,-n_1}-\frac{4\pi T\gamma}{D}(1-\delta_{s,0})\delta^{\alpha_3\alpha_4}\delta^{\alpha_1\alpha_4}\delta^{\alpha_1\alpha_3}\delta_{n_3-n_4,n_1-n_4}\times\right.$$
$$\left.\times D^t_{q_3}(i\omega_{n_3-n_4})\right]D_{q_3}(i\omega_{n_3-n_4})=\sum_{n_4<0}\left[\mathrm{v}\delta^{\alpha_3\alpha_1}\delta_{n_3,-n_4}\delta_{n_4,-n_1}-\frac{12\pi T\gamma}{D}\delta^{\alpha_1\alpha_3}\delta_{n_3n_1}D^t_{q_3}(i\omega_{n_3-n_4})\right]D_{q_3}(i\omega_{n_3-n_4}). \tag{10.5}$$

Here we used the notation $\mathrm{v}=\sum_i \mathrm{v}_i$. Terms, proportional to v will vanish when we consider the crossover to class A, since $\mathrm{v}_0=-\mathrm{v}_3$, and $i=1,2$ aren't summed over.

We neglect every term, which is proportional to $N_r$. Now we do the same thing just for the first two lines in (10.4):

$$\sum_{r,\gamma,\alpha_2,k}\left[\delta^{\alpha\alpha_3}\delta^{\gamma\alpha_2}\delta_{nn_3}\delta_{kn_2}+\mathrm{v}_r\delta^{\alpha\alpha_2}\delta^{\gamma\alpha_3}\delta_{n,-n_2}\delta_{k,-n_3}-\frac{4\pi T\gamma}{D}(1-\delta_{r,0})\delta^{\alpha\gamma}\delta^{\alpha_2\alpha_3}\delta^{\alpha\alpha_2}\delta_{n-k,n_3-n_2}D^t_{q_1}(i\omega_{n-k})\right]$$
$$\left[\delta^{\alpha_1\alpha}\delta^{\alpha_2\gamma}\delta_{n_1n}\delta_{kn_2}+\mathrm{v}_r\delta^{\alpha_1\gamma}\delta^{\alpha_2\alpha}\delta_{n_1,-k}\delta_{n_2,-n}-\frac{4\pi T\gamma}{D}(1-\delta_{r,0})\delta^{\alpha\gamma}\delta^{\alpha_1\alpha_2}\delta^{\alpha\alpha_2}\delta_{n-k,n_1-n_2}D^t_{q_1}(i\omega_{n_1-n_2})\right]\times$$
$$\times D_{q_1}(i\omega_{n_1-n_2})D_{q_1}(i\omega_{n-k})=\delta^{\alpha_1\alpha_3}\delta_{n_1,n_3}\left[2\mathrm{v}\delta^{\alpha\alpha_1}\delta_{n_1,-n_2}\delta_{n,n_1}D^2_{q_1}(i\omega_{2n})(1-\delta_{n,0})+4\delta_{n,-n_2}D^2_{q_1}(i\omega_{n_1+n})-\right.$$
$$-\frac{24\pi T\gamma}{D}\delta^{\alpha\alpha_1}\delta_{n,n_1}D^t_{q_1}(i\omega_{n_1-n_2})D^2_{q_1}(i\omega_{n_1-n_2})-\frac{24\pi T\gamma}{D}\delta^{\alpha\alpha_1}\delta_{n,-n_2}D^t_{q_1}(i\omega_{n_1+n})D^2_{q_1}(i\omega_{n_1+n})+$$
$$\left.+\frac{48\pi^2T^2\gamma^2}{D^2}\delta^{\alpha\alpha_1}\left(\theta(n_1-n_2-n)\left(D^t_{q_1}\right)^2(i\omega_{n_1-n_2})D^2_{q_1}(i\omega_{n_1-n_2})\right)\right]. \tag{10.6}$$

Now, we can sum over $n_2$ for the terms without external frequencies:

The product of (10.5) and (10.6) is needed to be summed over the rest of the replica indices. Let me mention, that terms in (10.6), which don't comprise $\delta^{\alpha\alpha_1}$ could be eliminated since their contribution will be proportional to $N_r$:

$$\sum_{\alpha_1,\alpha_3}\dots=\delta_{n_1,n_3}\left[\left(\mathrm{v}\delta_{n_4,-n_1}-\frac{12\pi T\gamma}{D}D^t_{q_3}(i\omega_{n_3-n_4})\right)D_{q_3}(i\omega_{n_3-n_4})\times\left\{2\mathrm{v}\delta_{n_1,-n_2}\delta_{n,n_1}(1-\delta_{n,0})D^2_{q_1}(i\omega_{2n})-\right.\right.$$
$$-\frac{24\pi T\gamma}{D}\delta_{n,n_1}D^t_{q_1}(i\omega_{n_1-n_2})D^2_{q_1}(i\omega_{n_1-n_2})-\frac{24\pi T\gamma}{D}\delta_{n,-n_2}D^t_{q_1}(i\omega_{n_1+n})D^2_{q_1}(i\omega_{n_1+n})(1-\delta_{n,0})(1-\delta_{n_1,0})+$$
$$\left.\left.+\frac{48\pi^2T^2\gamma^2}{D^2}\left(\theta(n_1-n_2-n)\left(D^t_{q_1}\right)^2(i\omega_{n_1-n_2})D^2_{q_1}(i\omega_{n_1-n_2})\right)\right\}\right] \tag{10.7}$$

Before gathering the result of this contraction, let us comment on other contractions. There is a different way to contract $w$ matrices, which could not be reduced to the cyclic permutation under the trace:

$$\underline{\langle(w_{-q_2,r})^{\alpha\gamma}_{nk}(w^\dagger_{q_2,r})^{\alpha_2\alpha_3}_{n_2n_3}\rangle}\,\underline{\langle(w_{r,-q_2})^{\alpha_3\alpha_4}_{n_3n_4}(w^\dagger_{r,q_2})^{\gamma\alpha}_{kn}\rangle}\,\underline{\langle(w_{p,q_1})^{\alpha_1\alpha_2}_{n_1n_2}(w^\dagger_{p,-q_1})^{\alpha_4\alpha_1}_{n_4n_1}\rangle}=$$
$$=\underline{\langle(w_{-q_2,r})^{\alpha\gamma}_{nk}\mathrm{v}_r(w^\dagger_{q_2,r})^{\alpha_3\alpha_2}_{-n_3,-n_2}\rangle}\,\underline{\langle\mathrm{v}_r(w_{r,-q_2})^{\alpha_4\alpha_3}_{-n_4,-n_3}(w^\dagger_{r,q_2})^{\gamma\alpha}_{kn}\rangle}\,\underline{\langle(w_{p,q_1})^{\alpha_1\alpha_2}_{n_1n_2}(w^\dagger_{p,-q_1})^{\alpha_4\alpha_1}_{n_4n_1}\rangle}, \tag{10.8}$$

where we used the identity (3.5) twice. If we us the fact that green contraction implies $n_2=n_4$

and $\alpha_2 = \alpha_4 = \alpha_1$, and change the summation variables $\alpha_2 \leftrightarrow \alpha_3$, $n_2 \leftrightarrow -n_3$, we derive, that this contribution is identical to the one calculated above, since the prefactor is unchanged under these transformations. Multiplying (10.5) and (10.6) and summing over the spare indices, and accounting for the cyclic permutations under the trace, one derives (6.1.2).

Since the partition function is normalized to $Z = 1$, only connected diagrams contribute; vacuum bubbles cancel out upon normalization. In evaluating (6.4.1), one must also exclude contractions where two legs of the same three-point vertex are connected. Such contractions correspond to zero energy transfer, while the vertex itself is nonzero only when the energy transfer is finite. In principle, the derivation procedure is the same for all contributions: one chooses one of the contractions, the rest of them are either the same or could be obtained using (3.5) twice.

## 11 Integrals Evaluation

In this section, we introduce the results of integration and discuss ways in which these integrals should be evaluated. Contribution (6.4.1) is different from any other: integration over momenta could not be factorized. It makes calculations a little more complicated, which we discuss below. But first, let us introduce the detailed answer for contributions:

$$
\begin{aligned}
-\frac{1}{2}\,\mathrm{tr}\Big\langle (W^2)^{\alpha\alpha}_{nn} S_0^{(4)}\Big\rangle \to\; & \underset{①}{\frac{8A^2(\epsilon)h^{2\epsilon}}{g^2}} - \underset{②}{\frac{12\epsilon\gamma A^2(\epsilon)h^{2\epsilon}}{(1+\gamma)g^2}} - \underset{②}{\frac{24\ln(1+\gamma)A^2(\epsilon)h^{2\epsilon}}{g^2}} - \underset{③}{\frac{24\ln(1+\gamma)A^2(\epsilon)h^{2\epsilon}}{g^2}} \\
& + \underset{④}{\frac{36(-\gamma+(1+\gamma)\ln(1+\gamma))\epsilon A^2(\epsilon)h^{2\epsilon}}{(1+\gamma)g^2}} + \underset{④}{\frac{72\ln^2(1+\gamma)A^2(\epsilon)h^{2\epsilon}}{g^2}} - \underset{⑤}{\frac{12A^2(\epsilon)h^{2\epsilon}}{\epsilon^2 g^2}}\Big[\ln(1+\gamma) + \\
& + \underset{⑤}{\frac{\epsilon}{4}\ln(1+\gamma)(2\ln 2 - \ln(1+\gamma))}\Big] + \frac{36A^2(\epsilon)h^{2\epsilon}}{g^2}\Big[\underset{⑥}{-\gamma + \ln(1+\gamma)} + \\
& \underset{⑥}{\frac{\epsilon}{4}\cdot\frac{(1+\gamma)\ln^2(1+\gamma)+2(1-\gamma^2)\ln(1+\gamma)-2\gamma(1-\gamma)}{1+\gamma}}\Big] + \underset{⑥}{\frac{36\ln^2(1+\gamma)A^2(\epsilon)h^{2\epsilon}}{\epsilon^2 g^2}} \\
& + \frac{12A^2(\epsilon)h^{2\epsilon}}{2g^2}\Big[\frac{-\gamma+(1+\gamma)\ln(1+\gamma)}{1+\gamma} \underset{⑦}{+} \frac{\epsilon}{4(1+\gamma)}\Big[2\gamma(1-\ln 2) + \\
& \underset{⑦}{\ln(1+\gamma)(-2+2(1+\gamma)\ln 2-(1+\gamma)\ln(1+\gamma))}\Big]\Big] \\
& + \frac{18A^2(\epsilon)h^{2\epsilon}}{g^2}\Big[\frac{\gamma(2+\gamma)-2(1+\gamma)\ln(1+\gamma)}{(1+\gamma)} + \epsilon\cdot\underset{⑧}{\frac{(10-3\gamma)\gamma+2(\gamma^2-5)\ln(1+\gamma)-2(1+\gamma)\ln^2(1+\gamma)}{4(1+\gamma)}}\Big] \\
& - \frac{36A^2(\epsilon)h^{2\epsilon}}{g^2}\ln(1+\gamma)\Big[\underset{⑧}{\frac{-\gamma+(1+\gamma)\ln(1+\gamma)}{1+\gamma}} + \frac{\epsilon}{4}\cdot\frac{2\gamma-(2+\gamma)\ln(1+\gamma)}{1+\gamma}\Big].
\end{aligned} \tag{11.1}
$$

$$
\begin{aligned}
-\frac{1}{2}\,\mathrm{tr}\Big\langle (W^2)^{\alpha\alpha}_{nn} S^{(4)}_{\mathrm{int}}\Big\rangle = {} & \underset{\textcircled{1}}{\frac{12\gamma\epsilon A^2(\epsilon)h^{2\epsilon}}{g^2}} + \underset{\textcircled{2}}{\frac{12\gamma A^2(\epsilon)h^{2\epsilon}}{g^2}} + \underset{\textcircled{3}}{\frac{6\gamma A^2(\epsilon)h^{2\epsilon}}{g^2}\Big(1+\frac{\epsilon}{2}\ln 2\Big)} \\
& \underset{\textcircled{4}}{-\frac{48A^2(\epsilon)h^{2\epsilon}}{g^2}\Big(\gamma-\ln(1+\gamma)+\frac{\epsilon}{4}\left(2\ln(1+\gamma)-2\gamma+\ln^2(1+\gamma)\right)\Big)} \\
& \underset{\textcircled{5}}{+\frac{72A^2(\epsilon)h^{2\epsilon}}{g^2}\Big(\gamma-\ln(1+\gamma)+\frac{\epsilon}{4}\left(2\gamma-2(1+2\gamma)\ln(1+\gamma)+\ln^2(1+\gamma)\right)\Big)} \\
& \underset{\textcircled{6}}{-\frac{12A^2(\epsilon)h^{2\epsilon}}{g^2}\Big[\gamma-\ln(1+\gamma)+\frac{\epsilon}{4}\left(-2\gamma+2\ln(1+\gamma)+4\ln 2(\gamma-\ln(1+\gamma))+\ln^2(1+\gamma)\right)\Big]} \\
& \underset{\textcircled{7}+\textcircled{14}}{+\frac{24A^2(\epsilon)h^{2\epsilon}}{g^2(1+\gamma)}\Big[\gamma(2+\gamma)-2(1+\gamma)\ln(1+\gamma)+\frac{\epsilon}{4}\left(-\gamma(2+\gamma)+2(1+\gamma)^2\ln(1+\gamma)-2(1+\gamma)\ln^2(1+\gamma)\right)\Big]} \\
& \underset{\textcircled{8}}{-\frac{36A^2(\epsilon)h^{2\epsilon}}{g^2(1+\gamma)}\Big[\gamma(2+\gamma)-2(1+\gamma)\ln(1+\gamma)+\frac{\epsilon}{2}\Big((2+3\gamma)\gamma-(2+5\gamma+2\gamma^2)\ln(1+\gamma)+(1+\gamma)\ln^2(1+\gamma)\Big)\Big]} \\
& \underset{\textcircled{9}}{+\frac{6A^2(\epsilon)h^{2\epsilon}}{g^2(1+\gamma)}\Big[\gamma(2+\gamma)-2(1+\gamma)\ln(1+\gamma)+\frac{\epsilon}{4}\Big(-5\gamma(2+\gamma)+8\gamma(2+\gamma)\ln 2+(10+8\gamma-16(1+\gamma)\ln 2)\ln(1+\gamma)+} \\
& \underset{\textcircled{9}}{+2(1+\gamma)\ln^2(1+\gamma)\Big)\Big]} \underset{\textcircled{10}}{-\frac{18\gamma A^2(\epsilon)h^{2\epsilon}}{g^2}\Big[\ln(1+\gamma)-\frac{\epsilon}{4}\ln^2(1+\gamma)\Big]} \underset{\textcircled{11}}{-\frac{48\gamma A^2(\epsilon)h^{2\epsilon}}{g^2}\Big[\ln(1+\gamma)-\frac{\epsilon}{4}\ln^2(1+\gamma)\Big]} \\
& \underset{\textcircled{12}}{-\frac{24\epsilon A^2(\epsilon)h^{2\epsilon}}{g^2}\Big[\gamma-\ln(1+\gamma)\Big]\ln(1+\gamma)} \underset{\textcircled{13}+\textcircled{15}}{+\frac{48\gamma^2A^2(\epsilon)h^{2\epsilon}}{g^2}\Big[\frac{\ln(1+\gamma)}{1+\gamma}+\frac{\epsilon}{4}\cdot\frac{(2+\gamma)\ln^2(1+\gamma)}{\gamma(1+\gamma)}\Big]}. \qquad (11.2)
\end{aligned}
$$

$$
-\frac{1}{8}\,\mathrm{tr}\Big\langle (W^4)^{\alpha\alpha}_{n_1n_1}\Big\rangle = -\frac{4A^2(\epsilon)h^{2\epsilon}}{g^2}\Big[2-\frac{21}{2}\ln(1+\gamma)+\frac{45}{4}\ln^2(1+\gamma)-\frac{3\epsilon}{8}\ln(1+\gamma)\left(2\ln 2-\ln(1+\gamma)\right)\Big]. \qquad (11.3)
$$

The aforementioned contributions allow for the systematic analytical evaluation. Let us consider the fifth contribution in (6.2.3).

$$
\int_0^\infty dy \int_q D^t_q(y) D^2_{q_3} \int_y^\infty dx \int_p D_p(x) \qquad (11.4)
$$

One can notice that the integrals over momentum, as was mentioned above, could be factorized. Therefore, we perform this integration first, and then proceed with frequency integration. One of the useful identities we use in many of these contributions is:

$$
\frac{1}{A(\epsilon)}\frac{\partial^2}{\partial\alpha^2}\left[(\alpha-1)^2\int_{\boldsymbol{p}}\frac{1}{(p^2+1+x)^2(p^2+1+\alpha x)}\right] = \frac{\epsilon(\epsilon-2)}{4}(1+\alpha x)^{\epsilon/2-2}. \qquad (11.5)
$$

Similar identities hold for the product of any number of $D$ and $D^t$ on the same momentum and frequency. These identities are equivalent to using the Feynman parametrization trick we discuss below, when considering $S^{(3)}_{\text{int}}$ terms. The idea is to simplify the result of integration to a power of $1+\alpha x$, then use the equation:

$$\int_0^\infty dx(1+ax)^\alpha(1+bx)^\beta = -\frac{a^{-1-\beta}\left(1-\frac{a}{b}\right)^{1+\alpha+\beta} b^\beta \pi \,\mathrm{Csc}(\pi\beta)\Gamma(-1-\alpha-\beta)}{\Gamma(-\alpha)\Gamma(-\beta)} - \frac{{}_2F_1\left(1,-\alpha,2+\beta,\frac{a}{b}\right)}{b+b\beta} \tag{11.6}$$

The result of the integration over frequency can then be expanded in $\epsilon$ to the needed order, after which integration over additional variables, such as $\alpha$ in (11.5), can be performed. In this specific example we evaluate first:

$$\int_0^\infty dx(1+\alpha x)^{\epsilon/2-2}(1+x)^{\epsilon/2+1} = -\frac{1}{\alpha^2\epsilon} + \frac{-2+2\alpha+\ln(\alpha)}{2\alpha^2}. \tag{11.7}$$

Then one should perform the integration over $\alpha$ twice:

$$\int_1^{1+\gamma} d\alpha' \int_1^{\alpha'} d\alpha \left(-\frac{1}{\alpha^2\epsilon}+\frac{-2+2\alpha+\ln(\alpha)}{2\alpha^2}\right) = \frac{-2\gamma(2+3\epsilon)+\ln(1+\gamma)(4+(6+4\gamma)\epsilon-\epsilon\ln(1+\gamma))}{4\epsilon}. \tag{11.8}$$

Restoring the prefactors and reexpanding the answer in powers of $\epsilon$, one obtains the result from (11.2). The same procedure works for every integral in the aforementioned contributions.

Now we proceed to contributions with a less straightforward way to evaluate them.

$$
\begin{aligned}
S^{(3)}_{\text{int},1} = {} & \frac{12\gamma^2 A^2(\epsilon)h^{2\epsilon}}{g^2}\Big[\frac{-2\gamma+(2+\gamma)\ln(1+\gamma)}{\gamma^3} - \frac{\epsilon}{1+\gamma} \underset{\textcircled{1}}{+} \frac{\epsilon(2+\gamma)}{\gamma^3}\Big(\mathrm{Li}_2(-\gamma)+\ln(1+\gamma)+\frac{1}{4}\ln^2(1+\gamma)\Big)\Big] - \\
& - \underset{\textcircled{2}}{\frac{6\gamma^2 A^2(\epsilon)h^{2\epsilon}}{g^2}}\Big[\frac{\gamma-\ln(1+\gamma)}{\gamma^2} - \frac{\epsilon}{\gamma^2}\Big(\mathrm{Li}_2(-\gamma)+\frac{1}{2}(2+\gamma)\ln(1+\gamma)+\frac{1}{4}\ln^2(1+\gamma)\Big)\Big] + \\
& + \frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\big(\underset{\textcircled{3}}{\epsilon R_5(\gamma)} + \underset{\textcircled{4}+\textcircled{12}}{\epsilon E_5(\gamma)}\big) - \underset{\textcircled{5}+\textcircled{8}}{\frac{18\gamma^2 A^2(\epsilon)h^{2\epsilon}}{g^2}}\Big\{2\frac{\gamma^3+(\gamma^2+3\gamma+2)\ln^2(\gamma+1)-(3\gamma+2)\gamma\ln(\gamma+1)}{\gamma^3(\gamma+1)} \\
& \qquad - \frac{\epsilon}{\gamma^3(1+\gamma)}\Big[\gamma^2(2+3\gamma)\ln(1+\gamma) - \underset{\textcircled{5}+\textcircled{8}}{2(1+\gamma)(2+\gamma)\ln^2(1+\gamma)} + \\
& [\gamma^2-(1+\gamma)(2+\gamma)\ln(1+\gamma)]\underset{\textcircled{5}+\textcircled{8}}{[2\,\mathrm{Li}_2(-\gamma)+\frac{1}{2}\ln^2(1+\gamma)]}\Big]\Big\} + \frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\big(\underset{\textcircled{5}+\textcircled{8}}{\epsilon T_2(\gamma)} + \underset{\textcircled{6}}{\epsilon R_6(\gamma)} + \underset{\textcircled{7}+\textcircled{14}}{\epsilon E_7(\gamma)} + \underset{\textcircled{9}}{\epsilon R_7(\gamma)}\big) \\
& + \underset{\textcircled{10}}{\frac{36\gamma^2 A^2(\epsilon)h^{2\epsilon}}{g^2}}\Big\{-\frac{2}{\gamma}\ln(1+\gamma)\frac{\ln(1+\gamma)-\gamma}{\gamma} + \epsilon\frac{\ln(1+\gamma)}{\gamma^2}\Big[-(2+\gamma)\ln(1+\gamma) - \mathrm{Li}_2(-\gamma) + \mathrm{Li}_2\left(\frac{\gamma}{1+\gamma}\right)\Big]\Big\} \\
& + \frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\underset{\textcircled{10}}{\epsilon T_3(\gamma)} + \underset{\textcircled{11}+\textcircled{13}+\textcircled{15}}{\frac{96\gamma^3 A^2(\epsilon)h^{2\epsilon}}{g^2}}\Big\{2\frac{-\gamma+(1+\gamma)\ln(1+\gamma)}{\gamma^2(1+\gamma)^2} + \frac{\epsilon}{\gamma^2(1+\gamma)}\Big[\frac{2\gamma}{1+\gamma} + 2\,\mathrm{Li}_2(-\gamma) + \frac{1}{2}\ln^2(1+\gamma)\Big]\Big\} \\
& \qquad + \frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\underset{\textcircled{11}+\textcircled{13}+\textcircled{15}}{\epsilon E_6(\gamma)}\ .
\end{aligned} \tag{11.9}
$$

In the main text, we neglected the expressions for $E$ contributions due to their irrelevance in the final expression. Here we present the explicit expressions for them:

$$
\begin{aligned}
E_5(\gamma) &= -24\gamma\int_1^{1+\gamma} dz(1+\gamma-z)\int[]\frac{u_3^2}{(u_iu_j)(u_1(1+\gamma)+u_3z)(u_2(1+\gamma)+u_3z)^2} \\
E_6(\gamma) &= 24\gamma^2\int_1^{1+\gamma} dz\int[]\frac{u_3^2 z}{(u_iu_j)(1+\gamma)(u_1(1+\gamma)+u_3z)(u_2(1+\gamma)+u_3z)^2} \\
E_7(\gamma) &= 24\gamma\int_1^{1+\gamma} dz(1+\gamma-z)\int[]\frac{u_3^2(z-1)}{(u_iu_j)(u_1(1+\gamma)+u_3z)^2(u_2(1+\gamma)+u_3z)^2}.
\end{aligned} \tag{11.10}
$$

$$
\begin{aligned}
S^{(3)}_{\text{int},2} = {} & -\frac{6\gamma^2 A^2(\epsilon)\epsilon^2 h^{2\epsilon}}{g^2}\Big[-\frac{2}{\epsilon}\frac{\ln(1+\gamma)}{\gamma}\Big[\underset{\textcircled{1}}{\frac{\ln(1+\gamma)}{\gamma}} + \frac{\epsilon}{\gamma}\Big(\mathrm{Li}_2(-\gamma)+\frac{1}{4}\ln^2(1+\gamma)\Big)\Big]\Big] + \frac{A^2(\epsilon)h^{2\epsilon}}{g^2}\underset{\textcircled{1}}{\epsilon T_1(\gamma)} \\
& + \frac{6\gamma^2 A^2(\epsilon)\epsilon^2 h^{2\epsilon}}{g^2}\Big[-\frac{2}{\epsilon}\Big[\frac{\ln(1+\gamma)}{\gamma} - \frac{2}{1+\gamma}\Big]\underset{\textcircled{2}+\textcircled{4}}{\Big[}\frac{\ln(1+\gamma)}{\gamma} + \frac{\epsilon}{\gamma}\Big(\mathrm{Li}_2(-\gamma)+\frac{1}{4}\ln^2(1+\gamma)\Big)\Big] + \frac{\epsilon A^2(\epsilon)h^{2\epsilon}}{g^2}\big(R_3(\gamma)+\underset{\textcircled{2}+\textcircled{4}}{R_4(\gamma)}\big) \\
& + \frac{\epsilon A^2(\epsilon)h^{2\epsilon}}{g^2}\big(\underset{\textcircled{3}}{R_8(\gamma)} + \underset{\textcircled{5}}{E_4(\gamma)}\big) + \frac{48\gamma^3 A^2(\epsilon)\epsilon^2 h^{2\epsilon}}{g^2(1+\gamma)}\Big[-\frac{2}{\epsilon\gamma^2}\Big[\ln(1+\gamma) + \epsilon\underset{\textcircled{5}}{\Big[}\mathrm{Li}_2(-\gamma) + \frac{1}{4}\ln^2(1+\gamma)\Big]\Big] + \frac{2(1-\epsilon)}{\epsilon\gamma(1+\gamma)}\Big].
\end{aligned} \tag{11.11}
$$

Here we define $E_4$:

$$E_4(\gamma) = -48\gamma^2 \int_1^{1+\gamma} dz \int [] \frac{u_3^2 z}{(u_i u_j)(1+\gamma)(u_1(1+\gamma)+u_3 z)(u_2(1+\gamma)+u_3 z)^2}. \tag{11.12}$$

$$\begin{aligned}
S^{(3)}_{\text{int},3} = \frac{3\gamma^2 A^2(\epsilon)\epsilon^2 h^{2\epsilon}}{g^2\epsilon(1+\gamma)} \Big[ & -\frac{2}{\epsilon}\Big(\frac{6\ln(1+\gamma)}{\gamma}-5\Big) + \underset{①+③}{\frac{6\epsilon}{\gamma}}\Big(\mathrm{Li}_2(-\gamma)+\frac{1}{4}\ln^2(1+\gamma)\Big)+5\epsilon- \\
& -\frac{6}{1+2\gamma}\Big(\frac{\pi^2}{6}-\ln^2(1+\gamma)-2\,\mathrm{Li}_2(-\gamma)\Big)+\underset{①+③}{\frac{5}{1+\gamma}}\Big(\frac{\pi^2}{6}-\mathrm{Li}_2(-\gamma)\Big)\Big]+ \\
& +\underset{②}{\frac{3\gamma^2 A^2(\epsilon)\epsilon^2 h^{2\epsilon}}{g^2\epsilon(1+\gamma)}\Big[-\frac{2(1-\epsilon)}{\epsilon}\Big]}+\epsilon R_1(\gamma)- \\
& -\underset{④}{\frac{6\gamma^3 A^2(\epsilon)\epsilon^2 h^{2\epsilon}}{g^2\epsilon(1+\gamma)}\Big[-\frac{2}{\epsilon}\frac{\gamma-\ln(1+\gamma)}{\gamma^2}+\frac{2}{\gamma^2}\Big(\mathrm{Li}_2(-\gamma)+\frac{1}{4}\ln^2(1+\gamma)+\gamma\Big)\Big]}+\epsilon R_4(\gamma)- \\
& -\underset{⑤}{\frac{24\gamma^3 A^2(\epsilon)\epsilon^2 h^{2\epsilon}}{g^2\epsilon(1+\gamma)}\Big[-\frac{2}{\epsilon\gamma^2}\ln(1+\gamma)+\epsilon\Big(\mathrm{Li}_2(-\gamma)+\frac{1}{4}\ln^2(1+\gamma)\Big)+\frac{2(1-\epsilon)}{\epsilon\gamma(1+\gamma)}\Big]}+\epsilon E_3(\gamma).
\end{aligned} \tag{11.13}$$

$E_3$ is given by the expression:

$$E_3(\gamma) = 24\gamma^2 \int_1^{1+\gamma} dz \int [] \frac{u_3^2 z}{(u_i u_j)(1+\gamma)(u_1(1+\gamma)+u_3 z)(u_2(1+\gamma)+u_3 z)^2}. \tag{11.14}$$

$$S^{(3)}_{\text{int},4} = -\underset{①}{\frac{3\gamma^2 A^2(\epsilon)\epsilon^2 h^{2\epsilon}}{4g^2(1+2\gamma)}\Big(\frac{\pi^2}{6}-\ln^2(1+\gamma)-2\,\mathrm{Li}_2(-\gamma)\Big)}+\underset{②+③}{2\epsilon E_1(\gamma)}+\underset{④}{\epsilon E_2(\gamma)}. \tag{11.15}$$

We conclude a series of definitions with $E_1$ and $E_2$:

$$\begin{aligned}
E_1(\gamma) &= 12\gamma \int_1^{1+\gamma} dz(1+\gamma-z) \int [] \frac{u_3^2}{(u_i u_j)(u_1(1+\gamma)+u_3 z)(u_2(1+\gamma)+u_3 z)^2} \\
E_2(\gamma) &= -24\gamma \int_1^{1+\gamma} dz(1+\gamma-z) \int [] \frac{u_3^2(z-1)}{(u_i u_j)(u_1(1+\gamma)+u_3 z)^2(u_2(1+\gamma)+u_3 z)^2}.
\end{aligned} \tag{11.16}$$

The method of calculating these contributions requires Feynman parametrization, which is how we deal with non-factorizable momenta integration. We consider a contribution number $5+8$ in (6.4.4). This contribution is simplified by the relation similar to (6.4.6):

$$2-\gamma x D^t(x) = D^{-1}(x)\big(D(x)+D^t(x)\big). \tag{11.17}$$

The Feynman parametrization simplifies integration by introducing additional integrations:

$$\frac{1}{A_1^{\alpha_1}\cdots A_n^{\alpha_n}} = \frac{\Gamma(\alpha_1+\cdots+\alpha_n)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}\int_0^1 du_1\cdots\int_0^1 du_n\, \frac{\delta\left(1-\sum_{k=1}^n u_k\right)\, u_1^{\alpha_1-1}\cdots u_n^{\alpha_n-1}}{\left(\sum_{k=1}^n u_k A_k\right)^{\sum_{k=1}^n \alpha_k}} \tag{11.18}$$

When working with the loops we have here, one first simplifies the products of propagators on the same momentum, using (11.18). Here, for example, we use:

$$D_p(x)D_p^t(x)\big(D_p(x)+D_p^t(x)\big) = \int \frac{dz}{\gamma}\frac{2}{(p^2+h^2+xz)^3}. \tag{11.19}$$

The next step is to introduce a global Feynman parametrization with three parameters:

$$\begin{aligned} &\frac{1}{(\boldsymbol{p}+\boldsymbol{q})^2+1+(1+\gamma)y}\frac{1}{(p^2+1+x)^3}\frac{1}{q^2+1+x+y} = \\ &= \frac{\Gamma(5)}{\Gamma(2)}\int[]\frac{u_2^2}{(u_1((\boldsymbol{p}+\boldsymbol{q})^2+1+(1+\gamma)y)+u_2(p^2+1+x)+u_3(q^2+1+x+y))^5}, \end{aligned} \tag{11.20}$$

where we used the notation $\int[] = \left(\prod_i \int_0^1 du_i\right)\delta(1-\sum_i u_i)$ and made the integration dimensionless by factoring out $h$. The integration over momentum would always be the same: we start with the momentum shift $\boldsymbol{p}\to\boldsymbol{p}-\boldsymbol{q}\frac{u_3}{u_1+u_3}$, and then we perform the integration:

$$\int_{\boldsymbol{p},\boldsymbol{q}}\frac{1}{(ap^2+bq^2+c)^\alpha} = \frac{A^2(\epsilon)\epsilon^2}{4\Gamma^2(1-\epsilon/2)}\frac{\Gamma(\alpha-2-\epsilon)}{\Gamma(\alpha)}\frac{1}{(ab)^{1+\epsilon/2}c^{\alpha-2-\epsilon}}. \tag{11.21}$$

Now we are ready to perform integration over frequencies, which can be done exactly. What we are left with is an integral over Feynman variables. In this example we have an integral:

$$\frac{1}{\gamma}\int_1^{1+\gamma} dz\int[]\frac{u_2^2}{(u_iu_j)^{1+\epsilon/2}(u_1+u_2z)(u_1+u_3(1+\gamma))^2}, \tag{11.22}$$

where $u_iu_j = u_1u_2+u_2u_3+u_1u_3$. There is also a numerical prefactor, which is always the same in $\epsilon$ expansion and is equal to $-\epsilon A^2(\epsilon)/4$, and higher-order terms are irrelevant to the final answer. We use a change of variables, that satisfies the condition under $\delta$-function:

$$u_1 = \frac{u}{1+s},\ u_2 = \frac{s}{1+s},\ u_3 = \frac{1-u}{1+s}, \quad J = \frac{1}{(1+s)^3}, \quad u\in[0,1], \quad s\in[0,\infty]. \tag{11.23}$$

We can arbitrarily permute $u_i$ in this substitution for the convenience of integration. Under that substitution:

$$u_iu_j = \frac{s+u(1-u)}{(1+s)^2}. \tag{11.24}$$

But how does one identify the contribution, which corresponds to the second-order pole in $\epsilon$? We propose this method: first, one chooses a configuration of $u_i$ for the substitution (11.23), that is the most divergent as $s\to\infty$. This divergence is regularized by $\epsilon$, therefore, the integral will have a pole in $\epsilon$. In this example, the parametrization coincides with (11.23). Since the integrand is a rational function, we can explicitly extract the most divergent term:

$$\frac{u_2^2}{(u_1+u_2z)} = \frac{u_2}{z} - \frac{u_1}{z^2} + \frac{u_1^2}{z^2(u_1+u_2z)}. \qquad (11.25)$$

The first and the second functions on the right-hand side produce poles in $\epsilon$ and the integral amounts to the hypergeometric function, which could be expanded in $\epsilon$ up to the necessary order, like we did in (6.2.3). The integral with the third fraction is convergent for $\epsilon = 0$. It is a universal statement that for the integrals we are to calculate, this procedure always produces the correct pole terms, and the residue is always convergent. In principle, the integral over $u$ could be divergent, but it is not the case because it is regularized by $1+\gamma > 0$.

The capital-letter-denoted contributions in the main text are nothing but integrals with residues of these expansions. Their analysis is not as straightforward as it was for $\epsilon^{-2}$ contributions. Nevertheless, we obtained their asymptotics, as $\gamma \to -1$, using both numerical and analytical methods. Below, we present the expressions for them:

$$
\begin{aligned}
R_1(\gamma) &= \left(\frac{3\left(-1+\ln(1+\gamma)-\ln 4\right)}{1+\gamma} + \frac{1}{4}\left(15+4\pi^2+48\ln 2-24\ln 4-6\ln(1+\gamma)-12\ln 4\ln(1+\gamma)\right)\right), \\
R_2(\gamma) &= \left(\frac{1}{1+\gamma}\left(6-\pi^2+12\ln 2+6\ln^2 2-6\ln 2\ln 4-6\ln(1+\gamma)+6\ln 4\ln(1+\gamma)-6\ln^2(1+\gamma)\right)\right. \\
&\quad +\frac{1}{2}\left(-12+\pi^2-48\ln 2-6\ln^2 2+6\ln 4+12\ln 2\ln 4+12\ln(1+\gamma)-12\ln 2\ln(1+\gamma)\right. \\
&\quad \left.\left.-12\ln 4\ln(1+\gamma)+12\ln^2(1+\gamma)\right)+2.28-1.72\ln(1+\gamma)-1.84\ln^2(1+\gamma)\right), \\
R_3(\gamma) &= \left(3\ln^3(1+\gamma)+8.10+22.6\ln(1+\gamma)-4.158\ln^2(1+\gamma)\right), \\
R_4(\gamma) &= \left(\frac{3\left(4\ln^2(1+\gamma)-4\ln 2\ln(1+\gamma)+2\ln^2 2+\pi^2\right)}{1+\gamma}\right. \\
&\quad +\left(-24-2\pi^2-12\ln 2+6\ln^2 2-3\ln^2 4+24\ln(1+\gamma)-12\ln 2\ln(1+\gamma)\right. \\
&\quad \left.\left.+12\ln 4\ln(1+\gamma)-6\ln^2(1+\gamma)\right)\right), \\
R_5(\gamma) &= \left(-1.30+0.92\ln(1+\gamma)\right), \\
R_6(\gamma) &= \left(+9.5+9.04\ln(1+\gamma)+1.84\ln^2(1+\gamma)\right), \\
R_7(\gamma)+R_8(\gamma) &= \left(\frac{3(\ln(1+\gamma)-1)}{1+\gamma} + \# + \#\ln(1+\gamma) + \#\ln^2(1+\gamma)\right), \\
T_1(\gamma) &= \left(-26.00-22.50\ln(1+\gamma)-1.0\ln^3(1+\gamma)\right), \\
T_2(\gamma) &= \left(\frac{3(6\ln^2(1+\gamma)+12\ln(1+\gamma)+6+\pi^2)}{1+\gamma}\right. \\
&\quad \left.-3\left(24+2\pi^2-6\ln(1+\gamma)+2\pi^2\ln(1+\gamma)+6\ln^2(1+\gamma)+3\ln^3(1+\gamma)+6\zeta(3)\right)\right), \\
T_3(\gamma) &= \left(53.70+33.79\ln(1+\gamma)+1.50\ln^3(1+\gamma)\right). \qquad (11.26)
\end{aligned}
$$

This result has several notable features. First, we are able to analytically obtain terms that diverge as fast or faster than $(1+\gamma)^{-1}$, which allows us to exactly show that in (8.5) there is no such divergence. As for terms that diverge as $\ln^3(1+\gamma)$, which is of the most significant interest due to its relevance for the ZBA correction, we are able to obtain these constants numerically with high precision. That allows us to conclude their presence in the RG equation (7.12). Coefficients with a decimal point were obtained numerically, in contribution $R_7 + R_8$, the accuracy is insufficient to determine numbers. Our next goal is to find an analytical way to extract these logarithmic divergences, but that is the subject of future work.

## 12 Conclusion

Using perturbation theory within the framework of the class C Finkelstein NL$\sigma$M model, we computed the two-loop correction to the disorder-averaged LDoS. We derived a general expression for this correction valid for arbitrary energies and temperatures and subsequently analyzed the asymptotic behavior in the limits $E \to 0$ and $T \to 0$ . An analytical form of the correction was obtained in $\epsilon$-regularization, and numerical evaluations were performed in the limit, $\gamma \to -1$, which corresponds to the Coulomb interaction in class A. While this limit has no direct physical realization in class C—where repulsive interactions imply $\gamma > 0$—it serves as a useful benchmark for comparing the structure of divergences. Using a minimal subtraction scheme, we derived RG equations for LDoS.

The presence of terms proportional to $\ln^3(1+\gamma)$ indicates a parametric region of energies, defined by the inequality (9.3), where the two-loop correction dominates over the previously established one-loop result. Consequently, the asymptotic behavior of the LDoS at low energies is altered, leading to a correction of the previously known double-log-squared dependence.

Some asymptotic expressions were obtained numerically using the least squares method. Future research could involve developing an analytical approach, possibly employing a Mellin-Barnes transform, to systematically capture logarithmically divergent terms.

Furthermore, as demonstrated in [25], introducing a Zeeman term in the action induces a crossover from class C to class A, where systems exhibit the quantum Hall effect. The magnetic field makes certain modes in Nambu space massive, thus constraining the $W$-matrix structure primarily to components proportional to $\mathrm{s}_0$ and $\mathrm{s}_3$. This simplification allows for a straightforward calculation of the ZBA correction by slightly adjusting the summation over Nambu space indices. Notably, similar divergent terms proportional to $\ln^3(1+\gamma)$ could emerge in class A computations, indicating qualitative parallels between the corrections in both symmetry classes. A detailed investigation of these parallels is necessary and will be addressed in future work.

# References


[1] Alexander Altland and Martin R. Zirnbauer. Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures. Physical Review B, 55:1142–1161, 1996.

[2] G E Volovik and V M Yakovenko. Fractional charge, spin and statistics of solitons in superfluid $^3$He film. J. Phys.: Condens. Matter, 1(31):5263, aug 1989.

[3] G.E. Volovik. On edge states in superconductors with time inversion symmetry breaking. JETP Lett., 66:522, 1997.

[4] Ilya A. Gruzberg, Andreas W. W. Ludwig, and N. Read. Exact exponents for the spin quantum Hall transition. Phys. Rev. Lett., 82:4524, 1999.

[5] V. Kagalovsky, B. Horovitz, Y. Avishai, and J. T. Chalker. Quantum Hall plateau transitions in disordered superconductors. Phys. Rev. Lett., 82:3516, 1999.

[6] T. Senthil, J. B. Marston, and Matthew P. A. Fisher. Spin quantum Hall effect in unconventional superconductors. Phys. Rev. B, 60:4245, 1999.

[7] Oguzhan Can, Tarun Tummuru, Ryan P. Day, Ilya Elfimov, Andrea Damascelli, and Marcel Franz. High-temperature topological superconductivity in twisted double-layer copper oxides. Nature Physics, 17(4):519–524, Apr 2021.

[8] Heng Wang, Yuying Zhu, Zhonghua Bai, Zechao Wang, Shuxu Hu, Hong-Yi Xie, Xiaopeng Hu, Jian Cui, Miaoling Huang, Jianhao Chen, Ying Ding, Lin Zhao, Xinyan Li, Qinghua Zhang, Lin Gu, X. J. Zhou, Jing Zhu, Ding Zhang, and Qi-Kun Xue. Prominent josephson tunneling between twisted single copper oxide planes of bi2sr2-xlaxcuo6+y. Nature Communications, 14(1):5201, Aug 2023.

[9] S. Y. Frank Zhao, Xiaomeng Cui, Pavel A. Volkov, Hyobin Yoo, Sangmin Lee, Jules A. Gardener, Austin J. Akey, Rebecca Engelke, Yuval Ronen, Ruidan Zhong, Genda Gu, Stephan Plugge, Tarun Tummuru, Miyoung Kim, Marcel Franz, Jedediah H. Pixley, Nicola Poccia, and Philip Kim. Time-reversal symmetry breaking superconductivity between twisted cuprate superconductors. Science, 382(6677):1422–1427, 2023.

[10] Mickey Martini, Yejin Lee, Tommaso Confalone, Sanaz Shokri, Christian N. Saggau, Daniel Wolf, Genda Gu, Kenji Watanabe, Takashi Taniguchi, Domenico Montemurro, Valerii M. Vinokur, Kornelius Nielsch, and Nicola Poccia. Twisted cuprate van der waals heterostructures with controlled josephson coupling. Materials Today, 67:106–112, 2023.

[11] Pavel A. Volkov, Étienne Lantagne-Hurtubise, Tarun Tummuru, Stephan Plugge, J. H. Pixley, and Marcel Franz. Josephson diode effects in twisted nodal superconductors. Phys. Rev. B, 109:094518, Mar 2024.

[12] Patrick A. Lee. Localized states in a d-wave superconductor. Phys. Rev. Lett., 71:1887–1890, Sep 1993.

[13] J. M. ROWELL, W. L. McMillan, and W. L. FELDMANN. Phonon spectra in pb and $pb_{40}tl_{60}$ determined by tunneling and neutron scattering. Phys. Rev., 178:897–899, Feb 1969.

[14] Stuart Bermon and C. K. So. Determination of the kondo scattering amplitude and coherence distance from tunneling in doped-electrode junctions. Phys. Rev. Lett., 40:53–56, Jan 1978.

[15] R. C. Dynes and J. P. Garno. Metal-insulator transition in granular aluminum. Phys. Rev. Lett., 46:137–140, Jan 1981.

[16] A. Mottaghizadeh, Q. Yu, P. L. Lang, A. Zimmers, and H. Aubin. Metal oxide resistive switching: Evolution of the density of states across the metal-insulator transition. Phys. Rev. Lett., 112:066803, Feb 2014.

[17] W. L. McMillan and Jack Mochel. Electron tunneling experiments on amorphous $ge_{1-x}au_x$. Phys. Rev. Lett., 46:556–557, Feb 1981.

[18] B.L. Altshuler and A.G. Aronov. Zero bias anomaly in tunnel resistance and electron-electron interaction. Solid State Communications, 30(3):115–117, 1979.

[19] A L Efros and B I Shklovskii. Coulomb gap and low temperature conductivity of disordered systems. Journal of Physics C: Solid State Physics, 8(4):L49, feb 1975.

[20] B. L. Altshuler, A. G. Aronov, and P. A. Lee. Interaction effects in disordered fermi systems in two dimensions. Phys. Rev. Lett., 44:1288–1291, May 1980.

[21] S. Levitov and A. V. Shytov. Erratum: Semiclassical theory of the coulomb anomaly [jetp lett. 66, no. 3, 214–221 (10 august 1997)]. Journal of Experimental and Theoretical Physics Letters, 66(7):546–546, Oct 1997.

[22] A.M. Finkel'stein. Influence of coulomb interaction on the properties of disordered metals. Journal of Experimental and Theoretical Physics Letters, 57(1):168, Jan 1983.

[23] Yu.V. Nazarov. Anomalous current-voltage characteristics of tunnel junctions. Sov. Phys. JETP, 68(3):561–566, 1989.

[24] Alex Kamenev and Anton Andreev. Electron-electron interactions in disordered metals: Keldysh formalism. Phys. Rev. B, 60:2218–2238, Jul 1999.

[25] Maksim Parfenov and Igor Burmistrov. Bulk-edge correspondence at the spin-to-integer quantum hall effect crossover in topological superconductors, 2025.

[26] P. W. Anderson. Absence of diffusion in certain random lattices. Phys. Rev., 109:1492–1505, Mar 1958.

[27] Daniel Höf and Franz Wegner. Calculation of anomalous dimensions for the nonlinear sigma model. Nuclear Physics B, 275(4):561–579, 1986.

[28] E. J. Beamond, John Cardy, and J. T. Chalker. Quantum and classical localization, the spin quantum Hall effect, and generalizations. Phys. Rev. B, 65:214301, 2002.

[29] F. Evers, A. Mildenberger, and A. D. Mirlin. Multifractality at the spin quantum Hall transition. Phys. Rev. B, 67:041303, 2003.

[30] A. D. Mirlin, F. Evers, and A. Mildenberger. Wavefunction statistics and multifractality at the spin quantum Hall transition. J. Phys. A: Math. and Gen., 36:3255, 2003.

[31] A. R. Subramaniam, I. A. Gruzberg, and A. W. W. Ludwig. Boundary criticality and multifractality at the two-dimensional spin quantum Hall transition. Phys. Rev. B, 78:245105, 2008.

[32] Jonas F. Karcher, Ilya A. Gruzberg, and Alexander D. Mirlin. Generalized multifractality at the spin quantum Hall transition: Percolation mapping and pure-scaling observables. Phys. Rev. B, 105:184205, 2022.

[33] S. S. Babkin and I. S. Burmistrov. Generalized multifractality in the spin quantum hall symmetry class with interaction. Phys. Rev. B, 106:125424, Sep 2022.

[34] C. Castellani, C. DiCastro, P. A. Lee, M. Ma, S. Sorella, and E. Tabet. Enhancement of the spin susceptibility in disordered interacting electrons and the metal-insulator transition. Phys. Rev. B, 33:6169–6176, May 1986.

[35] A.A. Abrikosov, L.P. Gorkov, I.E. Dzyaloshinski, and R.A. Silverman. Methods of Quantum Field Theory in Statistical Physics. Dover Books on Physics. Dover Publications, 2012.

[36] M A Baranov, A M M Pruisken, and B Škoric. The problem of coulomb interactions in the theory of the quantum hall effect. Physics-Uspekhi, 44(10S):44, oct 2001.

[37] D.J. Amit and V. Martin-mayor. Field Theory, The Renormalization Group, And Critical Phenomena: Graphs To Computers (3rd Edition). World Scientific Publishing Company, 2005.